

Title: Summer Undergrad 2020 - Symmetries (A) - Lecture 1

Speakers: Giuseppe Sellaroli

Collection: Summer Undergrad 2020 - Symmetries

Date: May 25, 2020 - 2:00 PM

URL: <http://pirsa.org/20050045>

Abstract: Overview/definition of symmetry, elements of group theory



## Today's plan

### Lectures

- What is a symmetry?
- A crash course in group theory

### Activities

- GeoGebra
- Brainstorming
- Breakout rooms

**Note:** I will be writing on top of these slides. I'll send you a link to the blank slides for now and upload the pdf with my written notes later today.



Giuseppe



Assistant: Iván

The image shows a Zoom meeting window. At the top, there is a toolbar with various icons for navigation and interaction. The main content area displays a slide with the title "How to interact during the lectures" and a bulleted list of instructions. On the left side, a vertical pane shows a series of slide thumbnails. In the top right corner, a small video window shows the speaker, Giuseppe Sellaroli, wearing glasses and a patterned shirt.

## How to interact during the lectures

- This class is a safe (virtual) place. Questions are always welcome, no matter how trivial you may think they are.
- You can ask questions at any time during the lecture. You have a few options:
  - “Raise your hand” through Zoom and ask in person
  - Use the Zoom chat
  - Ask on sli.do (#W613)
- I will often ask you questions during the lectures. I **do not know** how well this is going to work online, but I’ll still do it.



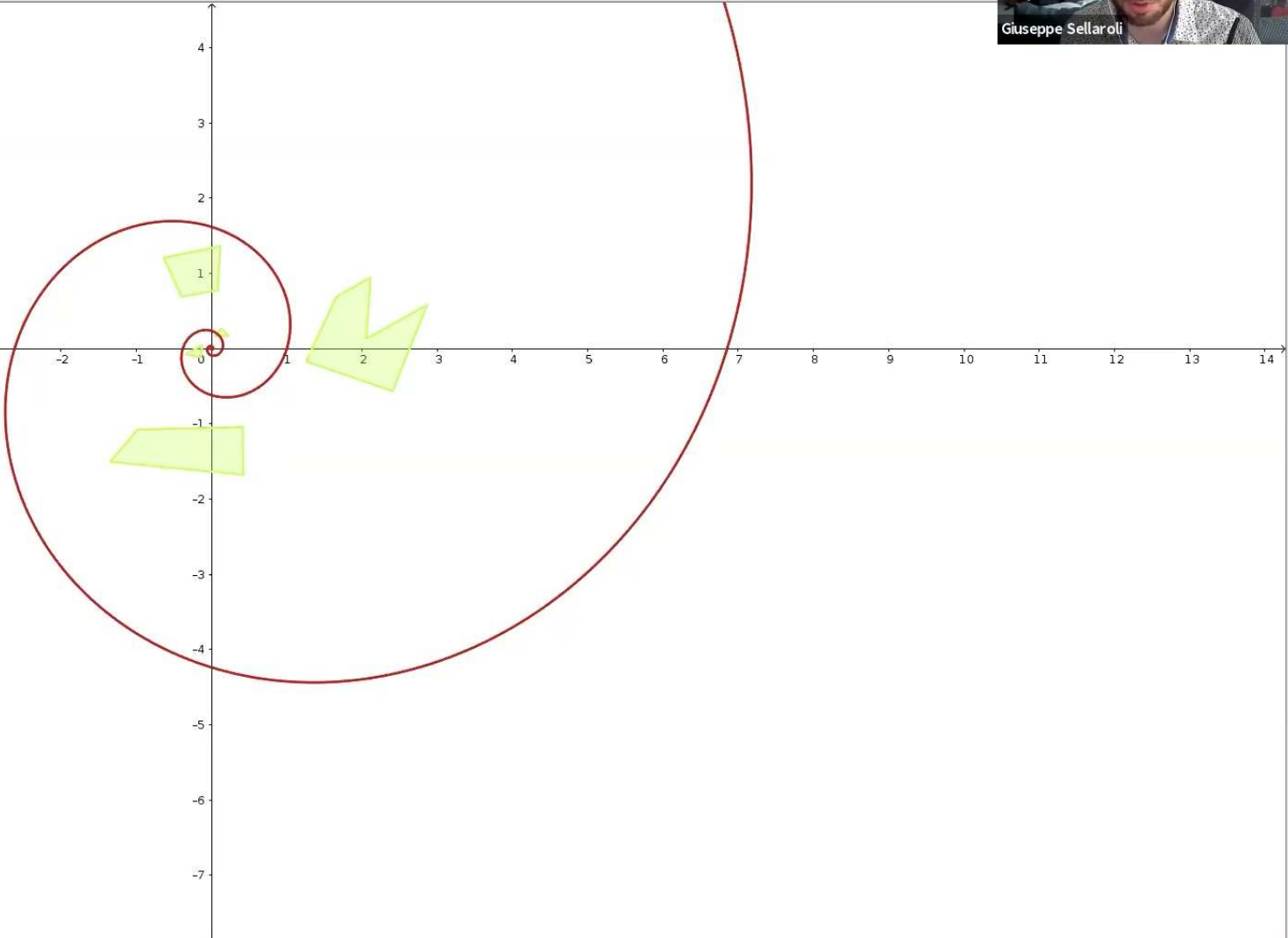
$$\begin{cases} x = e^{b\theta} \cos \theta \\ y = e^{b\theta} \sin \theta \\ \theta \in \mathbb{R} \end{cases}$$

$\varepsilon = 0$   
 $t = 0$

scale factor:  $e^{2\pi b\varepsilon} \sim 1$   
rotation by angle  $2\pi t \sim 0^\circ$

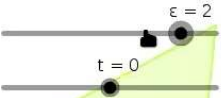
- show auxiliary objects
- link the sliders

reset





$$\begin{cases} x = e^{b\theta} \cos \theta \\ y = e^{b\theta} \sin \theta \\ \theta \in \mathbb{R} \end{cases}$$

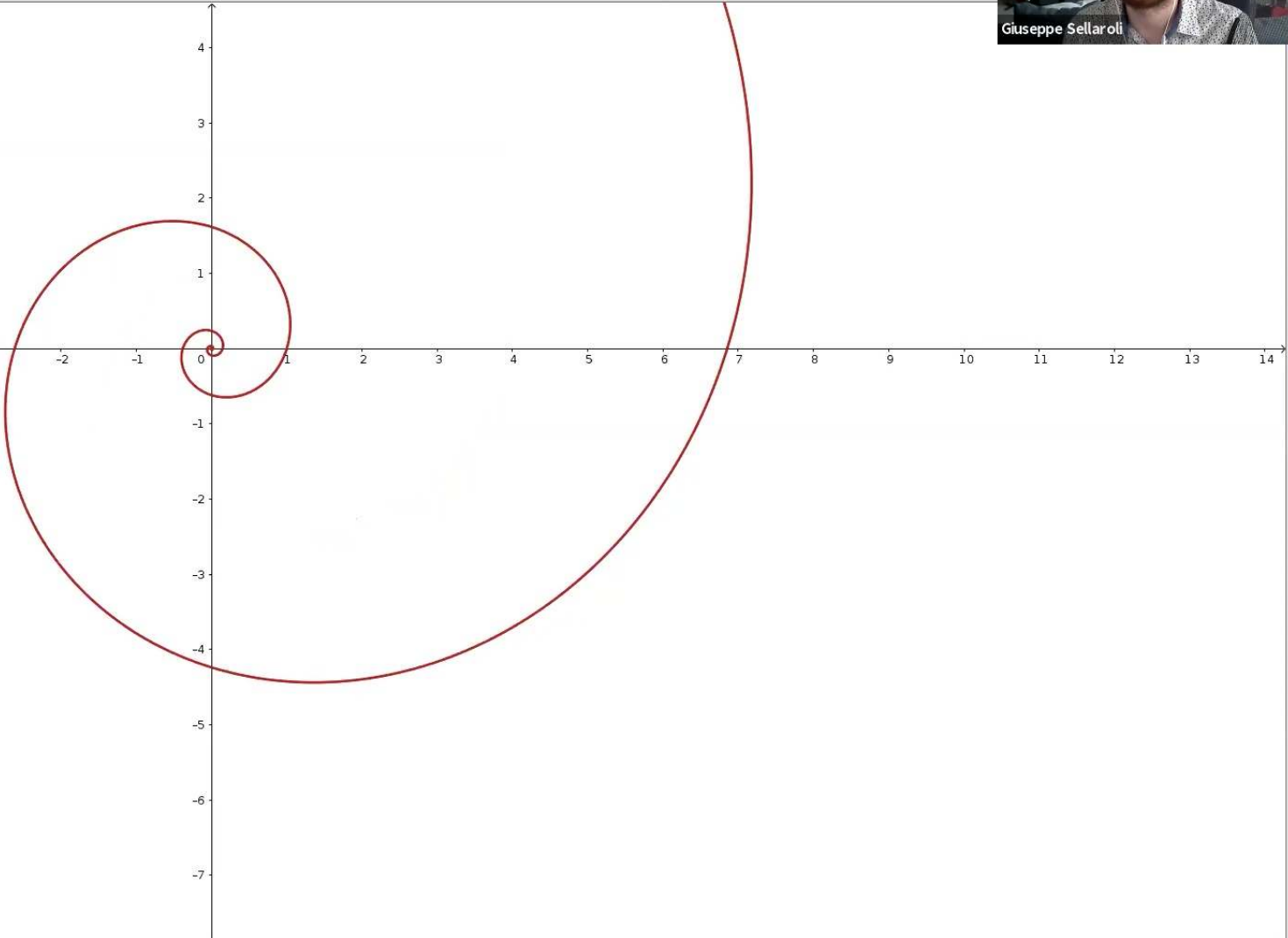


scale factor:  $e^{2\pi b\epsilon} \sim 46.98$   
rotation by angle  $2\pi t \sim 0^\circ$

show auxiliary objects

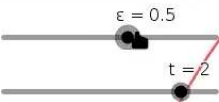
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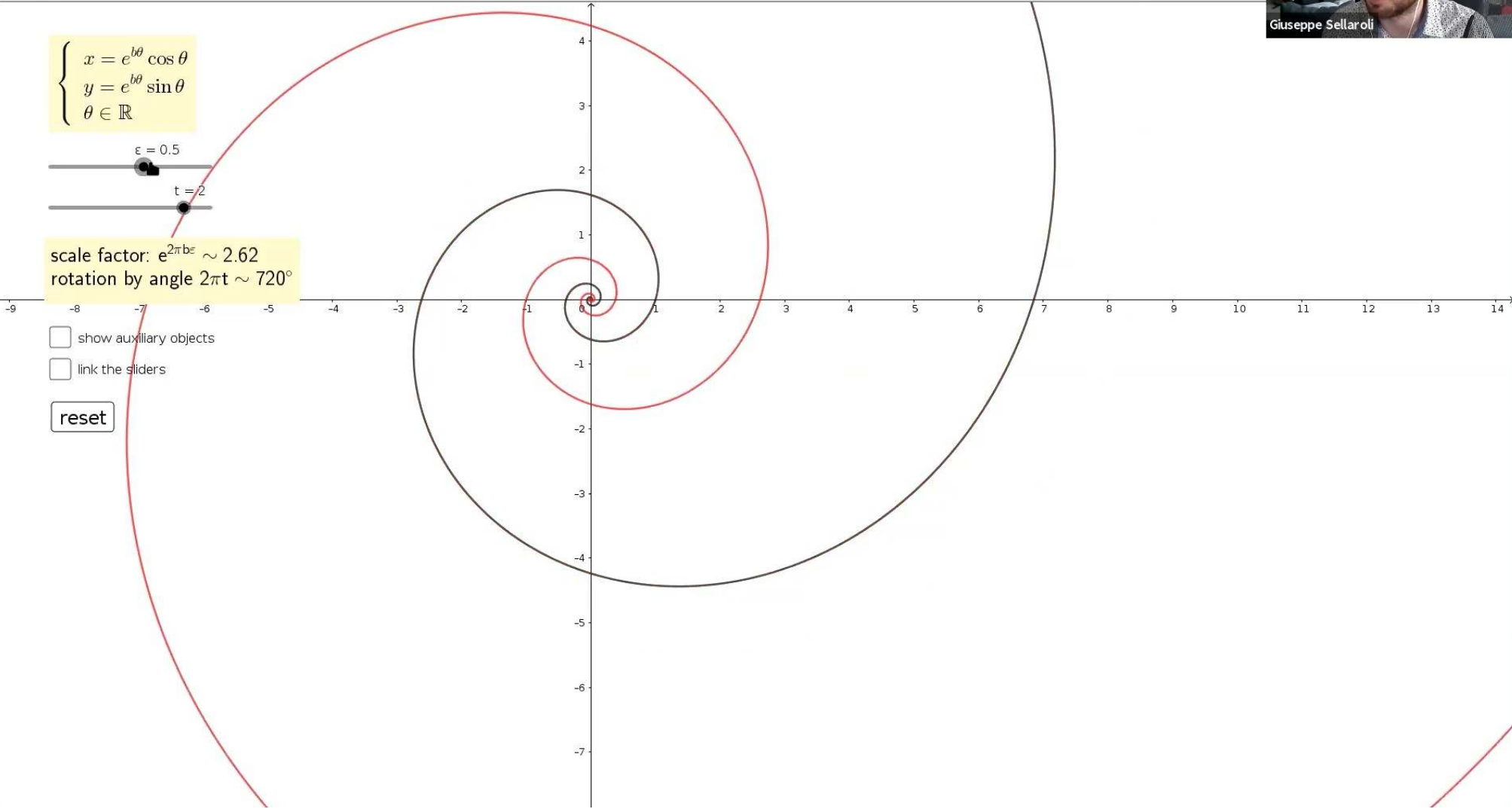
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scale factor:  $e^{2\pi b\epsilon} \sim 2.62$   
rotation by angle  $2\pi t \sim 720^\circ$

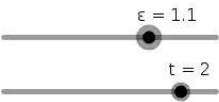
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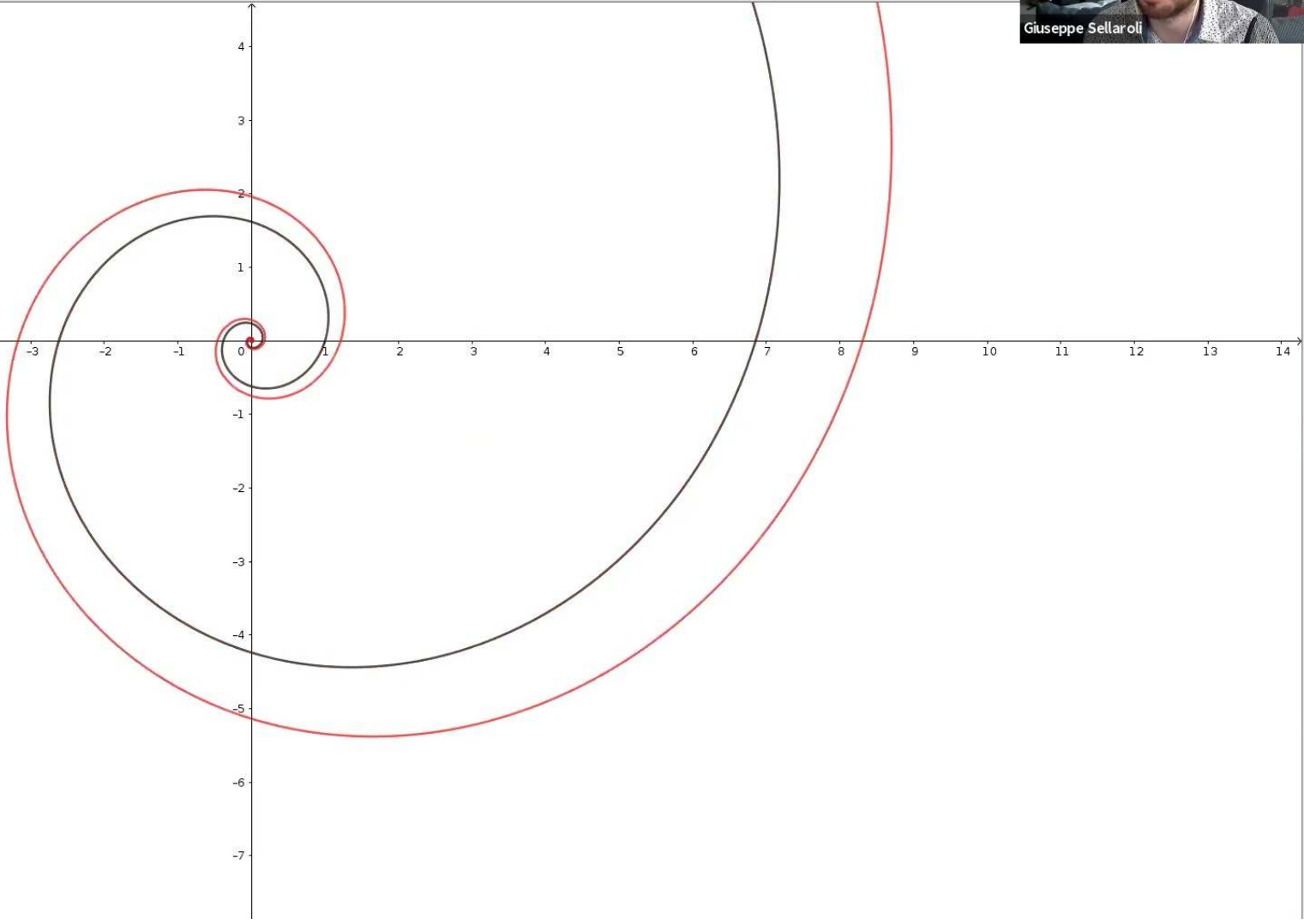
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scale factor:  $e^{2\pi b\epsilon} \sim 8.31$   
rotation by angle  $2\pi t \sim 720^\circ$

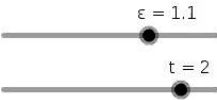
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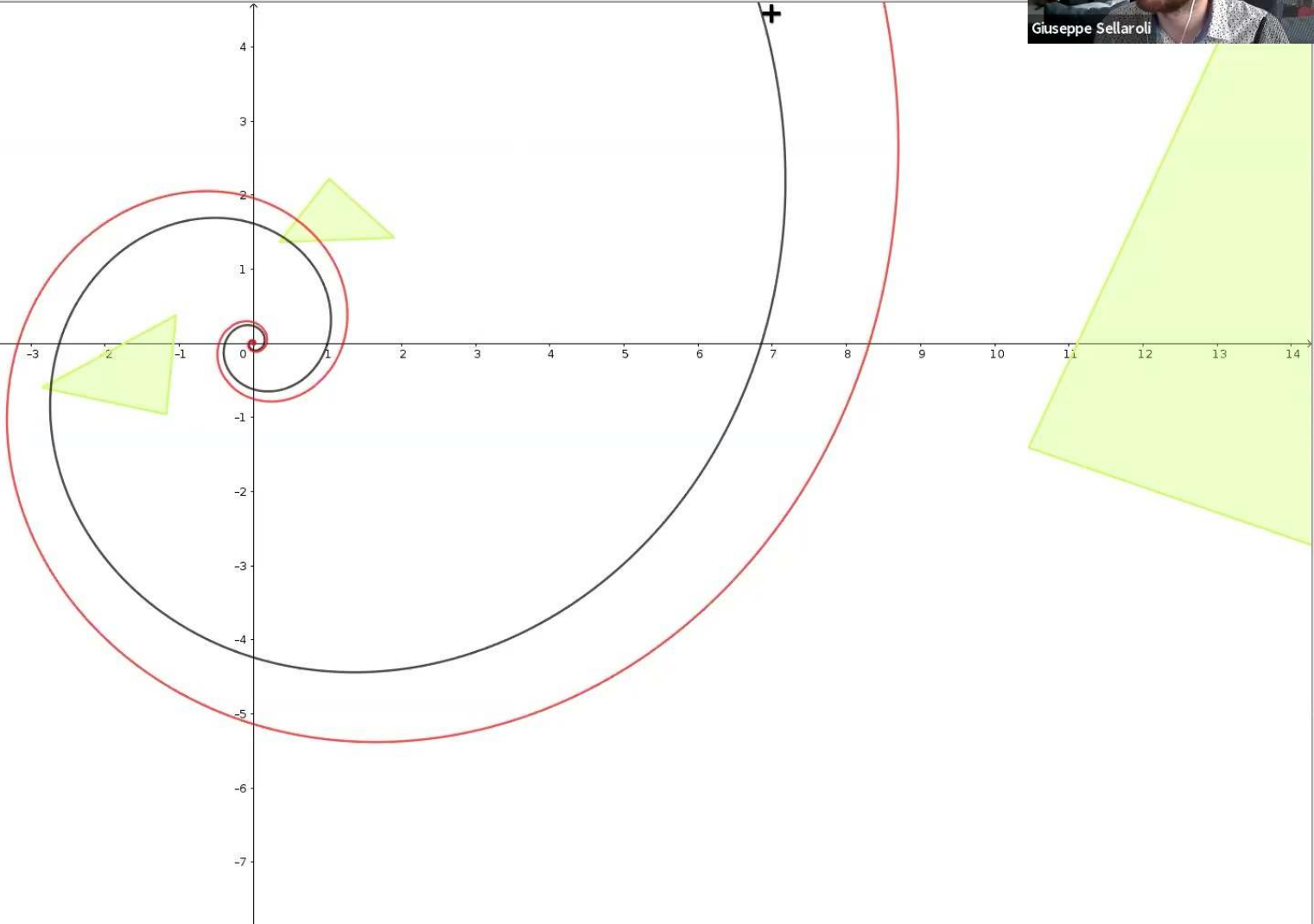
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reset



What is a symmetry?

a transformation of "A" that leaves "B" invariant

ex:  $B = \text{spiral}$   $A = \mathbb{R}^2$

ex:  $S = \int L dt$   $A = \text{time}$   $B = S$

- 
- "doing nothing" should be a symmetry (identity)
  - undo symmetries (invertible)
  - compose symmetries

## Groups and subgroups

### Definition (group)

A group is a set  $G$  together with an operation  $*$ :  $G \times G \rightarrow G$  satisfying the following properties:

- there is a special element  $e \in G$ , called the identity, such that

$$g * e = e * g = g, \quad \forall g \in G$$

- each element of  $G$  has an inverse, that is for each  $g \in G$  there is an element  $g^{-1} \in G$  such that

$$\underline{g^{-1}} * g = g * \underline{g^{-1}} = e$$

- the operation  $*$  is associative, that is

$$a * (b * c) = (a * b) * c, \quad \forall a, b, c \in G.$$

Additionally, we say that the group  $G$  is abelian or commutative if

$$\underline{a * b} = \underline{b * a}, \quad \forall a, b \in G.$$

composition of functions  
 $f \circ (g \circ h) = (f \circ g) \circ h$

Notation:

• we often use  $ab$  for  $a * b$

• technically the group is

$(G, *)$

↳ when there may be confusion

•  $(\mathbb{Z}, +)$ ,  $(\mathbb{R}, +)$ ,  $(\mathbb{C}, +)$  are abelian groups  
 $e = 0$      $a * b = a + b$      $a^{-1} = -a$

•  $(\mathbb{R} \setminus \{0\}, \cdot)$      $e = 1$      $a * b = ab$      $a^{-1} = \frac{1}{a}$

•  $n \times n$  matrices (invertible)     $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$   
with matrix multiplication     $e = \mathbb{1}_n$  (identity matrix)

• Circle group     $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

identify  $(x, y) \in S^1$  with a complex number  $z = x + iy$

then we use complex number multiplication

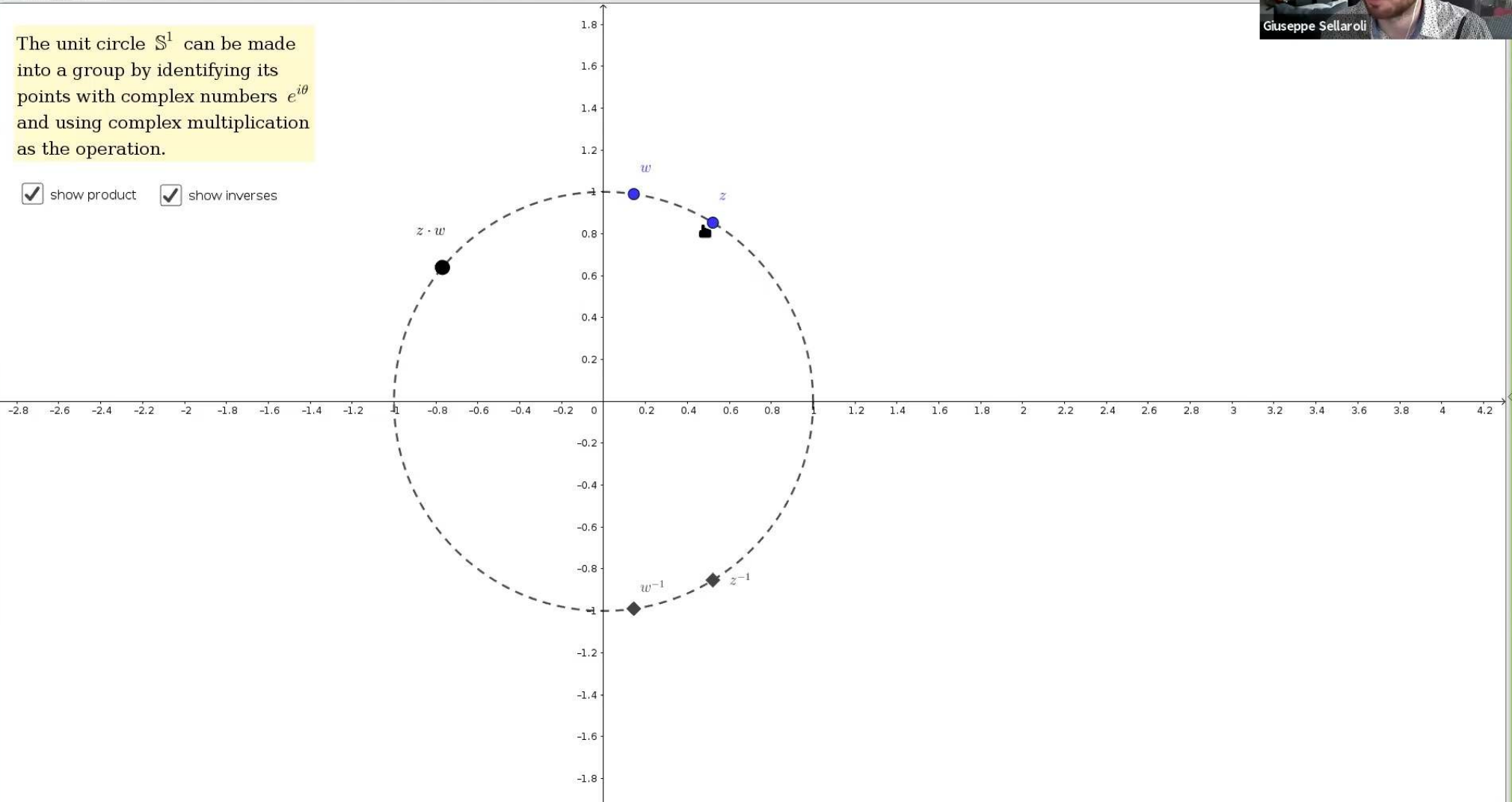
→ works because  $|zw| = |z||w| = 1$





The unit circle  $S^1$  can be made into a group by identifying its points with complex numbers  $e^{i\theta}$  and using complex multiplication as the operation.

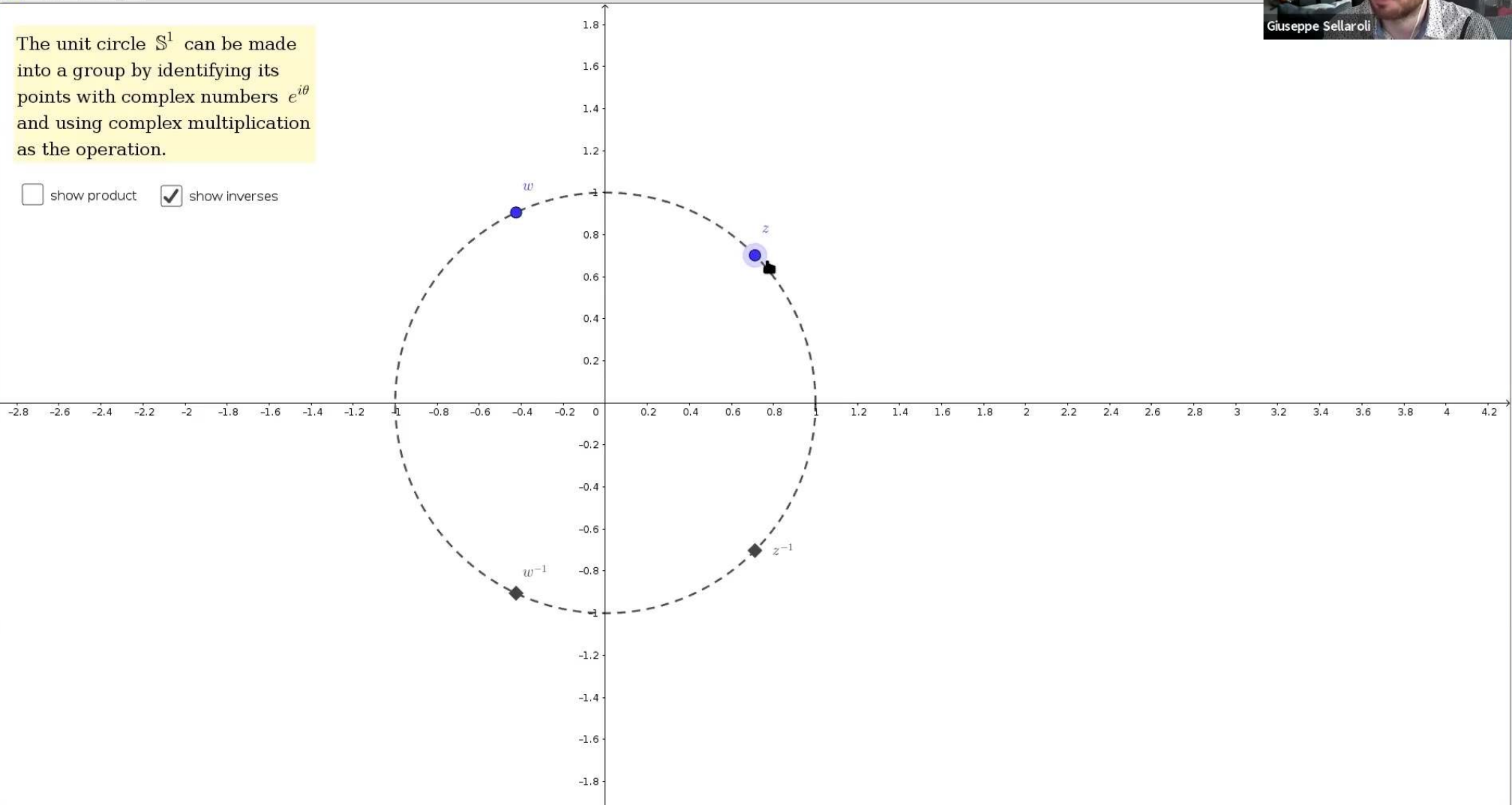
- show product
- show inverses





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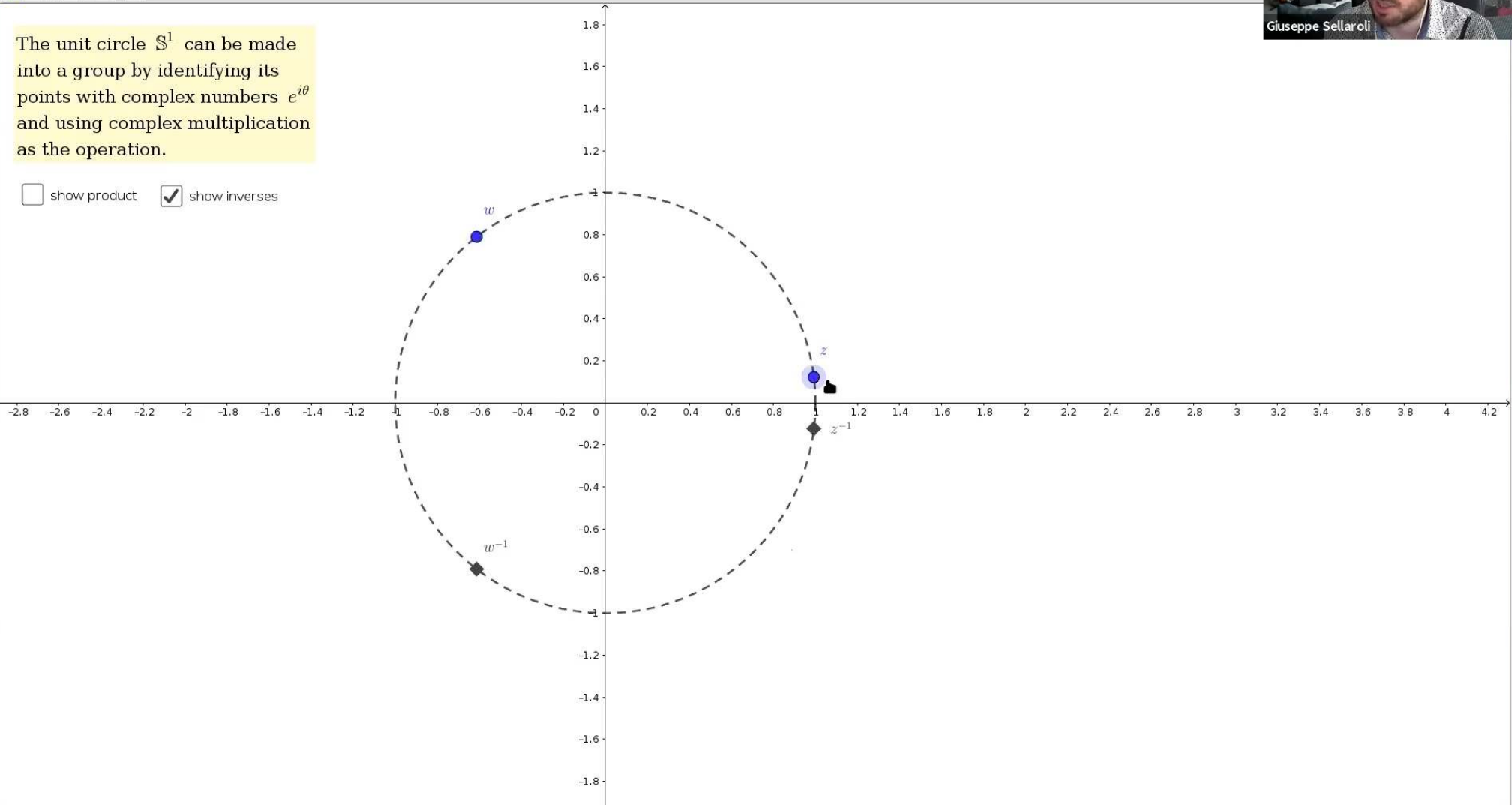
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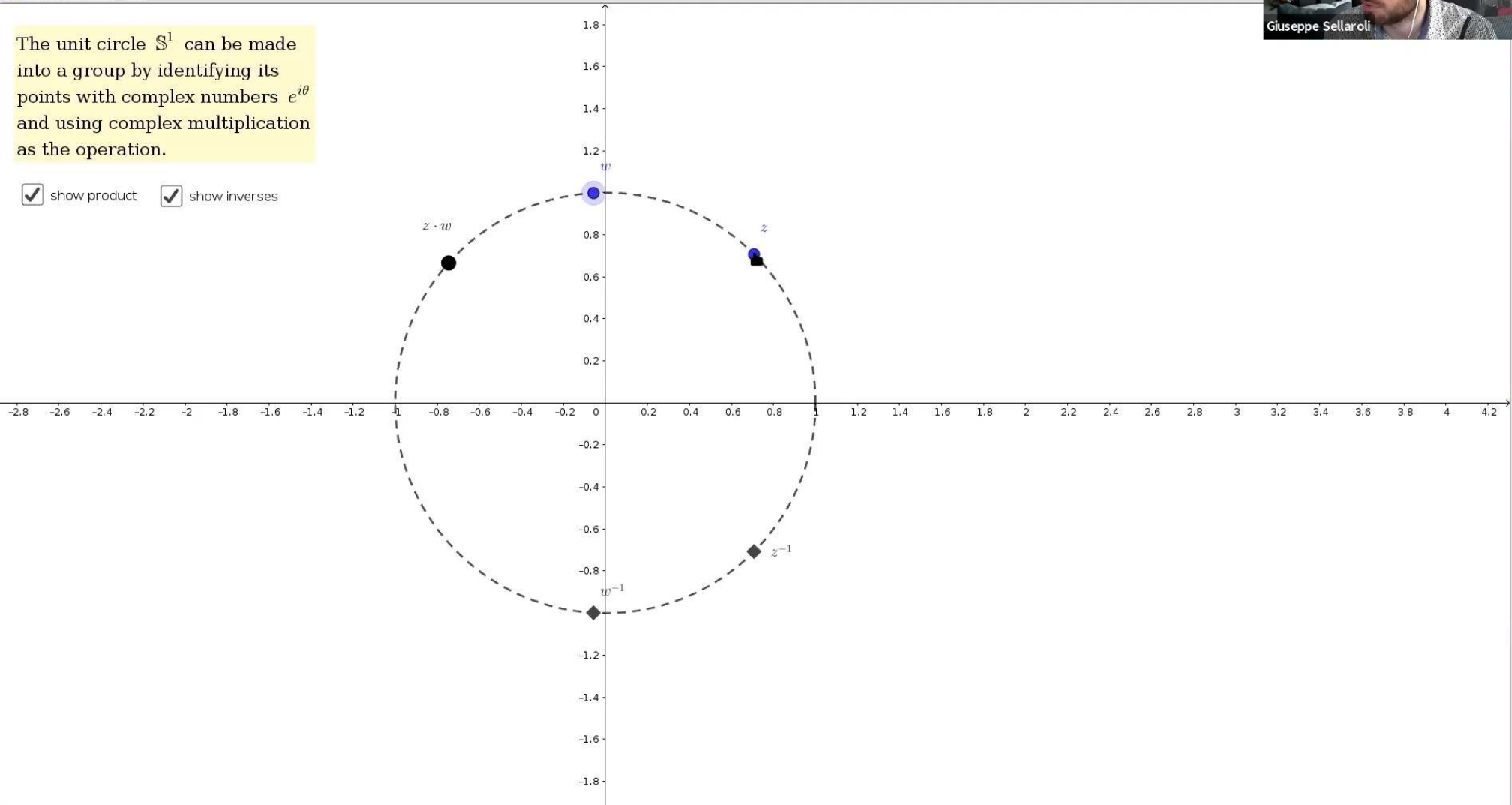
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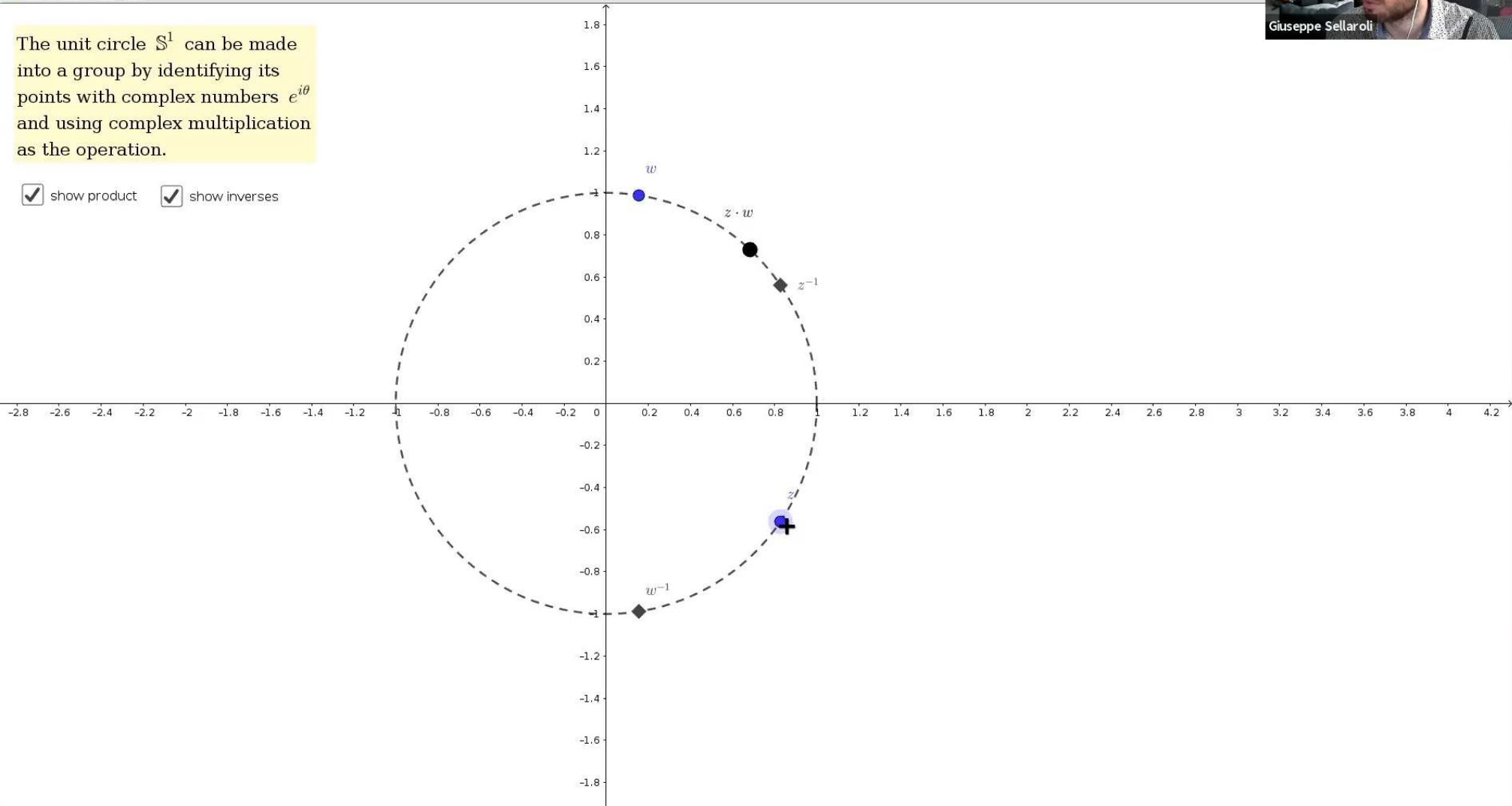
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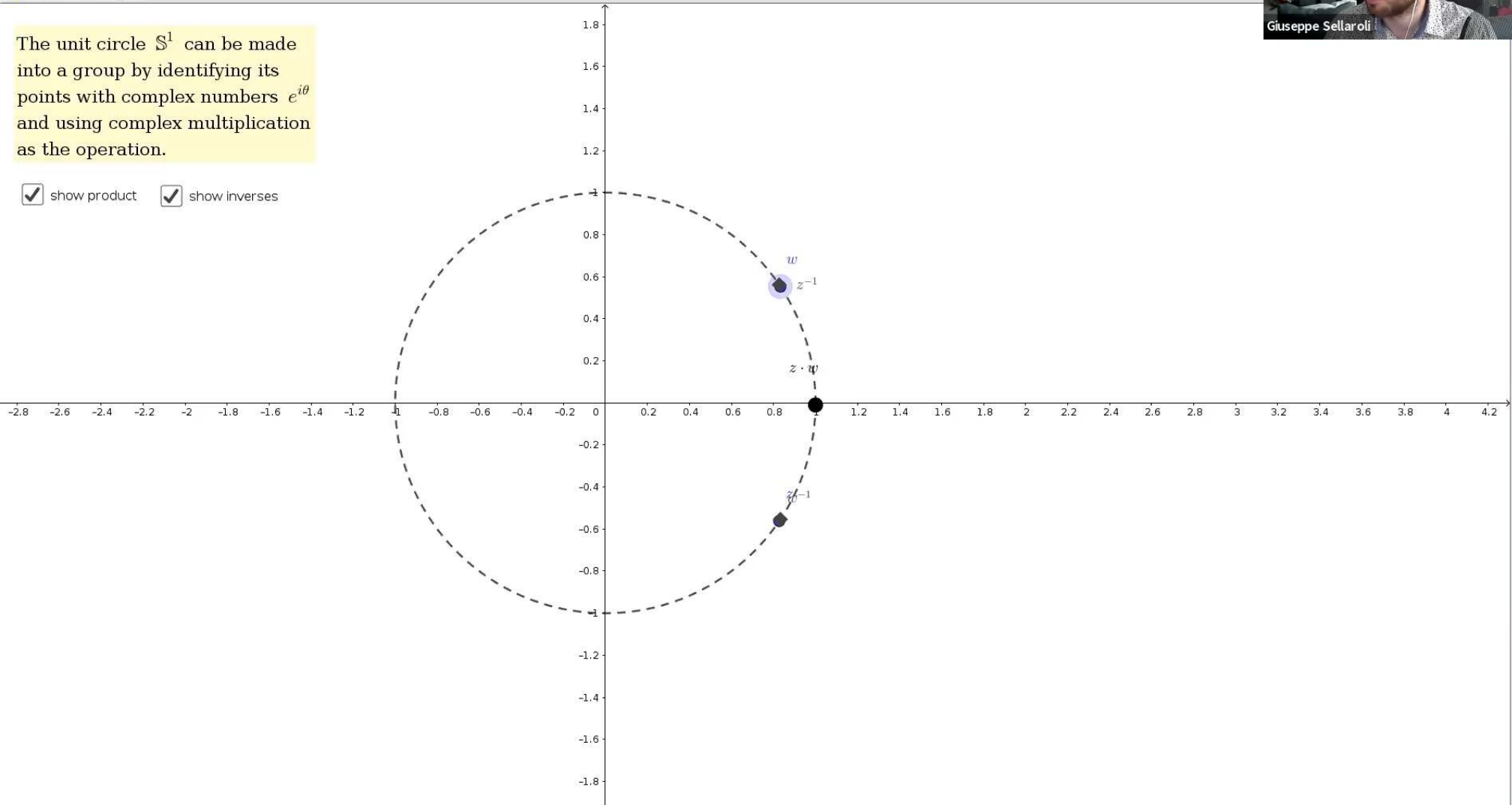
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### Definition (subgroup)

Let  $G$  be a group with operation  $*$ . A subgroup of  $G$  is a subset  $H \subseteq G$  that contains the identity element and is closed under the operation  $*$  and under inversion, that is

- $e \in H$
- $a * b \in H$  for all  $a, b \in H$
- $a \in H \implies a^{-1} \in H$

The notation  $H \leq G$  is commonly used to indicate that  $H$  is a subgroup of  $G$ .

### Exercise

Prove the following consequences of the definition of a group:

1. the identity element is unique (if two elements satisfy the identity property, they are necessarily equal)
2. for each  $g \in G$  the inverse  $g^{-1}$  is unique
3.  $(g^{-1})^{-1} = g$
4.  $(gh)^{-1} = h^{-1}g^{-1}$

↙ equiv. of subspace of  
vector space  
↓  
vector spaces ARE groups!

## Homomorphisms and isomorphisms

### Definition (group homomorphism)

A group homomorphism is a map  $\varphi : G \rightarrow H$  between two groups  $G$  and  $H$  such that

$$\varphi(a *_{G} b) = \varphi(a) *_{H} \varphi(b), \quad \forall a, b \in G.$$

### Exercise

Prove that if  $\varphi : G \rightarrow H$  is a group homomorphism, then

1.  $\varphi(e_G) = e_H$  (Hint: look at  $\varphi(e_G e_G)$ )
2.  $\varphi(g^{-1}) = \varphi(g)^{-1}$  for all  $g \in G$ .

← equiv to linear maps

→ multiply before or

after

$\varphi$  preserves group structure



### Definition (kernel)

The kernel of a group homomorphism  $\varphi : G \rightarrow H$  is the set

$$\ker \varphi = \{g \in G \mid \varphi(g) = e_H\}$$

of all the elements of  $G$  that are sent to the identity in  $H$ .

### Proposition

A group homomorphism  $\varphi : G \rightarrow H$  is injective if and only if its kernel is trivial, that is

$$\ker \varphi = \{\underline{e_G}\}.$$

⚡ Note:  $\varphi(e_G) = e_H$

$$e_G \in \ker \varphi$$

} same as vect. spaces

proof:  $(\Rightarrow)$  if  $\varphi$  injective, at most one thing can be sent to  $e_H$

$$\ker \varphi = \{e_G\}$$

$(\Leftarrow)$  let  $a, b \in G$  with  $\varphi(a) = \varphi(b) \Rightarrow e_H = \varphi(a)^{-1} \varphi(b)$

$\Rightarrow e_H = \varphi(a^{-1}) \varphi(b) = \varphi(a^{-1}b) \Rightarrow a^{-1}b \in \ker \varphi \Rightarrow a^{-1}b = e_H \Rightarrow a = b$

### Definition (isomorphism)

Two groups  $G$  and  $H$  are *isomorphic* (denoted by  $G \cong H$ ) if there exists an invertible group homomorphism  $\varphi : G \rightarrow H$ . Such a map is called an *isomorphism* between  $G$  and  $H$ .

### Exercise

Prove that  $\mathbb{Z}_2$  is isomorphic to the subgroup  $\{\mathbb{I}_n, -\mathbb{I}_n\} \leq \text{GL}(n, \mathbb{C})$ . While you are at it, prove that the latter is indeed a subgroup!

### Exercise

I'll do you one better: prove that any group with only two elements is isomorphic to  $\mathbb{Z}_2$ .

← same as v. spele version

Consider  $(\mathbb{R}, +)$  and  $(\mathbb{R}_{>0}, \cdot)$   
↓  
subgroup of  
 $(\mathbb{R} \setminus \{0\}, \cdot)$

$$\exp: x \in \mathbb{R} \rightarrow e^x \in \mathbb{R}_{>0}$$

- invertible
- $\exp(x+y) = \exp(x) \exp(y)$   
→ group homomorphism

$$(\mathbb{R}, +) \cong (\mathbb{R}_{>0}, \cdot)$$