Title: Conformal Geometry of Null Infinity, including gravitational waves

Speakers: Yannick Herfray

Series: Quantum Gravity

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Abstract: Since the seminal work of Penrose, it has been understood that conformal compactifications (or "asymptotic simplicity") is the geometrical framework underlying Bondi-Sachs' description of asymptotically flat space-times as an asymptotic expansion. From this point of view the asymptotic boundary, a.k.a "null-infinity", naturally is a conformal null (i.e degenerate) manifold. In particular, "Weyl rescaling" of null-infinity should be understood as gauge transformations. As far as gravitational waves are concerned, it has been well advertised by Ashtekar that if one work with a fixed representative for the conformal metric, gravitational radiations can be neatly parametrized as a choice of "equivalence class of metric-compatible connections". This nice intrinsic description however amounts to working in a fixed gauge and, what is more, the presence of equivalence class tend to make this point of view tedious to work with.

I will review these well-known facts and show how modern methods in conformal geometry (namely tractor calculus) can be adapted to the degenerate conformal geometry of null-infinity to encode the presence of gravitational waves in a completely geometrical (gauge invariant) way: Ashtekar's (equivalence class of) connections are proved to be in 1-1 correspondence with choices of (genuine) tractor connection, gravitational radiation is invariantly described by the tractor curvature and the degeneracy of gravity vacua correspond to the degeneracy of flat tractor connections. The whole construction is fully geometrical and manifestly conformally invariant."

Intrinsic conformal geometry of Null-Infinity

Yannick Herfray

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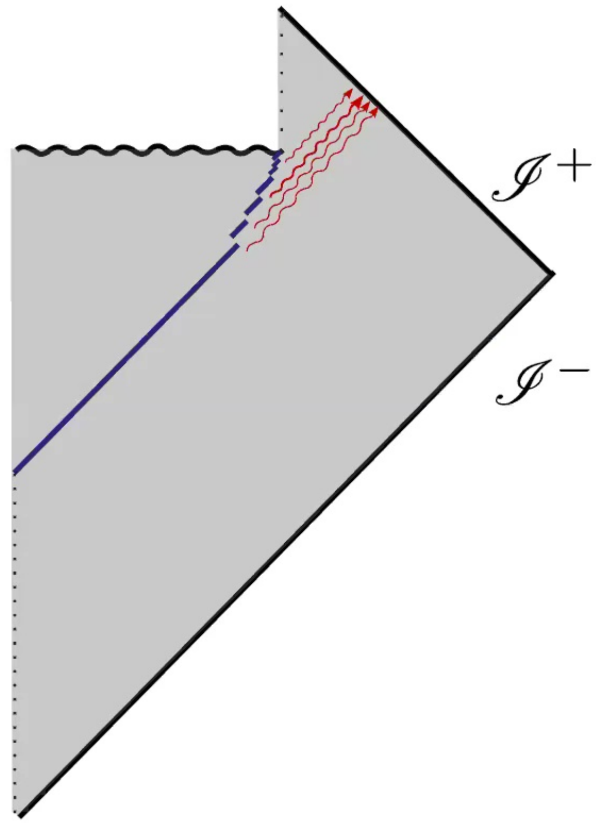
Based on arXiv:2001.01281

Black Hole Information Paradox



The ambiguity in the no-hair theorem, or How unique is Minkowski space?

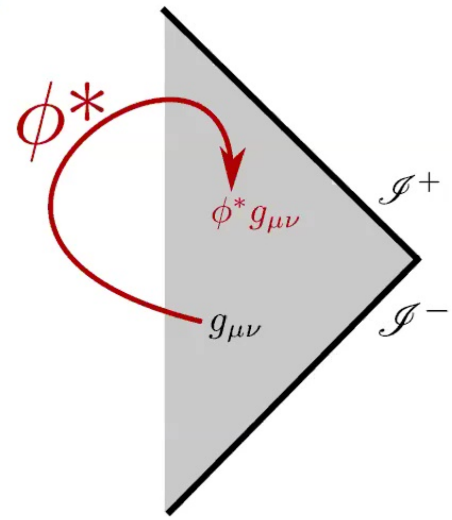
The "no-hair" theorem states that stationary black hole solutions must be diffeomorphic to Kerr space-times, however,

- Does the choice of Mass M and Angular momentum J uniquely defines a Kerr Space-time?

$\Rightarrow$ Not even true for Minkowski space ($M = 0$ and $J = 0$)

The ambiguity in the no-hair theorem, or How unique is Minkowski space?

Let this be "a" Minkowski space-time :



Is this unique?

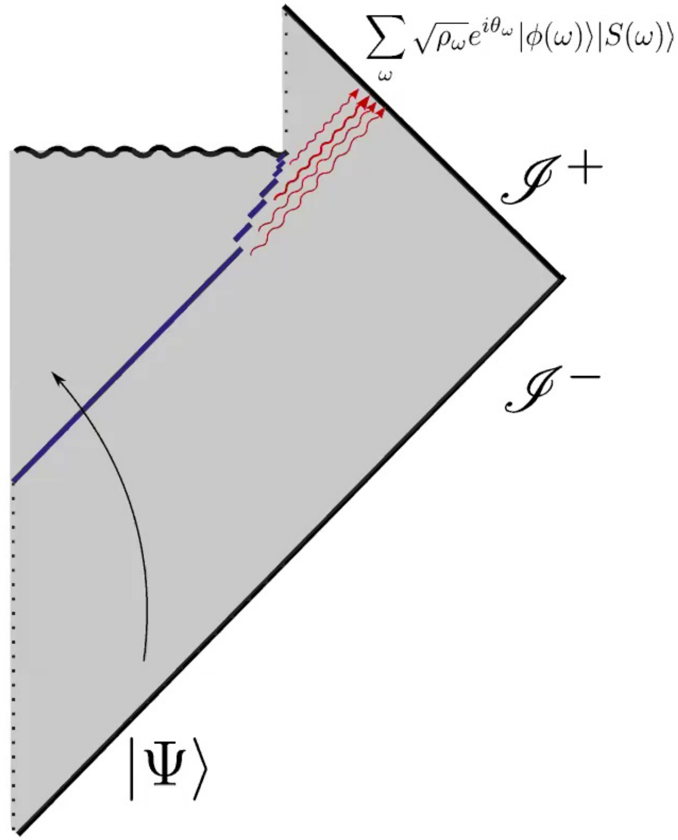
Surely no for any diffeomorphism ϕ will send such space-time $g_{\mu\nu}$ to another $\phi^* g_{\mu\nu}$.

What if we quotient by diffeomorphisms?

- quotienting by all diffeomorphisms will give you a unique remaining Minkowski space,
- quotienting only by diffeomorphisms fixing the conformal boundary $\phi\big|_{\mathscr{I}} = Id$ results in a moduli of "gravity vacua" Γ_0!

Black Hole Information Paradox revisited

(Hawking, Perry, Strominger)



Clearly, the space of “gravity vacua” is of utter importance.

I will today present a “fully geometrical” description of this moduli space.

“Fully geometrical” here means that it is

i) intrinsic at null-infinity (no reference to the bulk)

ii) coordinate-free

iii) manifestly conformally invariant

This results from a generalisation of the Tractor calculus from conformal geometry.

Asymptotically flat space-times and the tractor bundle of null-infinity

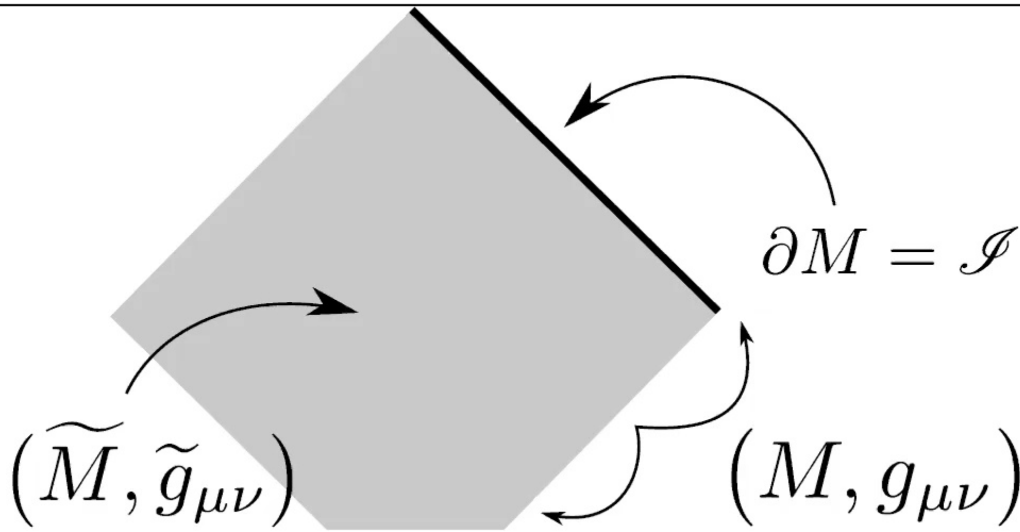
Asymptotically flat space-times

The space-time $\left(\widetilde{M}, \tilde{g}_{\mu\nu}\right)$ is **asymptotically simple** if there exists a space-time $(M, g_{\mu\nu})$ with boundary $\partial M = \mathscr{I}$ such that

- $\widetilde{M}$ is diffeomorphic to the interior $M \backslash \mathscr{I}$ of M
- there exists $\Omega \in C^\infty(M)$ a boundary defining function for $\mathscr{I}$ i.e

$$\Omega > 0 \text{ on } M, \qquad \Omega = 0,\ d\Omega \neq 0 \text{ on } \mathscr{I}$$

- $\tilde{g}_{\mu\nu} = \dfrac{1}{\Omega^2} g_{\mu\nu}$ on $\widetilde{M}$



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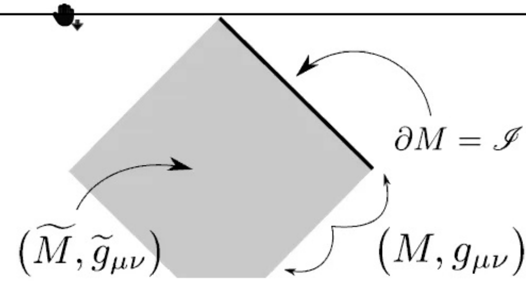
It is **asymptotically <u>flat</u> (resp AdS/dS)** if on top of this

- $\tilde{g}_{\mu\nu}$ is Einstein
- $g_{\mu\nu} n^{\mu} n^{\nu} = g^{\mu\nu}\left(d\Omega_{\mu}, d\Omega_{\nu}\right) = \underline{0}$ (resp ± 1) on $\mathscr{I}$



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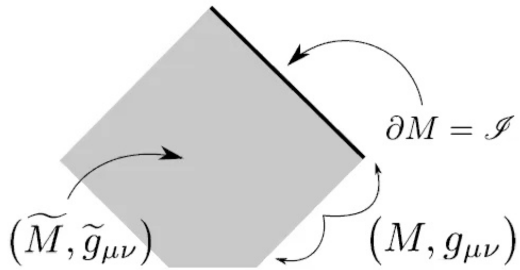
⚠ There is nothing unique about Ω nor $g_{\mu\nu}$! Rather one is working with an equivalence class:

$$(g_{\mu\nu}, \Omega) \sim \left(\lambda^2 g_{\mu\nu}, \lambda\Omega\right) \qquad \lambda \in C^\infty(M)$$

"Weak" or "zeroth order" structure of null-infinity

Let $(M, [g_{\mu\nu}], [\Omega])$ be an asymptotically flat space-time (in particular $\frac{1}{\Omega^2} g_{\mu\nu}$ is Einstein).



The "weak null-infinity structure" induced on the boundary $\mathscr{I}$ is

- a degenerate conformal metric $[h_{ab} \sim \lambda^2 h_{ab}]$ with one-dimensional kernel, obtained as
$$h_{ab} := g_{\mu\nu}\big|_{\mathscr{I}}$$
- an equivalence class of vector fields $\big[(n^a, h_{ab}) \sim (\lambda^{-1} n^a, \lambda^2 h_{ab})\big]$, obtained as
$$n^a := g^{\mu\nu} d\Omega_\nu \big|_{\mathscr{I}}$$
- with compatibility conditions $n^a h_{ab} = 0$ (following from $g^{\mu\nu} d\Omega_\mu d\Omega_\nu = 0$) and $\mathcal{L}_n h_{ab} \propto h_{ab}$ (following from Einstein equations).

"Universal" null-infinity structure

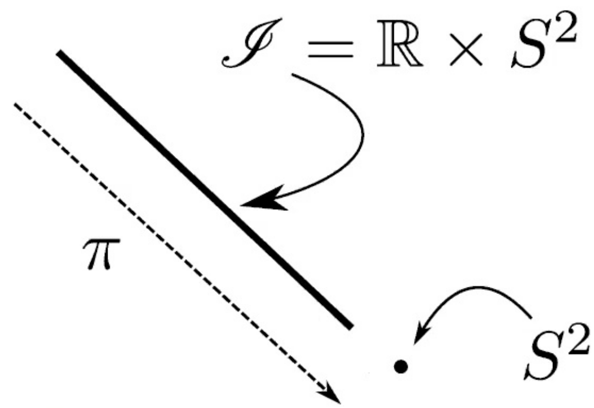
Let $\mathscr{I}$ be 3-dimensional manifold, we will say that it is equipped with the **universal null-infinity structure** if

- $\mathscr{I} = S^2 \times \mathbb{R}$ is the total space of a fibre bundle $\mathscr{I} \xrightarrow{\pi} S^2$

it is equipped with

- the conformal-sphere metric $[h^{(S^2)}_{AB}]$ on S^2
- an equivalence class $[n^a]$ of vertical vector fields $n^a d\pi_a = 0$

NB: then $h_{ab} = \pi^* h^{(S^2)}_{AB}$ automatically implies $n^a h_{ab} = 0$, $\mathcal{L}_n h_{ab} \propto h_{ab}$.



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Symmetry group

The group of diffeomorphism of $\mathscr{I}$ preserving the universal null-infinity structure is the BMS group:

$$BMS\,(4) = \mathcal{C}^\infty(S^2) \rtimes SO\,(3,1)$$

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The tractor bundle

The asymptotically simple geometry $(M, [g_{\mu\nu}], [\Omega])$ defines the null conformal geometry $(\mathscr{I}, [h_{ab}], [n^a])$.

This null geometry is enough to define a tractor bundle $\mathcal{T} \to \mathscr{I}$ at null-infinity.

<u>In a trivialisation,</u> a tractor $Y^I \in \mathcal{C}^\infty(\mathcal{T})$ can be written as:

$$\begin{pmatrix} Y^+ \\ Y^A \\ Y^- \\ Y^u \end{pmatrix}$$

with Y^+, $Y^- \in \mathcal{C}^\infty(\mathscr{I})$
$Y^A \partial_A + Y^u \partial_u \in \mathcal{C}^\infty(T\mathscr{I})$

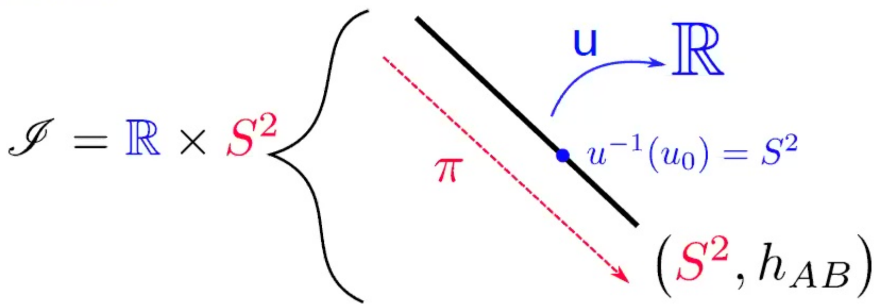
This is a 5-dimensional vector bundle over null infinity, canonically defined from $([h_{ab}], [n^a])$ and equipped with a degenerate metric :

$$Y^2 = 2Y^+Y^- + Y^AY^Bh_{AB}$$

and a preferred degenerate direction $I^I = (0, 0^A, 0, 1)$.

Trivialisations

Let $\left(\mathscr{I} \to S^2, [h_{ab}^{(S^2)}], [n^a]\right)$ be a manifold equipped with the universal null-infinity structure.



A **well-adapted trivialisation** (u, h_{AB}) is a choice

- of trivialisation $u : \mathscr{I} \to \mathbb{R}$ of $\mathscr{I} \xrightarrow{\pi} S^2$

$$(u, \pi) : \left| \begin{array}{ccc} \mathscr{I} & \to & \mathbb{R} \times S^2 \\ x & \mapsto & (u(x), \pi(x)) \end{array} \right.$$

- of representative $h_{AB} \in [h_{AB}^{(S^2)}]$
  (since $(n^a, h_{ab}) \sim (\lambda n^a, \lambda^2 h_{ab})$, this also gives a representative $n^a \in [n^a]$)

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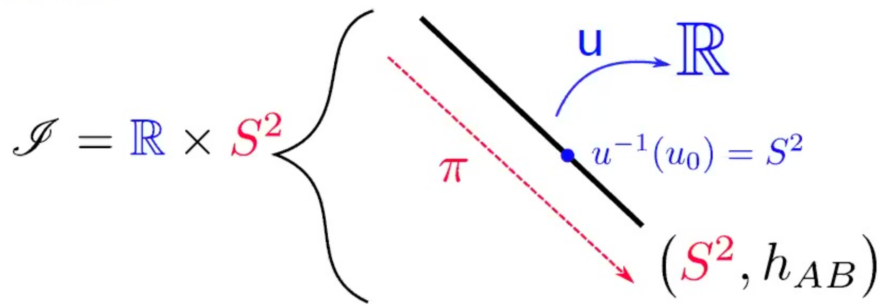

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- with compatibility condition $n^a du_a = 1$ (i.e "$n^a = \partial_u$")

Tractor "transformation rules"

A **well-adapted trivialisation** (h_{AB}, u) allows to represent a tractor field $Y^I \in \Gamma[\mathcal{T}]$ as

$$Y^I = \begin{pmatrix} Y^+ \\ Y^A \\ Y^- \\ Y^u. \end{pmatrix}$$

If $\left(\hat{h}_{AB} = \lambda^2 h_{AB}, \hat{u} = \lambda(u - \xi)\right)$ is any other well adapted trivialisation we have the transformation rules

$$\hat{Y}^I = P^I{}_J Y^J$$

where

$$P^I{}_J = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ \lambda^{-1}\Upsilon^A & \lambda^{-1}\delta^A{}_B & 0 & 0 \\ -\frac{\lambda^{-1}}{2}\Upsilon^2 & -\lambda^{-1}\Upsilon_B & \lambda^{-1} & 0 \\ \frac{1}{n-1}\left(\Delta\xi + (\Upsilon^2 - \nabla_C\Upsilon^C)(u-\xi)\right) & \Upsilon_B(u-\xi) - d\xi_B & 0 & 1 \end{pmatrix}$$

with $\Upsilon_A = \lambda^{-1} d_A \lambda$

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Gravitational radiation as a tractor connection at null-infinity

BMS coordinates

Choices of well-adapted trivialisation (u, h_{AB}) on $(\mathscr{I}, [h_{ab}], [n^a])$ are in one-to-one correspondence with BMS-coordinates on $(M, [g], [\Omega])$. These are local coordinates

$$(u, \Omega, \pi) \left| \begin{array}{ccc} M & \to & \mathbb{R} \times \mathbb{R} \times S^2 \\ x & \to & (u(x), \Omega(x), y^A(x)) \end{array} \right.$$

on a neighbourhood of $\mathscr{I}$ in M such that

$$\tilde{g}_{\mu\nu} = \frac{1}{\Omega^2} \left(2d\Omega du + h_{AB}(y) + \Omega C_{AB}(u, y) + \mathcal{O}\left(\Omega^2\right) \right)$$

Asymptotic shear and gravitational waves

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The "asymptotic shear" C_{AB} is known to encode the gravitational radiation reaching null-infinity.

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The "asymptotic shear" C_{AB} is known to encode the gravitational radiation reaching null-infinity.
$\Rightarrow$ this is however nothing like a tensor on $\mathscr{I}$!

Had we chosen another well-adapted trivialisation $\left(\hat{u} = \lambda\left(u - \xi\right), \hat{h}_{AB} = \lambda^2 h_{AB}\right)$ on $(\mathscr{I}, [h_{ab}], [n^a])$ with $\xi, \lambda \in \mathcal{C}^\infty\left(S^2\right)$ we would have

$$\begin{aligned} h_{AB} &\mapsto \hat{h}_{AB} = \lambda^2 h_{AB} \\ n^a &\mapsto \hat{n}^a = \lambda^{-1} n^a \\ C_{AB} &\mapsto \hat{C}_{AB} = \lambda C_{AB} - 2\lambda\left(\nabla_A \nabla_B\big|_0 \xi + \hat{u}\, \nabla_A \nabla_B\big|_0 \lambda^{-1}\right) \end{aligned}$$

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What is the (invariant) geometrical objects whose coordinates transform as the asymptotic shear?

Brief answer

The "asymptotic shear" C_{AB} parametrizes
a choice of "tractor connection" on $(M, [h_{AB}], [n^a])$.

$$d_b + A^I_b{}_J = d_b + \begin{pmatrix} 0 & -\theta_{bC} & 0 & 0 \\ -\xi_b{}^A & \Gamma_b{}^A{}_C & \theta_b{}^A & 0 \\ 0 & \xi_{bC} & 0 & 0 \\ -\psi_b & -\frac{1}{2}C_{bC} & du_b & 0 \end{pmatrix}$$

with

$$C_{bA} = C_{AB}\,\theta^B_b,$$

$$\xi_{bA} = \left(\frac{1}{2}\partial_u C_{AB} - \frac{R}{4}h_{AB}\right)\theta^B_b,$$

$$\psi_b = \frac{1}{4}R\,du_b - \frac{1}{2}\nabla^C C_{BC}\,\theta^B_b.$$

and transformation rules

$$A^I{}_J \mapsto \hat{A}^I{}_J,$$

$$\hat{A}^I{}_J = P^I{}_K A^K{}_L P^L{}_J - dP^I{}_K P_J{}^K$$

under change of well-adapted trivialisation

$$(h_{AB}, u) \mapsto \left(\hat{h}_{AB}, \hat{u}\right)$$

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What kind of object is this tractor connection ?

Brief answer,

> The tractor connection is a “gauge” connection
> for the Poincaré group $\mathrm{Iso}(3,1) = \mathbb{R}^4 \rtimes \mathrm{SO}(3,1)$

In a well-adapted trivialisation (u, h_{AB}) we have

$$D_b = d_b + \begin{pmatrix} 0 & -\theta_{bC} & 0 & 0 \\ -\xi_b{}^A & \Gamma_b{}^A{}_C & \theta_b{}^A & 0 \\ 0 & \xi_{bC} & 0 & 0 \\ -\psi_b & -\frac{1}{2}C_{bC} & du_b & 0 \end{pmatrix} \in \mathbb{R}^4 \rtimes SO(3,1)$$

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More precisely...

- the tractor bundle is a natural vector bundle canonically associated to conformal manifolds (here adapted to *degenerate* conformal manifolds)
- in conformal geometry the "normal" connection on this bundle is unique (for n$\geq$3)

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- in conformal geometry the "normal" connection on this bundle is unique (for n$\geq$3)
- however, for the degenerate conformal geometry of null-infinity, <u>"null-normal" connections</u> on the tractor bundle are not unique
- rather these null-normal tractor connections form an affine space modelled on trace-free symmetric tensor on S^2 (i.e "C_{AB}")
- **this is an invariant description** but choices of well-adapted trivialisation (u, h_{AB}) (equivalently BMS coordinates) acts as a trivialisation for this bundle, the tractor connection is then explicitly parametrized as a function of C_{AB}

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An enlightening comparison:

Gravitational radiation at null-infinity
vs
Maxwell theory on Minkowski space

Maxwell's equation on Minkowski space

Background: $(M = \mathbb{R}^4, g_{\mu\nu})$ where $g_{\mu\nu}$ is a flat metric.

Symmetry group: Poincaré group
(= subgroup of diffeomorphism preserving the background)

Well-adapted coordinates: 3+1 orthonormal splitting (t, x^i)
$\Rightarrow$ the Poincaré group sends a well-adapted set of coordinates to another.

Potential (in coordinates): (ϕ, A^i)

Field (in coordinates):
$$E^i = -(\nabla\phi)^i - \partial_t A^i$$
$$B^i = (\nabla \times A)^i$$

Field eqs (in coordinates):
$$\nabla . E = \rho$$
$$(\nabla \times B)^i - \partial_t E^i = j^i$$

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$$(\nabla \times B)^i - \partial_t E^i = j^i$$

$\Rightarrow$ Changing the set of adapted coordinates mixes the fields
$\Rightarrow$ This however preserve the "form" of Maxwell equations
$\Rightarrow$ If we fix a coordinate system, the Poincaré group takes solutions of the fields equations to others

Maxwell's equation on Minkowski space

Background: $(M = \mathbb{R}^4, g_{\mu\nu})$ where $g_{\mu\nu}$ is a flat metric.

Symmetry group: Poincaré group
(= subgroup of diffeomorphism preserving the background)

Well-adapted coordinates: 3+1 orthonormal splitting (t, x^i)
$\Rightarrow$ the Poincaré group sends a well-adapted set of coordinates to another.

Potential (invariant form) : a 1-form A_μ on M

Field (invariant form): $F_{\mu\nu} = (dA)_{\mu\nu}$

Field eqs (invariant form): $(d \star F)_{\mu\nu\rho} = J_{\mu\nu\rho}$

$\Rightarrow$ This is a "Poincaré invariant" point of view (i.e does not depend on the choice of adapted coordinates)
$\Rightarrow$ The Poincaré group takes solutions of the fields equations to others
$\Rightarrow$ Gives a "4D-type" of intuition, allows to easily construct invariants, suggest Yang-Mills as generalisation, etc

Gravitational radiations at Null-infinity

Background: $\left(\mathscr{I} = \mathbb{R} \times S^2, [h_{AB}], [n^a]\right)$, i.e "universal null-infinity structure".

Symmetry group: BMS group, $BMS(3) = C^\infty\left(S^2\right) \rtimes \mathrm{SO}(3,1)$
(= subgroup of diffeomorphism preserving the background)

Well-adapted coordinates: (u, h_{AB})
$\Rightarrow$ the BMS group sends a well-adapted set of coordinates to another.

Potential (in coordinates): C_{AB}

Field (in coordinates): ψ_4,ψ_3,$Im\left(\psi_2\right)$

Field eqs (in coordinates): ψ_4,ψ_3,$Im\left(\psi_2\right)$ are choices of outcoming gravitational radiations

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Potential (invariant form) : a tractor connection $D = d + A^I{}_J$ on M

Field (invariant form): $F^I{}_J = dA^I{}_J + A^I{}_K \wedge A^K{}_J$

Field eqs (invariant form): The curvature encodes the outcoming gravitational radiations "$F^I{}_J = J^I{}_J$"

$\Rightarrow$ This is a “BMS invariant” point of view (i.e does not depend on the choice of well-adapted coordinates)
$\Rightarrow$ The BMS group takes solutions of the fields equations to others
$\Rightarrow$ Gives a “conformally invariant” type of intuition, allows to easily construct invariants etc

Gravity vacua

The presence of gravitational wave at null-infinity is encoded in the curvature of the tractor connection.

The space Γ_0 of "gravity vaccua" is therefore the space of

flat tractor connections.

This space isn't a point, rather the BMS group act transitively on it with stabilisers isomorphic to the Poincaré group:

$$\Gamma_0 = {}^{BMS}\!\big/_{\mathrm{Iso}\,(3,1)}$$

Therefore the "gravity vacuum", Minkowski space, is not unique but rather we have a space of "gravity vacua" corresponding to all the possible flat tractor connections.

More on the geometrical nature of gravity vacua

What kind of object is this tractor connection ?

Brief answer,

> The tractor connection is a “gauge” connection
> for the Poincaré group $\mathrm{Iso}(3,1) = \mathbb{R}^4 \rtimes \mathrm{SO}(3,1)$

In a well-adapted trivialisation (u, h_{AB}) we have

$$D_b = d_b + \begin{pmatrix} 0 & -\theta_{bC} & 0 & 0 \\ -\xi_b{}^A & \Gamma_b{}^A{}_C & \theta_b{}^A & 0 \\ 0 & \xi_{bC} & 0 & 0 \\ -\psi_b & -\frac{1}{2}C_{bC} & du_b & 0 \end{pmatrix} \in \mathbb{R}^4 \rtimes SO(3,1)$$

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Why is the Poincaré group showing up here ?

- We all know that Minkowski space $\mathbb{M}_4$ is an homogenous space for the Poincaré group,

$$\mathbb{M}_4 = {}^{\mathrm{Iso}(3,1)}\!\big/_{\mathrm{SO}(3,1)}$$

- A lesser known fact is that the conformal boundary $\mathscr{I}_{flat}$ of this homogeneous space is also an homogeneous space for the Poincaré group,

$$\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\!\big/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$$

Wait a minute...

we said that the symmetry group of the "universal null-infinity structure" $\left(\mathscr{I} \to S^2, [h_{AB}^{(S^2)}], [n^a]\right)$ is the (infinite dimensional) BMS group,

...now we are saying that the conformal boundary $\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\!/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$ of Minkowski space $\mathbb{M}_4 = {}^{\mathrm{Iso}(3,1)}\!/_{\mathrm{SO}(3,1)}$ comes with a preferred action of the Poincaré group.

How does these two facts stick together?

The conformal boundary $\mathscr{I}_{flat}$ of Minkowski space $\mathbb{M}_4$ comes equipped with more than the universal structure, it is equipped with a set $\{s : S^2 \to \mathscr{I}\}_{s \in \mathcal{H}}$ of *good cuts* on top of the universal structure

= "a choice of gravity vacua".

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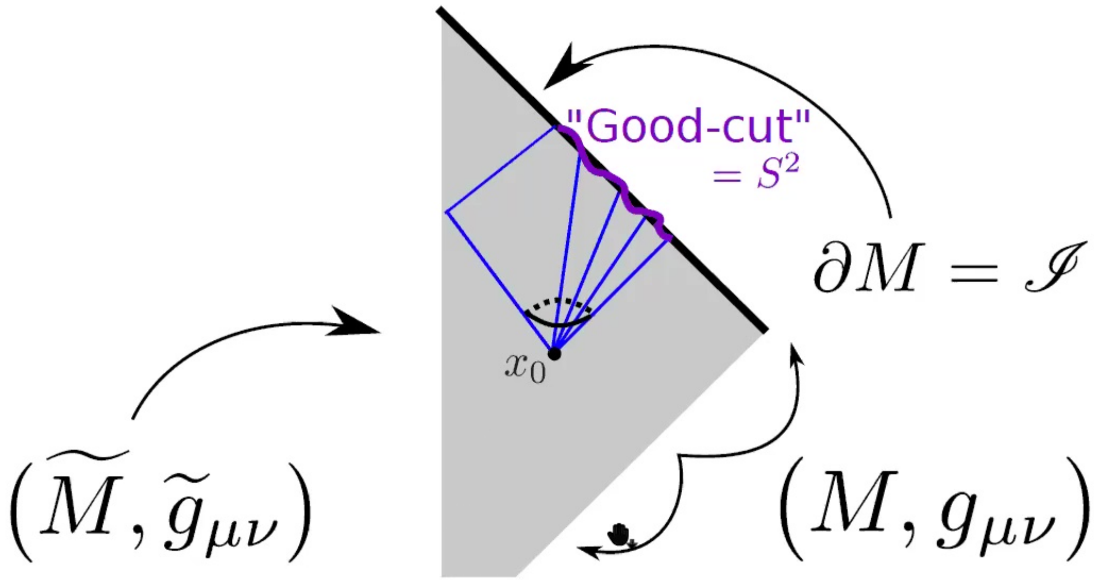

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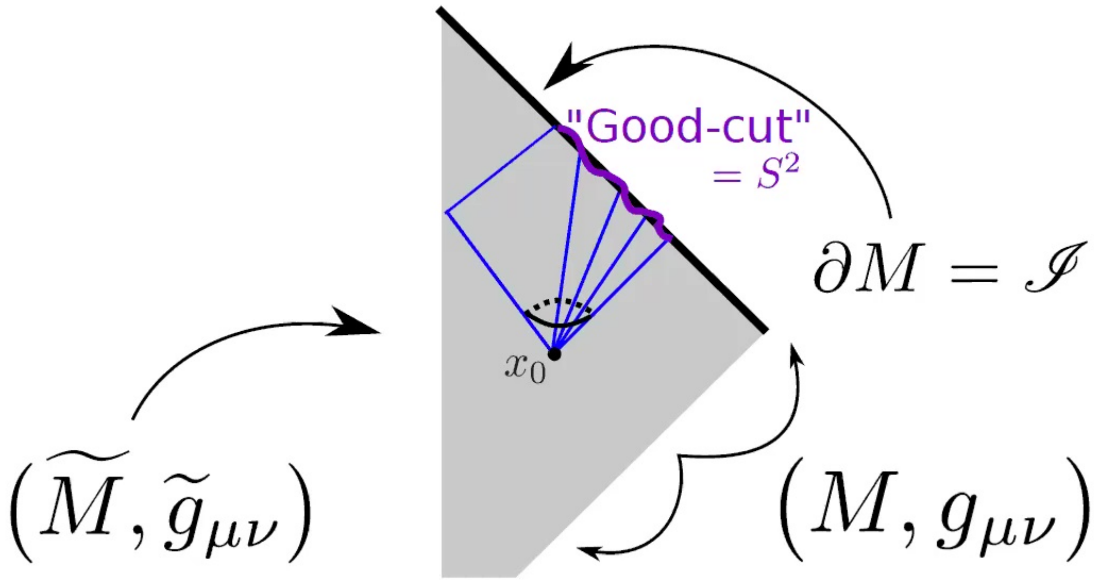

- Each of the null-cones emanating from $\mathbb{M}^4$ intersects $\mathscr{I}_{flat}$ along a "cut" $s\colon \mathscr{I} \to S^2$ (really the image of the section). (There is thus a 4-dimensional space $\mathcal{H}$ of these "good-cuts", $s \in \mathcal{H}$)
- The subgroup of BMS stabilizing these cuts is isomorphic to the Poincaré group

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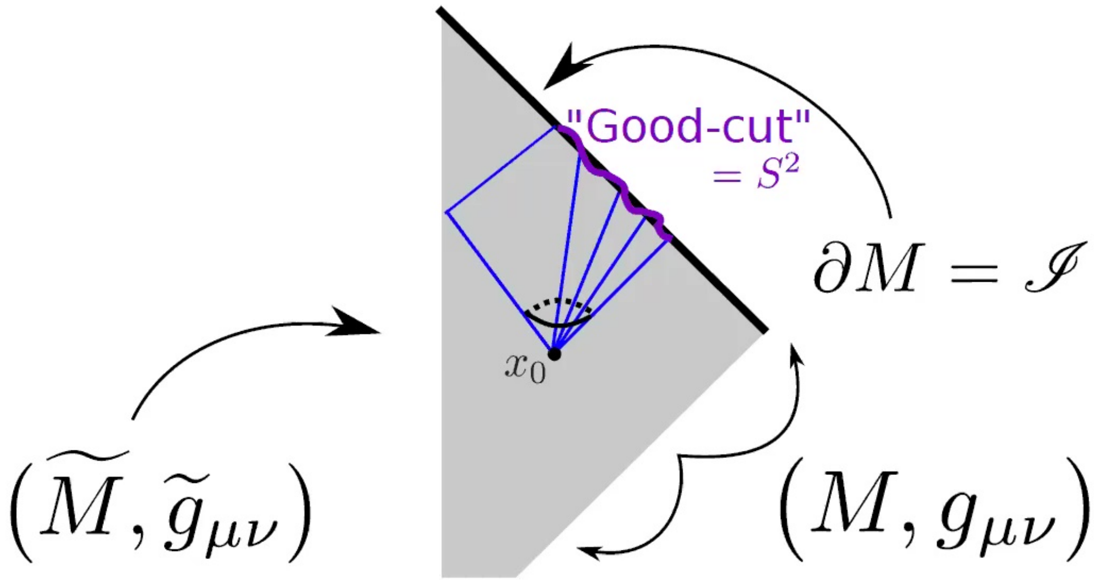

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More on the geometrical nature of gravity vacua

What kind of object is this tractor connection ?

Precise answer,

> the tractor connection is a Cartan connection
>
> modelled on the homogenous space $\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\!\diagup_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$

Recall that, the essential property of a Cartan connection modelled on G/H is that it is flat if and only if one can find a local diffeomorphism $\phi : M \to G/H$.

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Recall that, the essential property of a Cartan connection modelled on G/H is that it is flat if and only if one can find a local diffeomorphism $\phi : M \to G/H$.

A choice of gravity vacua is equivalent to...

a choice of flat Cartan connection
$D = d + A^I{}_J$ s.t $F^I{}_J = dA^I{}_J + A^I{}_K \wedge A^K{}_J = 0$

and therefore equivalent to...

to an isomorphism $\phi : \mathscr{I} \to {}^{\mathrm{Iso}(3,1)}\!\big/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$.

More on the geometrical nature of gravity vacua

A tractor connection is a Cartan connection

modelled on the homogenous space $\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\!/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$

A choice of Gravity vacua therefore amounts to ...

- a flat Cartan connection D modelled on ${}^{\mathrm{Iso}(3,1)}\!/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$,
- an isomorphism $\phi \colon \mathscr{I} \to {}^{\mathrm{Iso}(3,1)}\!/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$ to the model homogenous space,
- a 4-dimensional space of good-cuts $\mathcal{H}_D = \{s \colon S^2 \to \mathscr{I} \mid GCEq\,(s) = 0\}$,
- a copy of the Poincaré group $\mathrm{Iso}\,(3,1)$ inside the BMS group.

More on the geometrical nature of gravity vacua

A tractor connection is a Cartan connection

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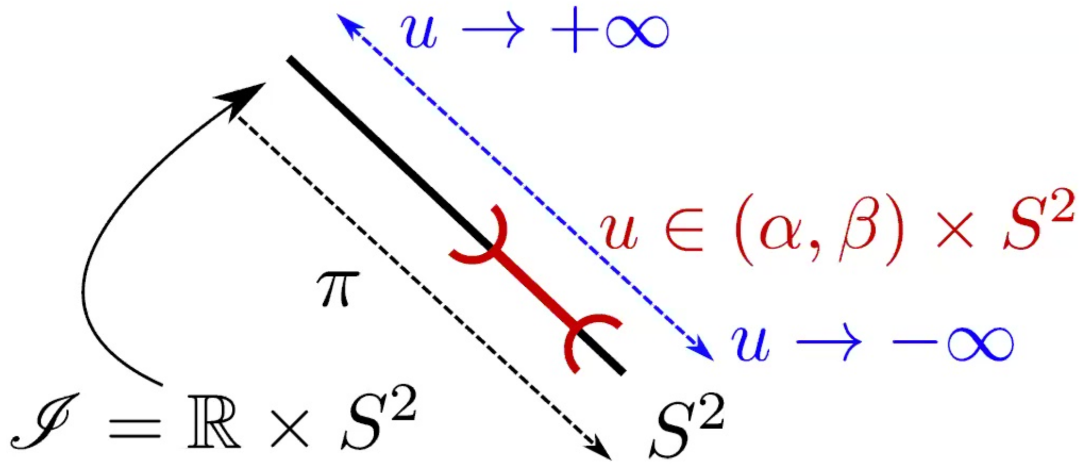
What is more, a good-cut then is equivalent to a covariantly constant section of the tractor bundle. i.e

$$\left\{\, s\colon S^2 \to \mathscr{I} \mid s \in \mathcal{H}_D \right\} \qquad \Leftrightarrow \qquad \left\{\, \Phi^I \in \Gamma\left[\mathcal{T}\right] \mid D\Phi^I = 0 \right\}$$

Relations to Carroll manifolds (and others)

Memory effect

Gravity vacua have the following interesting property: they are completely defined by there value on an open set of the form $(\alpha, \beta) \times S^2$.



i.e if D is flat on $U = (\alpha, \beta) \times S^2$ there is a unique flat extension D_0^U on the whole of $\mathscr{I}$.

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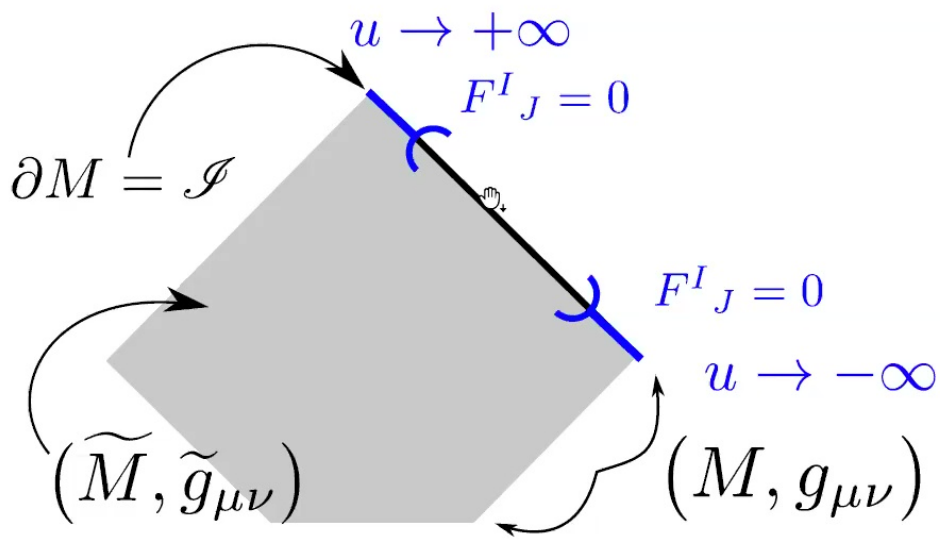

Let D be a null-normal tractor connection corresponding to a "burst" of gravitational waves
i.e such that it is both flat in the "far future" and "far past"
(i.e its curvature is compactly supported on $\mathscr{I}$.)

Memory effect

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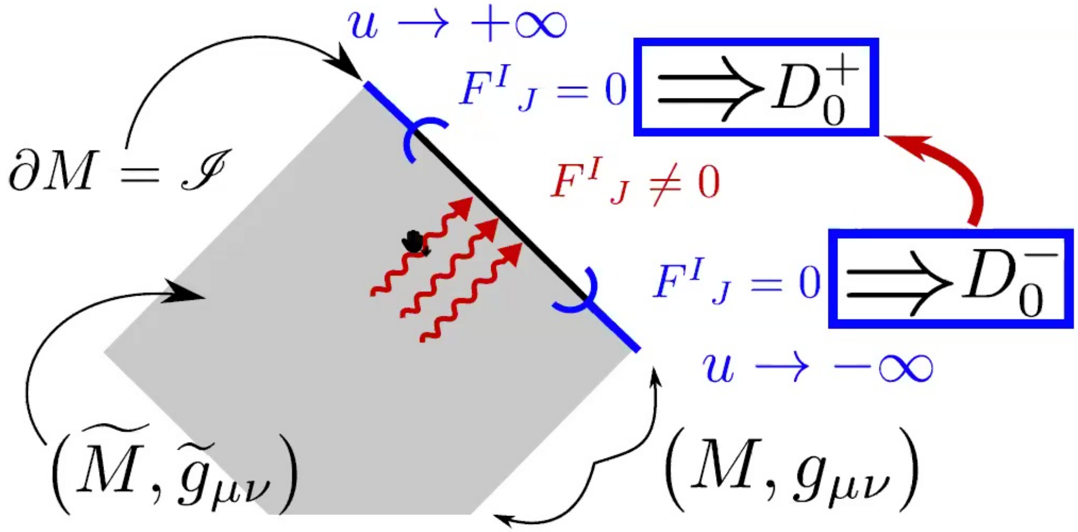

Therefore gravitational radiation

has sent one gravity vacua D_0^- to another one D_0^+.

The difference $D_0^+ - D_0^-$ is an invariant of the underlying space-times.

Null-tractor formalism

The “null-tractors” I presented here are, I believe, a very adequate set of tools when dealing with null-infinity:

- The geometry of null-infinity is intrinsically **conformal** (but degenerate).
- This is a rather weak structure but one can always define a tractor bundle (generalised from usual conformal geometry to adapt degeneracy)
- This gives a “null-tractor calculus” best adapted to deal with the geometry of null-infinity in a **manifestly conformally invariant** way.

Tractor connection

The use of tractors at null-infinity give a natural and satisfying answer to an old question:

What is the geometrical (i.e invariant) structure induced at null-infinity by the presence of gravitational waves?

$\Rightarrow$ This is a choice of "null-normal" tractor connection.
(as opposed to usual conformal geometry there is no unique normal tractor connection at null-infinity but an affine space modelled on C_{AB})

- Gravitational radiation is neatly encoded in the curvature of null-normal tractor connections
- Gravity vacua correspond to the degeneracy of flat tractor connections.
- The memory effect is completely transparent in these terms

Tractor connection

The use of tractors at null-infinity give a natural and satisfying answer to an old question:

What is the geometrical (i.e invariant) structure induced at null-infinity by the presence of gravitational waves?

Equivalently, this is a choice of Cartan geometry (modelled on $\mathrm{Iso}(3,1)/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$)

- gravity vacua amounts to maps $\phi : \mathscr{I} \to \mathrm{Iso}(3,1)/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$
- flat Cartan connections automatically reduce the symmetry group to the Poincaré group $\mathrm{Iso}(3,1)$,
- a choice of gravity vacua correspond to a choice of good-cuts in a geometrically transparent way.

Outlook

Neat, but what is it good for?

- Probably the only formalism that allows to describe physics at null-infinity in a fully invariant way
- We[1] have an Einstein-Hilbert variational principle in terms of tractor variables:
  - In principle all physics at null-infinity can thus be reformulated in this way!
  - We[2] are working on computing BMS charges and fluxes.
- Application to holographic duality: The null-normal tractor connection describes the geometrical background to which the boundary theory should be coupled.
- Very versatile formalism: it unifies all cosmological constant and both 3D and 4D space-times. Raise the hope to import ideas from one of these to others.

[1] Upcoming work with C.Scarinci
[2] Upcoming work with R.Ruzziconi

Thank You