

Title: Summer Undergrad 2020 - Symmetries (A) - Lecture 2

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Collection: Summer Undergrad 2020 - Symmetries

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Abstract: Continuous and discrete symmetries, infinitesimal symmetries



Today:

- group theory (continued)
- Discrete, continuous, infinitesimal symmetries
- Symmetries of an action





Quotient group and isomorphism theorem

Definition (normal subgroup)

Let G be a group. A subgroup $N \leq G$ is called normal if

$$gng^{-1} \in N, \quad \forall g \in G, \quad \forall n \in N.$$

The notation $N \trianglelefteq G$ is commonly used to indicate that N is a normal subgroup of G .

Definition (quotient group)

Let N be a normal subgroup of a group G . We can define an equivalence relation on G as

$$g \sim h \iff h^{-1}g \in N,$$

with equivalence classes

$$[g] = \{h \in G \mid h^{-1}g \in N\}.$$

The quotient group G/N (pronounced "G mod N") is the set of equivalence classes

$$G/N = \{[g] \mid g \in G\}$$

which is made into a group by defining

$$[g][h] = [gh], \quad [g]^{-1} = [g^{-1}], \quad e_{G/N} = [e_G].$$

closed under conjugation

$\Rightarrow g \sim h$ if $\exists a \in N$ st. $h = ga$

can go from g to h by multiplying with something in N

- we did not use the fact that N is normal

need to make sure that $[gh] = [g][h]$ does not depend on representatives

\Rightarrow is ok if N normal



Exercise

Show that the $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ is a normal subgroup of $(\mathbb{Z}, +)$ and that $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

$$\begin{array}{l} \tilde{h} \sim h \\ \tilde{h}^{-1}h \in N \end{array} \quad \begin{array}{l} \tilde{g} \sim g \\ \tilde{g}^{-1}g \in N \end{array}$$

$$[gh] = [\tilde{g}\tilde{h}] \Leftrightarrow gh \sim \tilde{g}\tilde{h} \Leftrightarrow (\tilde{g}\tilde{h})^{-1}gh \in N$$

$$\begin{aligned} \tilde{h}^{-1}\tilde{g}^{-1}gh &= \underbrace{\tilde{h}^{-1}h}_{\in N} \underbrace{h^{-1}\tilde{g}^{-1}gh}_{\in N} \in N \rightarrow \text{because } N \text{ is normal} \\ &\in N \end{aligned}$$



Theorem (first isomorphism theorem)

Let $\varphi : G \rightarrow H$ be a group homomorphism. Then:

- $\text{Im } \varphi$ is a subgroup of H
- $\ker \varphi$ is a normal subgroup of G
- $\text{Im } \varphi$ is isomorphic to the quotient group $G / \ker \varphi$

Exercise

Prove the first two points of the isomorphism theorem.

$$\Rightarrow \varphi(g) = \varphi(h)$$

$$\{g\} = \{h \in G \mid \varphi(h) = \varphi(g)\}$$

$$\boxed{\text{Im } \varphi \cong G / \ker \varphi}$$

$$N = \ker \varphi$$

$$\{g\} = ? = \{h \in G \mid h^{-1}g \in \ker \varphi\}$$

$$h^{-1}g \in \ker \varphi \Leftrightarrow \varphi(h^{-1}g) = e_H$$

$$\Leftrightarrow \varphi(h)^{-1}\varphi(g) = e_H$$



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Discrete vs continuous symmetries



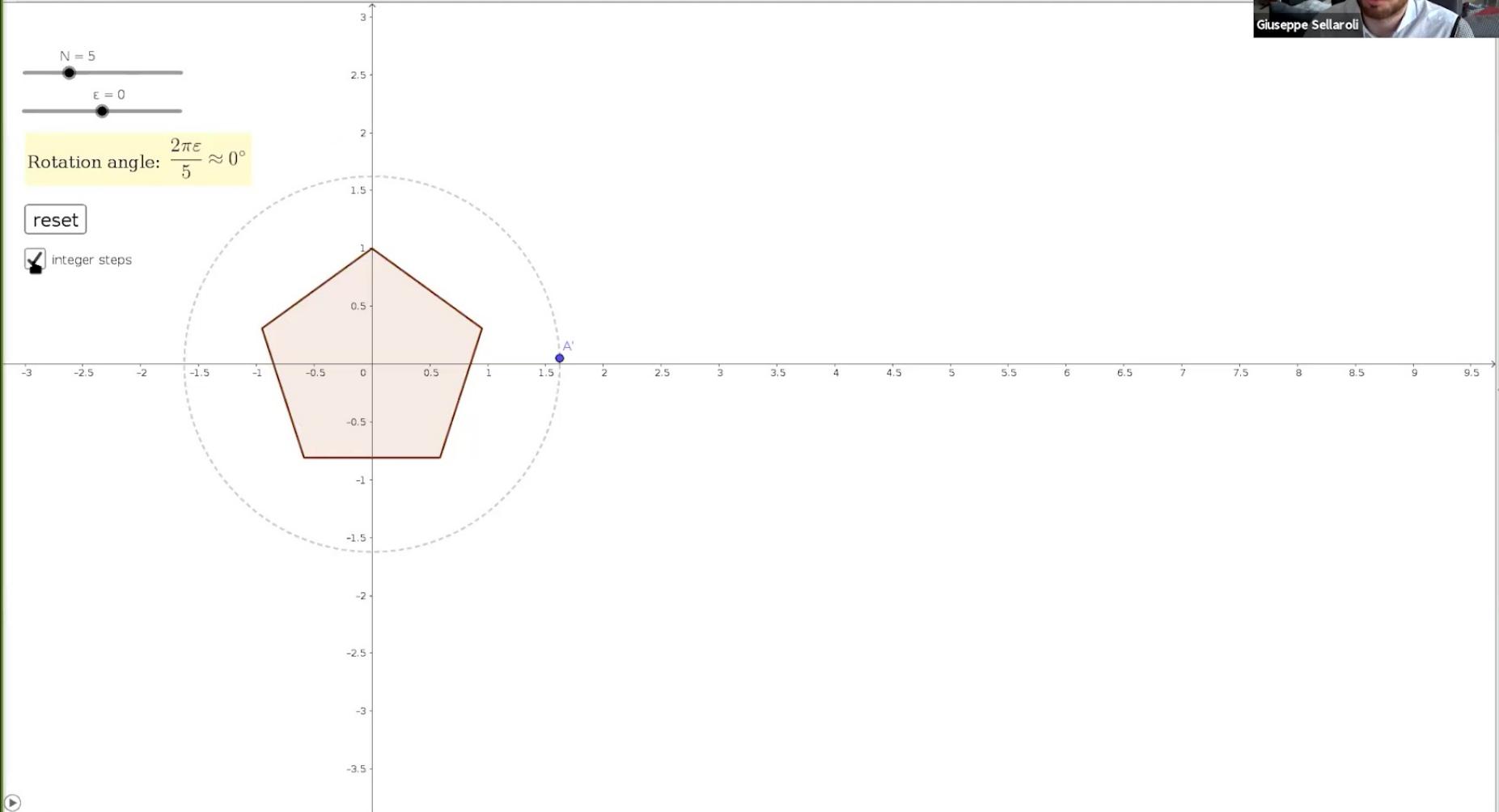


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N = 5
 ε = 0

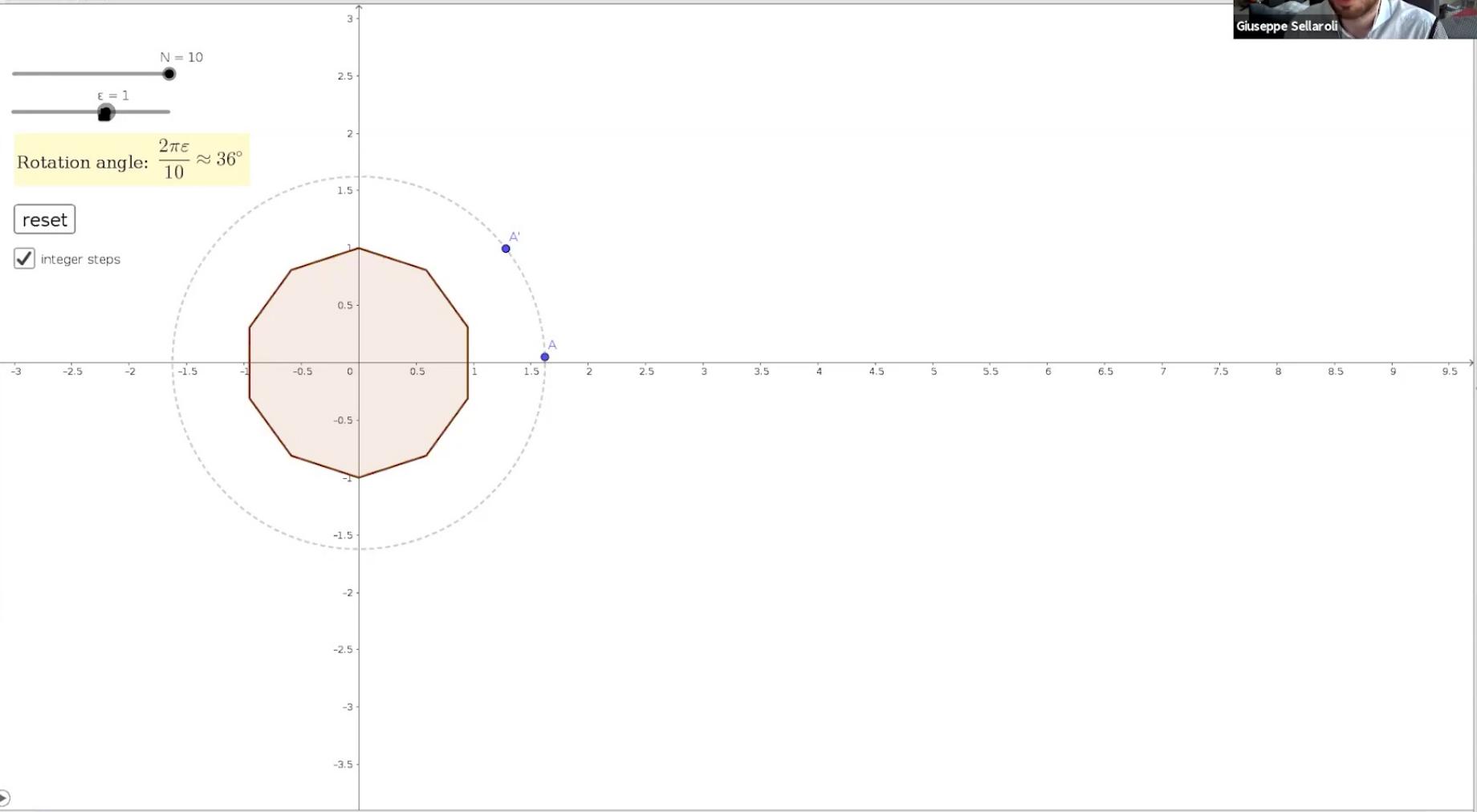
Rotation angle: $\frac{2\pi\epsilon}{5} \approx 0^\circ$

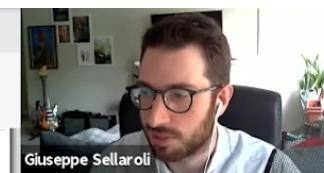
integer steps





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$$\mathbb{Z}_2 = \{[0], [1]\}$$

$$[0] = \{2k \mid k \in \mathbb{Z}\}$$

$$[1] = \{2k+1 \mid k \in \mathbb{Z}\}$$

$$[0] + [1] = [1] + [0] = [1]$$

$$[1] + [1] = [0]$$

$$[0] + [0] = [0]$$

$$2\mathbb{Z} = \{2k \mid k \in \mathbb{Z}\} \trianglelefteq \mathbb{Z}$$

$$\text{let } n = 2k \in 2\mathbb{Z}, \quad a \in \mathbb{Z}$$

$$a + n + (-a) = a - a + n = n \in 2\mathbb{Z}$$

$$2\mathbb{Z}/2\mathbb{Z} = \square$$

gn8⁻¹
↓



$$[1] + [1] = [2]$$

$$[0] + [0] = [0]$$

$$2\mathbb{Z} = \{2k \mid k \in \mathbb{Z}\} \trianglelefteq \mathbb{Z}$$

let $n = 2k \in 2\mathbb{Z}$, $a \in \mathbb{Z}$

ghg⁻¹



$$a + n + (-a) = a - a + n = n \in 2\mathbb{Z}$$

$$\mathbb{Z}/2\mathbb{Z} = \{[n] \mid n \in \mathbb{Z}\}$$

$$[n] + [m] = [n+m]$$

$$[n] = \{m \in \mathbb{Z} \mid m - n \in 2\mathbb{Z}\} = \{m \in \mathbb{Z} \mid m - n \text{ is even}\}$$

h
gh

only distinct ones: $[0] = \{2k \mid k \in \mathbb{Z}\}$

$$[1] = \{2k+1 \mid k \in \mathbb{Z}\}$$



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continuous if there is a continuous parameter for the group elements
 discrete otherwise

$\psi: \varepsilon \in (\mathbb{R}, +) \mapsto g_\varepsilon \in G$ (group homomorphism)

if there is a topology on G such that ψ is continuous

$\Rightarrow \{g_\varepsilon \mid \varepsilon \in \mathbb{R}\} = \text{Im } \psi$ is continuous

one-parameter subgroup

$g_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$

$g_0 = e_G$
 $g_{-\varepsilon} = g_\varepsilon^{-1}$
 $g_{\varepsilon+\varepsilon'} = g_\varepsilon g_{\varepsilon'}$ \Rightarrow one



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$\Rightarrow \{g_\varepsilon \mid \varepsilon \in \mathbb{R}\} = \text{Im } \psi$ is continuous

one-parameter subgroup

$$g_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$



$$g_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g_{\theta+\theta'} = g_\theta g_{\theta'}$$

$$g_{-\theta} = g_\theta^{-1}$$

$$g_0 = e_G$$

$$g_{-\varepsilon} = g_\varepsilon^{-1}$$

$\underline{g_{\varepsilon+\varepsilon'}} = g_\varepsilon g_{\varepsilon'}$ \Rightarrow one-parameter subgroup is abelian

infinitesimal

ensure that

$\varepsilon \in \mathbb{R} \mapsto g_\varepsilon \in G$ is "smooth"

as nice as we need it to be



(θ)

$$\delta_{\theta+\epsilon} = \delta_\theta \delta_\epsilon$$

$$\delta_{-\theta} = \delta_\theta^{-1}$$

infinitesimal

assume that

$\epsilon \in \mathbb{R} \mapsto \delta_\epsilon \in G$ is "smooth"

differentiable

as nice as
we need it
to be

linearise: $\delta_\epsilon = \delta_0 + \epsilon \frac{d}{d\epsilon} \delta_\epsilon \Big|_{\epsilon=0} + o(\epsilon^2)$

infinitesimal symmetry

inf symmetry: $e + \epsilon X + o(\epsilon^2)$

where $X = \frac{d}{d\epsilon} \Big|_{\epsilon=0} \delta_\epsilon$

derivative entry by entry

ex: $R_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$

$$\frac{d}{d\theta} \Big|_{\theta=0} R_\theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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A red-bordered box containing handwritten notes.

A white box with a thin black border.



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linearise: $\delta_{\varepsilon} = g_0 + \varepsilon \frac{d}{d\varepsilon} g_{\varepsilon} \Big|_{\varepsilon=0} + o(\varepsilon^2)$

infinitesimal symmetry

inf symmetry: $e + \varepsilon X + o(\varepsilon^2)$

where $X = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} g_{\varepsilon}$

derivative entry by entry

ex: $R_{\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\frac{d}{d\theta} \Big|_{\theta=0} R_{\theta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$R_{\theta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + o(\theta^2)$$

infinitesimal rotation