Title: Casimir and free energy for free fields and holographic theories in 2+1 on curved spaces

Speakers: Toby Wiseman

Series: Quantum Fields and Strings

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Abstract: We investigate putting 2+1 free and holographic theories on a product of time with a curved compact 2-d space. We then vary the geometry of the space, keeping the area fixed, at zero/finite temperature, and measure the Casimir/free energy respectively. I will begin by discussing the free theory for a Dirac fermion or scalar field on deformations of the round 2-sphere. I will discuss how the Dirac theory may arise in physical systems such as monolayer graphene. For small deformations we solve analytically using perturbation theory. For large deformations we use novel numerical methods to compute these energies for specific deformations, including ones that drive the space to become singular. I will give evidence that the round sphere globally maximises the Casimir/free energy, which may imply geometric instabilities for spheres of such mono layer materials, although probably not graphene. We then discuss the analog for a holographic theory by studying its gravity dual. Here we use gravity techniques to analytically prove at zero temperature the round sphere maximises the Casimir energy. We discuss attempts to show the same for free energy at finite temperature. And finally I will report on on-going numerical gravity calculations of the Casimir energy for specific deformations of the round sphere which yield an unexpected result.

Zoom Link: https://pitp.zoom.us/j/94306870420

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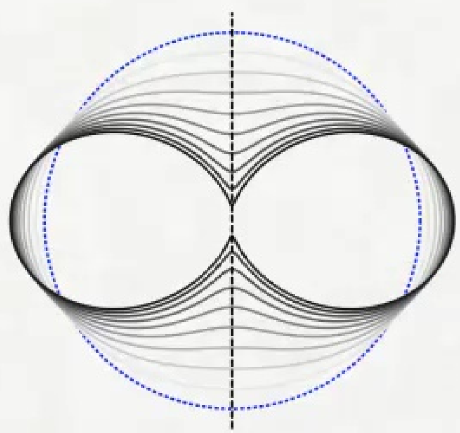
Casimir and free energy for free fields and holographic theories in 2+1 on curved spaces

Toby Wiseman (Imperial College)

with Sebastian Fischetti (McGill), Gary Gibbons, Krai Cheamsawat and Lucas Wallis



arXiv:2003.09428, 1906.07192, 1811.05995, 1803.04414, 1707.03825, 1508.04460

PLAN

- Setup – $(2+1)$-d QFT on $\mathbb{R}_t \times \Sigma$, with 2-d space Σ having prescribed area
- Physical setting - membranes with relativistic QFT d.o.f.
- Free theory; Dirac and scalar – small and large deformations of sphere
- Holographic theory; geometric proofs and on-going numerics

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CLASSICAL PRELUDE

- Classical matter gives a measure on geometry; famously bending energy of a 2-d membrane Σ of spherical topology

$$H = \kappa \int d^2x \sqrt{g} \left(K - \frac{2}{r} \right)^2$$

implies that **classically** the round sphere is energically the preferred geometry.

- Another way to phrase this – fix the area of your 2-geometry, Σ, and use the measure;

$$H' = \kappa \int d^2x \sqrt{g} K^2$$

essentially the 'Willmore functional' – again minimized by the round sphere.

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WHAT SHAPE DOES QFT PREFER?

- Quantum analog of this; take a relativistic $(2+1)$-d QFT on the space $\mathbb{R} \times \Sigma$
- Consider vacuum energy, or thermal energy at fixed temperature T.
- What geometry does the QFT energetically prefer?
- To compare we fix the area and consider varying the 'shape' and compare the (free) energy to a canonical geometry $\bar{\Sigma}$, eg. the round sphere.

See e.g. recent work for Euclidean 3-CFTs on squashed spheres

Anninos, Denef, Harlow '13; Bobev, Bueno, Vreys '17

Bueno, Cano, Hennigar, Mann '18 ; Bueno, Cano, Hennigar, Penas, Ruiperez '20

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WHAT SHAPE DOES QFT PREFER?

- Define (free) energy difference at a temperature $T = 1/\beta$;

$$-\beta \Delta F[\Sigma] = \ln Z[\beta, \Sigma] - \ln Z[\beta, \bar{\Sigma}]$$

This naively diverges, and we regulate it with a UV cut-off Λ.

- Introduce coordinates, and work in Euclidean signature; $\tau \sim \tau + \beta$;

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = +d\tau^2 + d\Sigma^2 \,, \quad d\Sigma = g_{ij} dx^i dx^j$$

- Analog of $\delta E = p\,\delta V$; $\qquad \delta \Delta F = \frac{1}{2} \int d^2x \sqrt{g} \langle T_{ij} \rangle_{\Sigma(\epsilon)} \delta g^{ij}$

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WHAT SHAPE DOES QFT PREFER?

- With a covariant regularization, cut-off Λ, the stress tensor one-point function suffers local divergences,

$$\langle T_{\mu\nu} \rangle_\Sigma = \left(c_1 \Lambda^3 + c_2 \Lambda \mu^2 \right) g_{\mu\nu} + c_3 \Lambda G_{\mu\nu} + O(\Lambda^0)$$

  with μ any mass scale in the theory. When renormalized by suitable counter-terms (c.c. and Einstein-Hilbert in the action) ambiguous finite local contributions will remain – scheme dependence.

- However in our context $G_{ij} = 0$, so,

$$\Delta F = -\frac{1}{2} \left(c_1 \Lambda^3 + c_2 \Lambda \mu^2 \right) \left(Vol(\Sigma) - Vol(\bar{\Sigma}) \right) + O(\Lambda^0)$$

  Since we are fixing the volume, ΔF is in fact UV finite and unambiguous.

- Why did this work? Einstein-Hilbert term in action is topological for our ultrastatic setting. This does **not** work in higher dimensions!

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COMMENTS

- For a classical membrane (at the beginning) the energy is depends on the extrinsic embedding of Σ into $\mathbb{R}^3$.
- For our QFT it depends only on the intrinsic geometry of Σ – this may arise from embedding of a membrane, or we can consider it more formally.

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SMALL DEFORMATIONS OF A SPHERE

- Consider $\bar{\Sigma}$ a homogeneous space – eg. a round sphere. Write Σ as;

$$ds^2_{\Sigma} = e^{2f(x;\epsilon)}\bar{g}_{ij}dx^i\,dx^j\ , \quad \int d^2x\sqrt{\bar{g}}\,e^{2f(x;\epsilon)} = \text{const}$$

- Consider now a small perturbation; $f(x;\epsilon) = \epsilon\, f^{(1)}(x) + O(\epsilon^3)$ so that,

$$\langle T_{ij}\rangle_{\Sigma(\epsilon)} = \bar{\sigma}_{ij} + \epsilon\,\delta\sigma_{ij}(x) + O(\epsilon^2)$$

- Then under assumption that stress tensor is static, $\bar{\sigma}^i{}_i$ =const.;

$$\Delta F(\epsilon) = \epsilon^2\int d^2x\sqrt{\bar{g}}\left(\left(f^{(1)}\right)^2\bar{\sigma}^i{}_i - \frac{1}{2}(f^{(1)})\,\delta\sigma^i{}_i\right) + O\left(\epsilon^3\right)$$

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SMALL DEFORMATIONS OF A SPHERE

- Consider $\bar{\Sigma}$ = round sphere; $d\bar{s}^2 = d\theta^2 + \sin^2\theta d\phi^2$.
- Decompose perturbation in spherical harmonics; $f^{(1)}(\theta,\phi) = \sum_{lm} f_{lm} Y_{lm}(\theta,\phi)$.
- Then symmetries imply;

$$\Delta F(\epsilon) = -\epsilon^2 \sum_{l,m} A_l \left|f_{lm}\right|^2 + O\left(\epsilon^3\right)$$

- Hence generally the round sphere is an extremal point of energy. Given a theory we can then compute A_l and hence the character of this extremal point...

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BASIC QUESTIONS

- Small deformations; what is the sign of A_l?

    1. Is the round sphere preferred as in the classical context?
    2. Or are some directions energetically ‘unstable’?

- What about large deformations of a round sphere? How would they behave?

    1. Is the round sphere a global extrema?
    2. Is the energy bounded for fixed area as the geometry gets highly deformed?

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SMALL DEFORMATIONS FOR A CFT AT $T = 0$

- We begin by considering a CFT. Then since the $\mathbb{R} \times S^2$ is Weyl flat, the 1-pt function $\langle T_{ij} \rangle$ to leading order is determined by the stress tensor two point function on flat space.

- It is then simple to show that for a sphere radius r;

$$\Delta F = -\epsilon^2 \frac{\pi^2 c_T}{48 r} \sum_{\ell,m} |f_{\ell,m}|^2 \frac{(\ell^2-1)(\ell+2)}{\ell} \left(\frac{\Gamma\left(\frac{\ell+1}{2}\right)}{\Gamma\left(\frac{\ell}{2}\right)} \right)^2 + \mathcal{O}(\epsilon^3),$$

  with universal form determined only by c_T, the effective central charge;
  Dirac $c_T = 3/(4\pi)^2$, scalar $c_T = (3/2)/(4\pi)^2$, holographic $c_T = \ell^2/(16\pi G) \times (\pi^2/48)$.

- Unitarity implies $c_T > 0$. Key observation: This is negative definite!?

- Does this mean QFT wants to destabilize a round sphere?

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PHYSICAL SETTING

- Consider a membrane material carrying relativistic QFT degrees of freedom.
- Treat this is a Born-Oppenheimer approximation;

  1. The membrane is non-relativistic and we treat it classically, and give it a usual membrane energy.
  2. We may integrate out the QFT (at zero or finite temperature) which lives on the geometry Σ induced from the membrane.
  3. This yields a non-local (free) energy.

- Consider deformations of the shape at fixed area; then,

$$H = \kappa \int d^2x \sqrt{g} \left(K - \frac{2}{r} \right)^2 + \Delta F(\Sigma)$$

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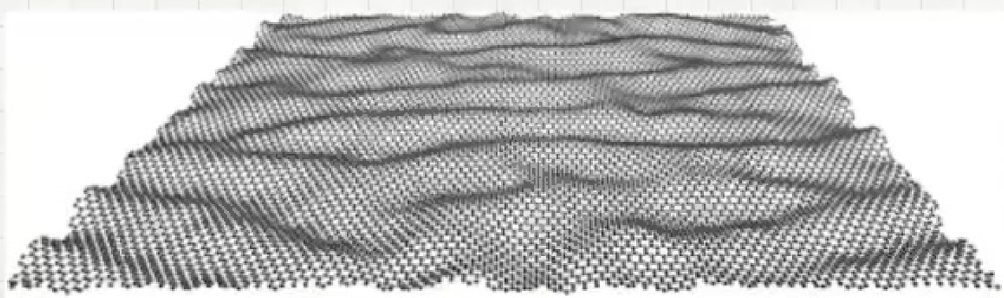
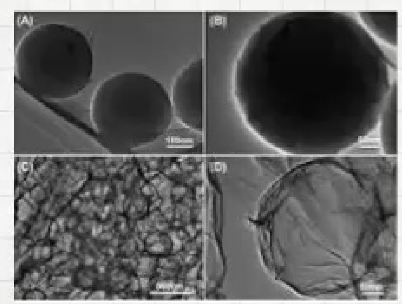
PHYSICAL EXAMPLE

- Monolayer graphene is an example. Its unit cell scale is ~ 2.5 Å. On larger scales it may be treated as a classical membrane carrying (two) Dirac fermions with $c \sim c_{\rm light}/100$.
- Being crystalline the UV breaks diff invariance and results in an additional gauge field in the Dirac theory – can choose deformations where this vanishes [de Juan et al '12]
- Its bending rigidity is estimated around $\kappa \sim 1$ eV.
- Most interestingly monolayer graphene is indeed seen to crumple or ripple on scales ~ 50 Å, with smallish amplitude ~ 5 Å [Meyer et al; Fasolino, Los, Katsnelson]
- Also there is considerable interest in constructing monolayer graphene spheres!





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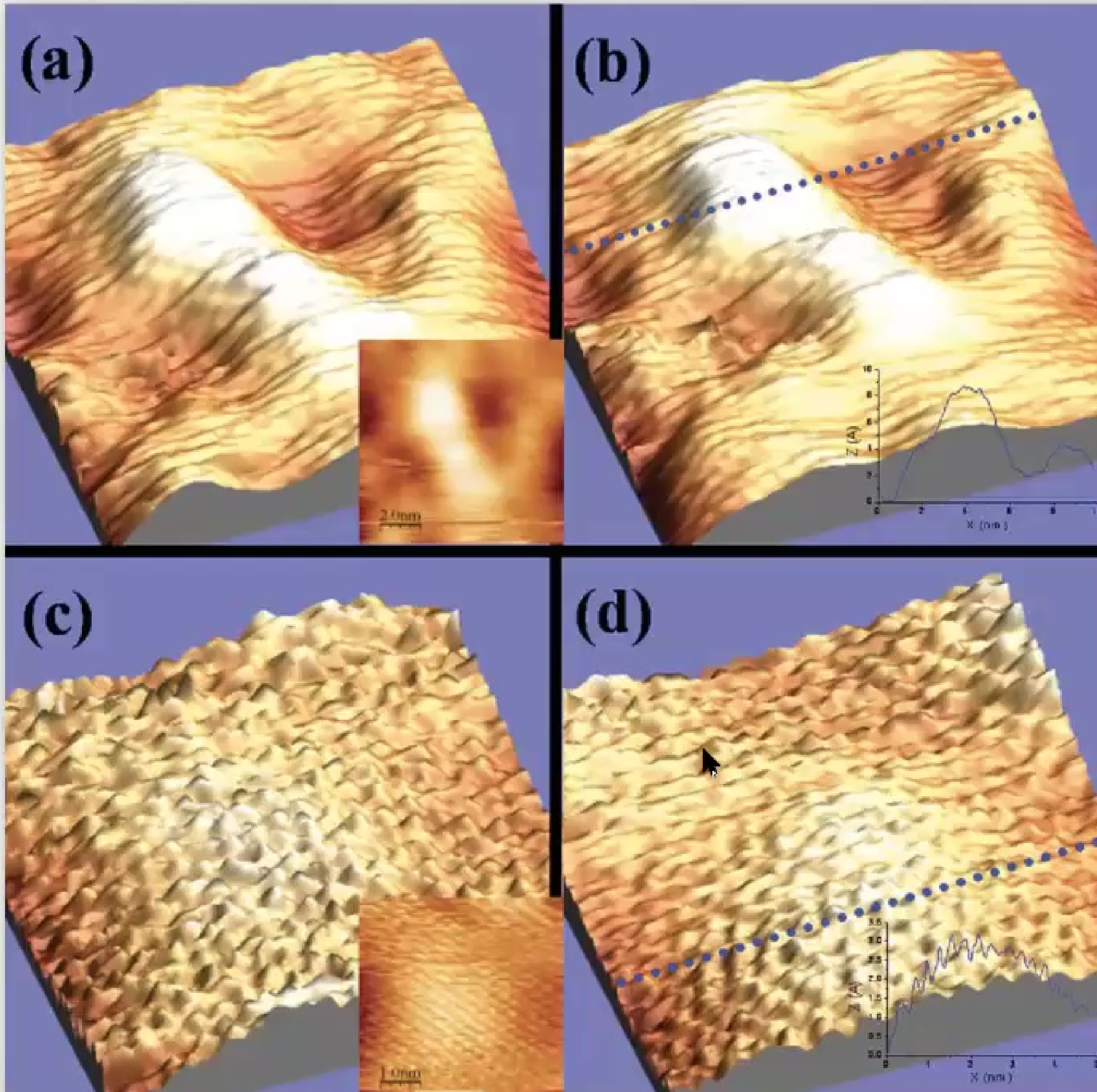

R Zan et al 2012

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PHYSICAL EXAMPLE

- Since Dirac is a CFT, we can estimate the competition between the classical membrane bending and the quantum ΔF at zero temperature.

- Taking the simple diff invariant membrane action (can do better...) one obtains;

$$\Delta F = \epsilon^2 \kappa \sum_{\ell,m} |f_{\ell,m}|^2 \left[A_\ell^{(c)} - \frac{\gamma}{r} A_\ell^{(q)} \right] + \mathcal{O}(\epsilon^3),$$

where

$$\gamma \equiv \frac{\pi^2 c_T \hbar c_{\text{eff}}}{48\kappa}$$

is some characteristic length scale and

$$A_\ell^{(c)} \equiv (\ell-1)^2(\ell+2)^2, \qquad A_\ell^{(q)} \equiv \frac{(\ell^2-1)(\ell+2)}{\ell} \left(\frac{\Gamma\left(\frac{\ell+1}{2}\right)}{\Gamma\left(\frac{\ell}{2}\right)} \right)^2 .$$

- Find $l = 2$ mode is unstable if; $\gamma \geq \gamma_{\text{crit}} \equiv \frac{32}{3\pi} r \simeq 3r$ – ie. small enough spheres.

- Might imagine that electron contribution is totally irrelevant – so $\gamma \ll$ unit cell size.

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PHYSICAL EXAMPLE

- Parametrically; $\gamma \sim c_{\text{eff}}\hbar\kappa \simeq 25\text{Å}$– this is surprisingly large?!
- But being careful, using $c_T = 2 \times \frac{3}{(4\pi)^2}$; $\gamma \simeq 0.3\text{Å}$
- So the effect is too weak to deform even small graphene spheres.
- Naturalness: one might expect $\gamma \sim$unit cell size.
- If one can engineer a monolayer material like graphene with an ('unnaturally') small κ this effect may act to crumple the membrane.

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FREE THEORIES

- We now consider free theories; Dirac fermion and the scalar (with curvature coupling).
- We will solve them perturbatively about the sphere and non-perturbatively using numerics.
- Then we wish to compute;

$$Z = (\det \mathcal{L})^{\sigma} \text{ with } \mathcal{L} = -\partial_\tau^2 + L + M^2,$$

where $\sigma = -1/2$ $(+1)$ for the scalar (fermion) and L is a differential operator on Σ. For scalar $L = -\nabla^2 + \xi R$. Similar for Dirac.

- Define heat kernel $K_{\mathcal{L}}(t) \equiv \mathrm{Tr}(e^{-t\mathcal{L}})$, in terms of which the free energy is

$$\beta F = -\ln Z = \sigma \int_0^\infty \frac{dt}{t} K_{\mathcal{L}}(t).$$

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FREE THEORIES

- Then at temperature T;

$$\beta \Delta F = \sigma \int_0^\infty \frac{dt}{t} e^{-M^2 t} \Theta_\sigma(T^2 t) \Delta K_L(t),$$

where $\Delta K_L(t) \equiv K_L(t) - K_{\overline{L}}(t)$ and,

$$\Theta_\sigma(\zeta) \equiv \sum_{n=-\infty}^{\infty} e^{-(2\pi)^2 (n-\sigma+1/2)^2 \zeta},$$

- The task is then to compute $\Delta K_L(t)$ – the 'differenced heat kernel'

- Note: if ΔK_L has definite sign, then ΔF does *for any* mass and temperature.

FREE THEORIES - SMALL DEFORMATIONS OF SPHERE

- Mercifully sparing you the detail;

$$\sigma\Delta K_L(t) = \epsilon^2 \sum_{\ell,m} a_\ell(t)|f_{\ell,m}|^2 + O(\epsilon^3), \qquad a_\ell(t) \equiv t \sum_{\ell'=0}^{\infty} e^{-\bar{\lambda}_{\ell'} t}(\alpha_{\ell,\ell'} + \beta_{\ell,\ell'} t)$$

For odd ℓ the expressions simplify; $\beta_{\ell,\ell'} = 0$ and $\alpha_{\ell,\ell'}$ is non zero only for $\ell' < \frac{\ell}{2}$. For the scalar;

$$\alpha_{\ell,\ell'} = -\frac{(2\ell'+1)(\bar{\lambda}_{\ell'} - \xi\ell(\ell+1))^2}{2\pi\ell(\ell+1)} \frac{\left(\frac{2+\ell}{2}\right)_{\ell'} \left(\frac{\ell}{2}\right)_{-\ell'}}{\left(\frac{3+\ell}{2}\right)_{\ell'} \left(\frac{1+\ell}{2}\right)_{-\ell'}}$$

and the fermion;

$$\alpha_{\ell,\ell'} = -\frac{(2\ell'+1)^3}{16\pi} \frac{\left(\frac{2+\ell}{2}\right)_{\ell'+1/2} \left(\frac{2+\ell}{2}\right)_{-(\ell'+1/2)}}{\left(\frac{1+\ell}{2}\right)_{\ell'+1/2} \left(\frac{1+\ell}{2}\right)_{-(\ell'+1/2)}}$$

where $(x)_n \equiv \Gamma(x+n)/\Gamma(x)$ are Pochhammer symbols.

- Key point: $\sigma\Delta K_L$ is negative definite so $\Delta F < 0$ for any mass and temperature.

- Note: specific to sphere – the Dirac fermion on a torus may have positive ΔF.

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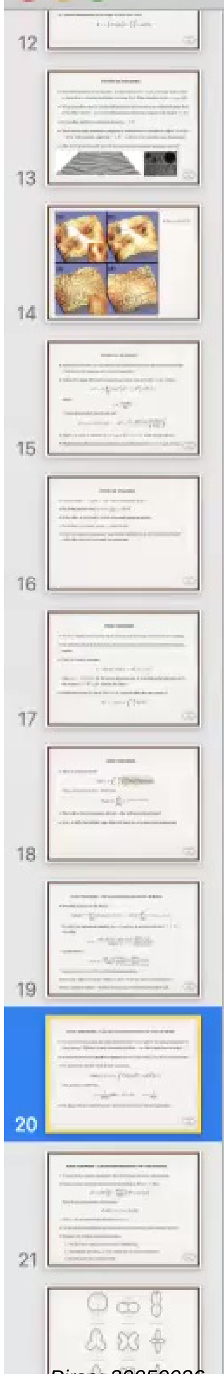

FREE THEORIES - LARGE DEFORMATIONS OF THE SPHERE

- Do these features persist for large deformations? Is the sphere the global maximum of (free) energy? This is a purely geometric problem – we don't know how to solve!

- Non-perturbatively at **small** and **large** t one sees that $\sigma \Delta K_L(t) \leq 0$ for these theories.

- For small t we use the 'heat kernel expansion';

$$\sigma \Delta K_L(t) \simeq -t\, c_{s,f} \int d^2x \left(\sqrt{g}\, R^2 - \sqrt{\bar{g}}\, \bar{R}^2 \right) \leq 0$$

with positive coefficients,

$$c_s = \frac{1}{2880\pi} (5(6\xi - 1)^2 + 1) \,, \qquad c_f = \frac{1}{960\pi}$$

- For large t we use known bounds on lowest (or second lowest) eigenvalues.

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FREE THEORIES - LARGE DEFORMATIONS OF THE SPHERE

- To proceed we compute numerically $\Delta K_L(t)$ for large fixed area deformations.

- Firstly consider geometry induced from embedding in $\mathbb{R}^3$ via $r = R(\theta)$,

$$ds^2 = R(\theta)^2 \left[\left(1 + \frac{R'(\theta)^2}{R(\theta)^2} \right) d\theta^2 + \sin^2\theta \, d\phi^2 \right] .$$

  Then choose axisymmetric deformations;

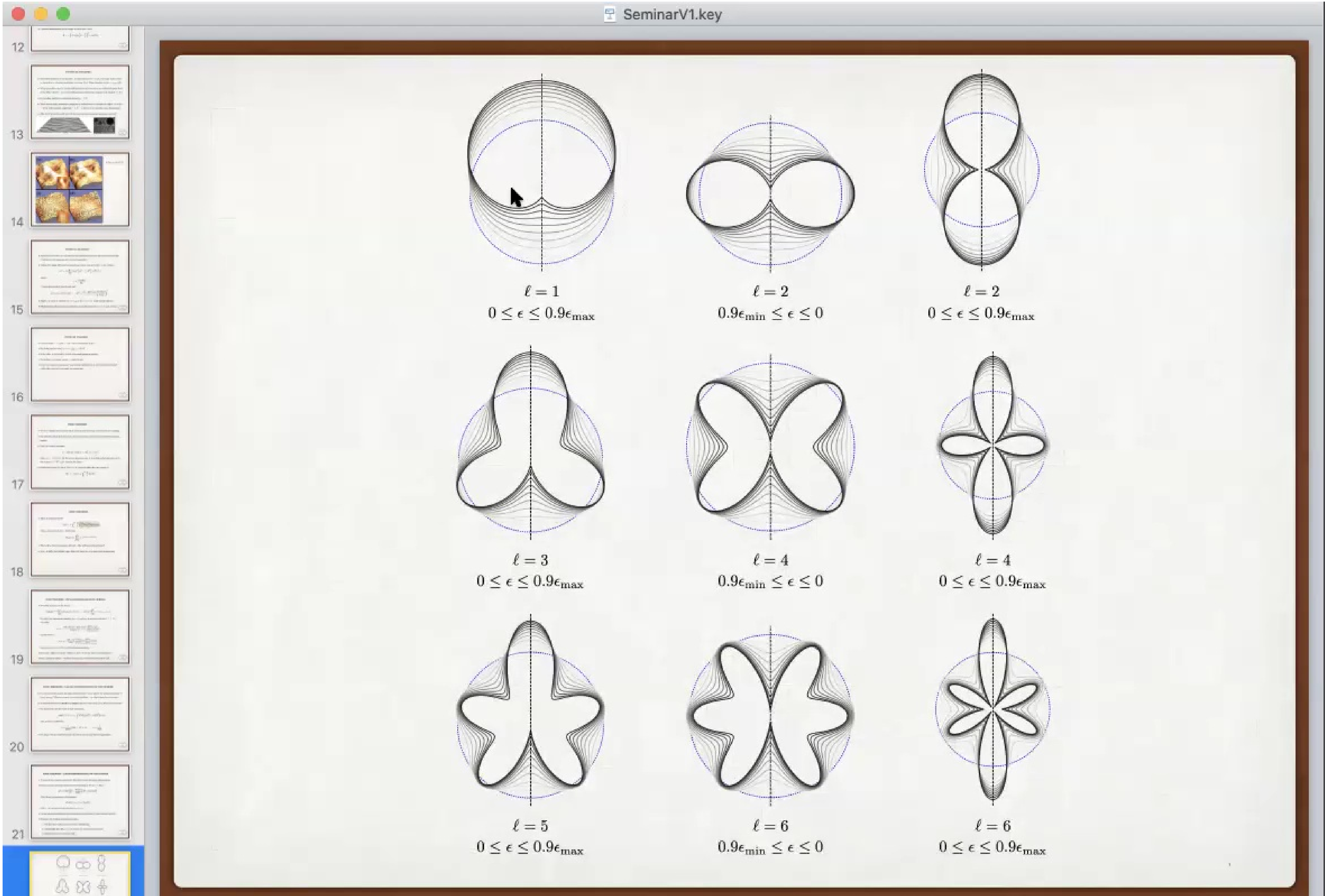
$$R_{\ell,\epsilon}(\theta) = c_{\ell,\epsilon} \left(1 + \epsilon \, Y_{\ell,0}(\theta)\right) ,$$

  with $c_{\ell,\epsilon}$ an area preserving constant, so $c_{\ell,0} = 1$.

- Use pseudo-spectral differencing to solve spectrum of L and $\bar{L}$; then construct $\Delta K_L(t)$.

- Summary; for fermion and minimal scalar...

  1. We find that $\sigma \Delta K_L(t)$ seems to have definite sign.
  2. Surprisingly $\Delta F / \Delta F_{pert}$ is very similar for the fermion and scalar.
  3. Energy grows more negative with ϵ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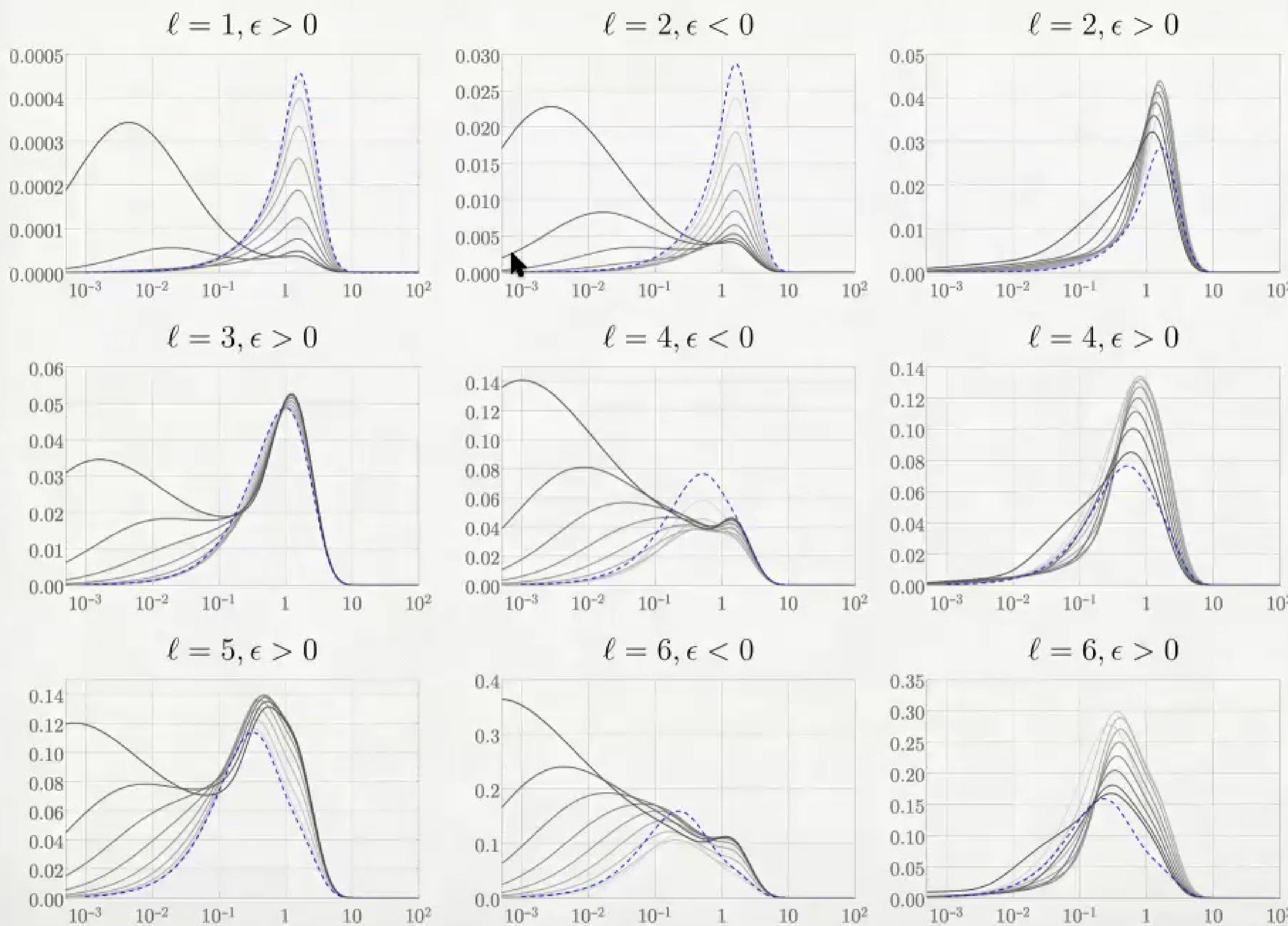
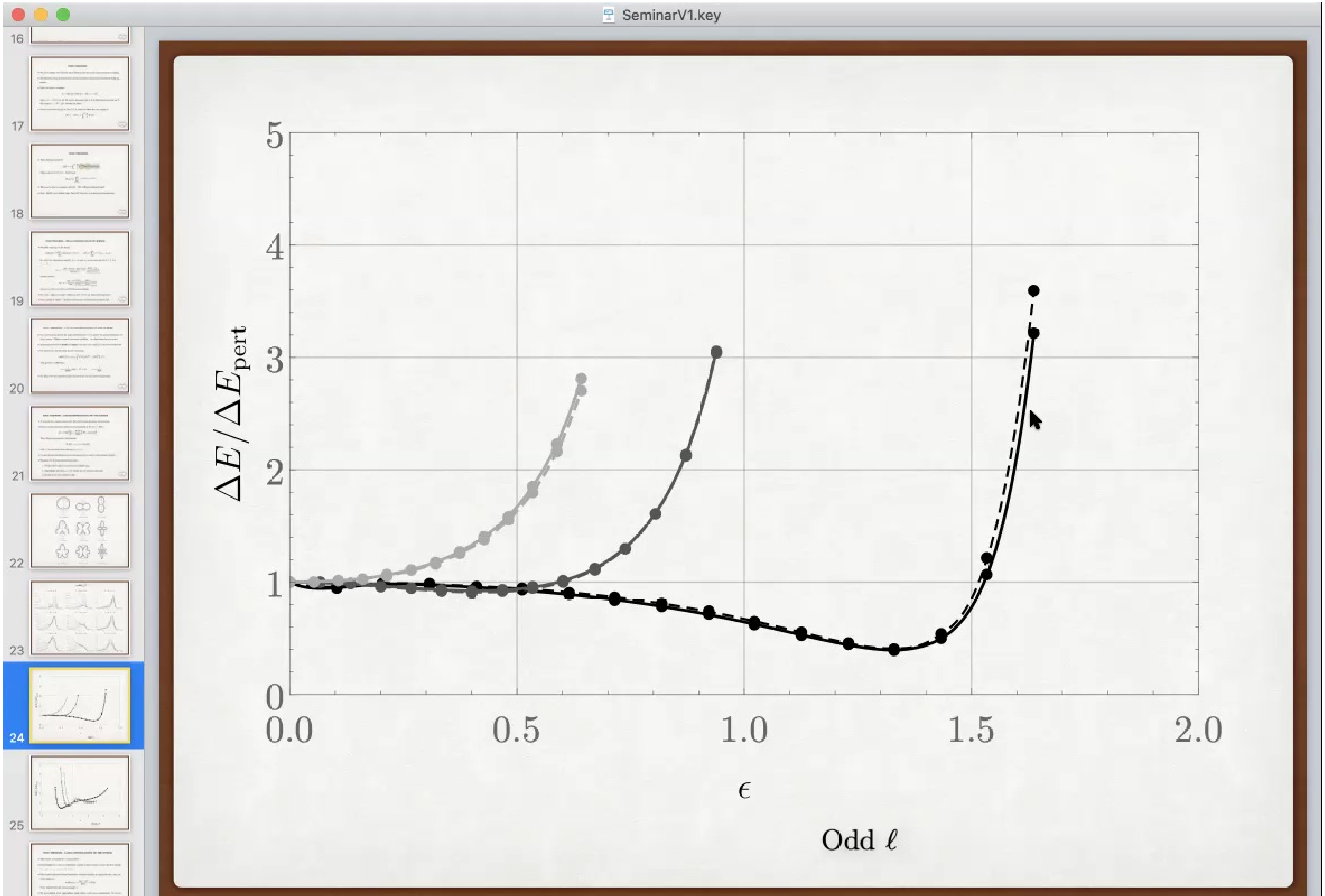
$\ell = 1$
$0 \leq \epsilon \leq 0.9\epsilon_{\max}$
$\ell = 2$
$0.9\epsilon_{\min} \leq \epsilon \leq 0$
$\ell = 2$
$0 \leq \epsilon \leq 0.9\epsilon_{\max}$
$\ell = 3$
$0 \leq \epsilon \leq 0.9\epsilon_{\max}$
$\ell = 4$
$0.9\epsilon_{\min} \leq \epsilon \leq 0$
$\ell = 4$
$0 \leq \epsilon \leq 0.9\epsilon_{\max}$
$\ell = 5$
$0 \leq \epsilon \leq 0.9\epsilon_{\max}$
$\ell = 6$
$0.9\epsilon_{\min} \leq \epsilon \leq 0$
$\ell = 6$
$0 \leq \epsilon \leq 0.9\epsilon_{\max}$

$$-\sigma \Delta K_L / \epsilon^2$$

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$\Delta E/\Delta E_{\text{pert}}$
0
1
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5
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1.0
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ϵ
Odd ℓ

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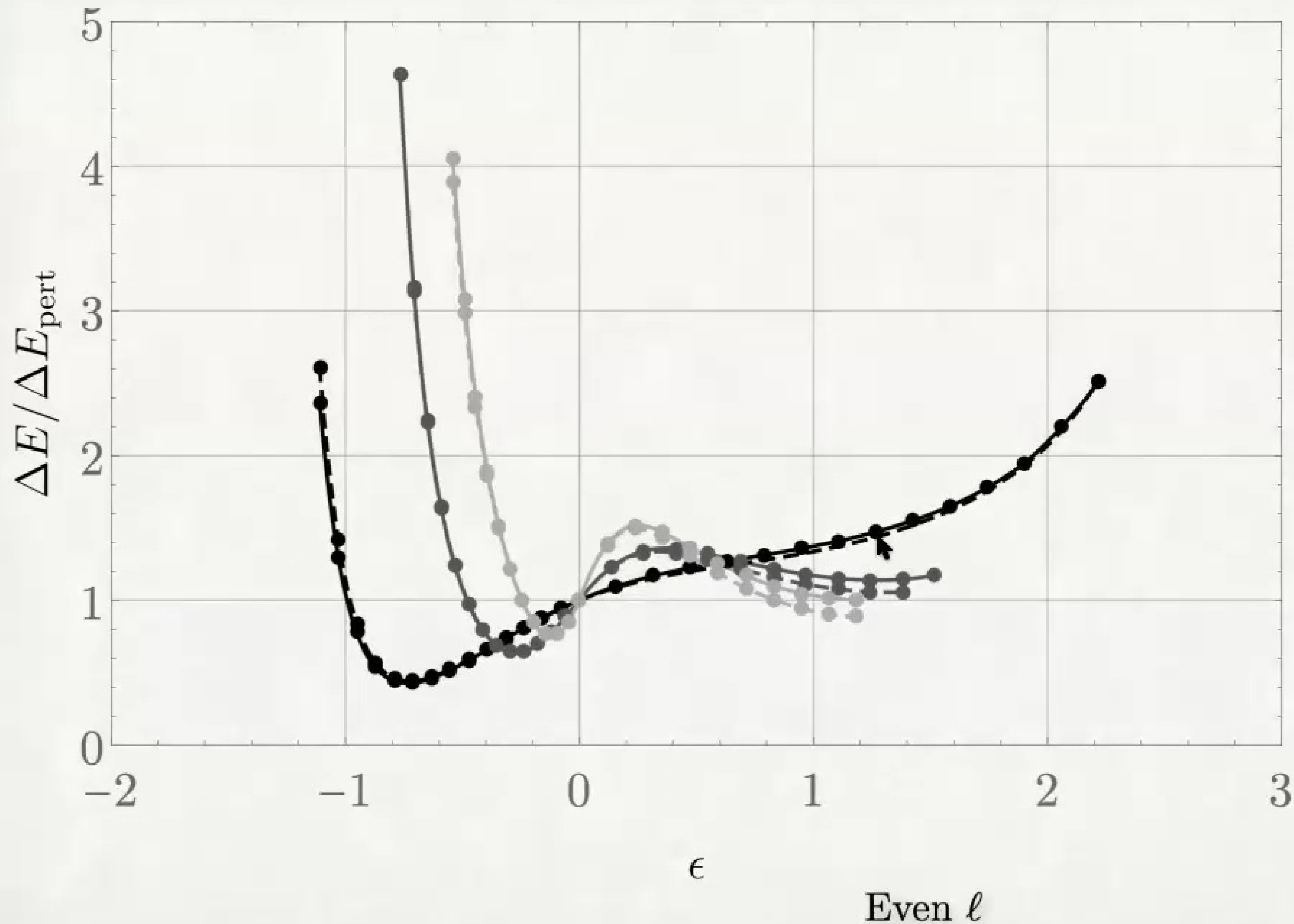

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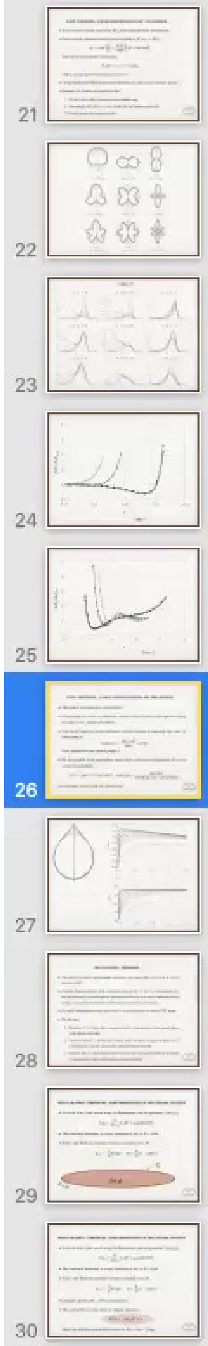

FREE THEORIES - LARGE DEFORMATIONS OF THE SPHERE

- Also looked at approach to conical defect.
- Interestingly for a cone, a conformally coupled scalar is known to have positive energy for angle excess, negative for deficit.
- Heat kernel expansion linear behaviour becomes singular as approach cone cone, instead going as,

$$\sigma \Delta K_L(t) = -\frac{(2\pi - \alpha)^2}{48\pi\alpha} + \mathcal{O}(t),$$

  Note: negative for any opening angle α.
- We investigated both approaching angle deficit and excess singularities; for excess cannot be embedded;

$$ds^2 = c_\epsilon \left(d\theta^2 + e^{2f_\epsilon} \sin^2\theta \, d\phi^2 \right), \text{ with } f_\epsilon(\theta) = \frac{\ln(\alpha/2\pi)}{\sec^2(\theta/2) + (1-\epsilon)^2 \csc^2(\theta/2)},$$

- Conclusion; $\Delta K_L(t)$ still has definite sign.

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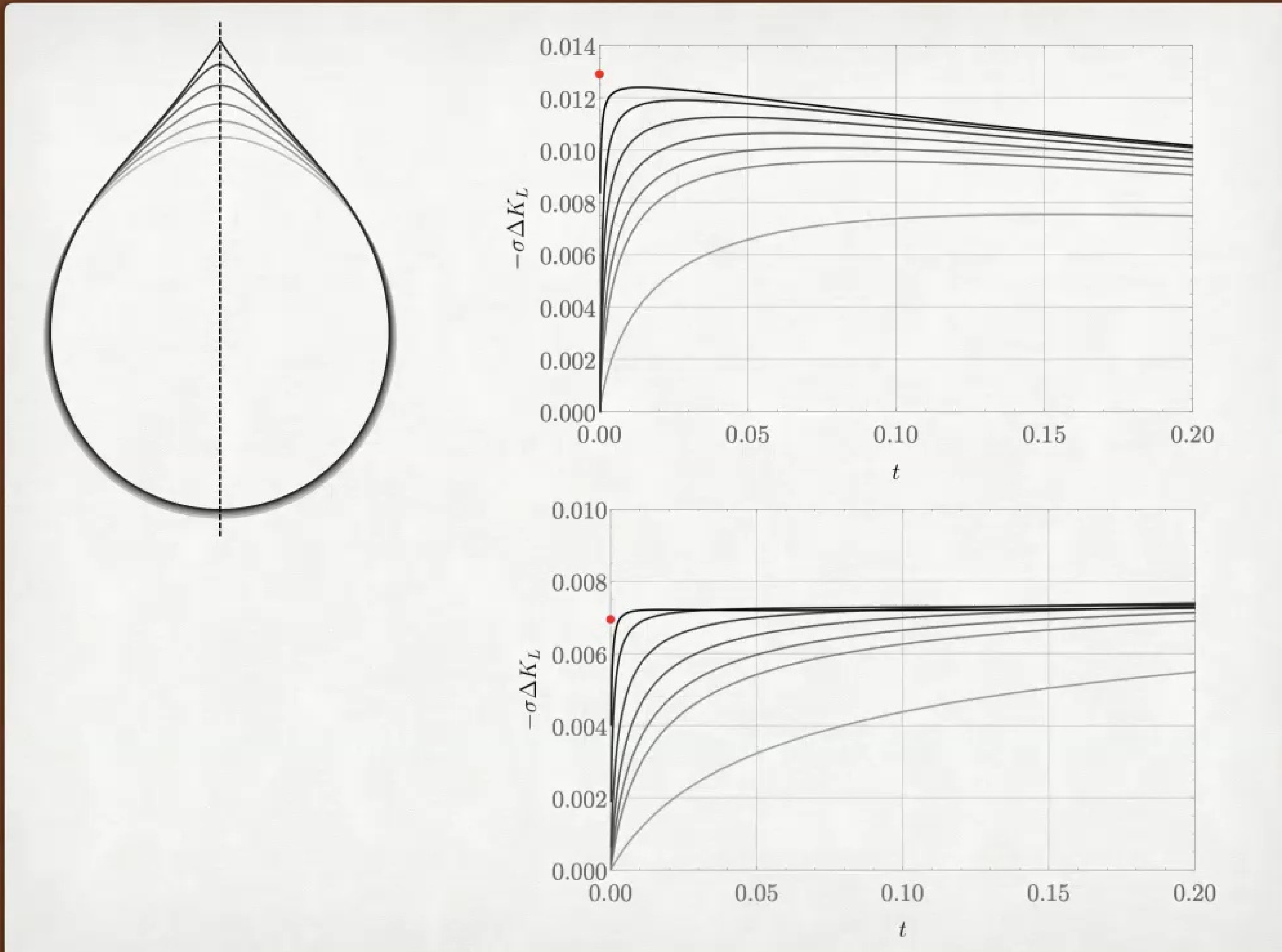


HOLOGRAPHIC THEORIES

- The geometric nature of holography provides a very powerful tool to look at curved spacetime QFT.
- Consider Einstein metrics with boundary given by $\mathbb{R} \times \Sigma$ for Σ a deformation of a fiducial geometry (eg round sphere) with prescribed area; then stress tensor determines energy, and at finite temperature bulk horizons contribute an entropy.
- For small deformations of the sphere at $T = 0$ it is given by the earlier CFT result.
- We will now;
  1. Prove for $T = 0$ that ΔE is negative for Σ a deformation of the round sphere (with global AdS bulk).
  2. Prove for finite $T > 0$ that ΔE (energy sadly *not* free energy) is negative for Σ a deformation of a flat torus (with AdS-Schwarzschild dual).
  3. Evaluate ΔE at zero temperature by numerically solving the bulk for the family of axisymmetric sphere deformations discussed earlier.

HOLOGRAPHIC THEORIES - DEFORMATIONS OF BOUNDARY SPHERE

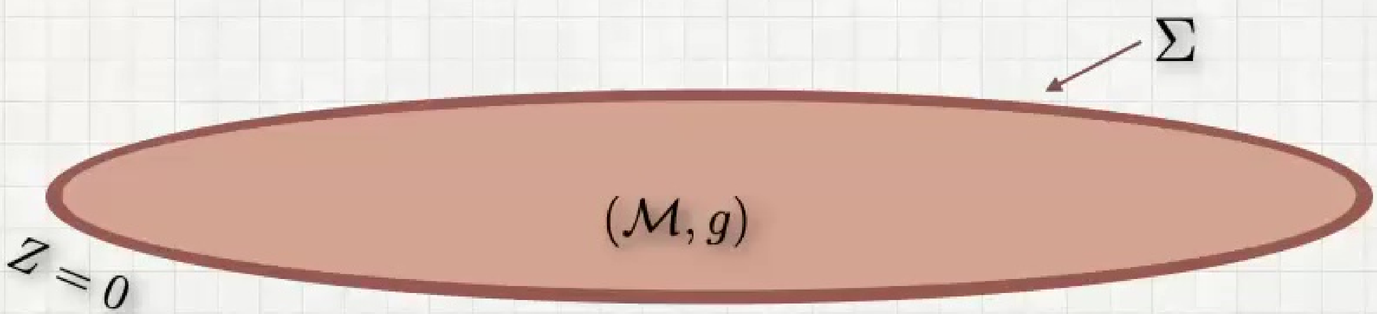
- Write 4d static bulk metric using 3d Riemannian 'optical geometry' $(\mathcal{M}, g_{ab})$;

$$ds^2_{(4)} = \frac{\ell^2}{Z^2(x)} \left(-dt^2 + g_{ab}(x) dx^a dx^b \right)$$

- The conformal boundary is a true boundary of $\mathcal{M}$, so $\Sigma = \partial \mathcal{M}$.

- Static bulk Einstein condition written covariantly over $\mathcal{M}$;

$$R_{ab} = -\frac{2}{Z} \nabla_a \partial_b Z \ , \quad R = \frac{6}{Z^2} \left(1 - (\partial Z)^2 \right)$$



HOLOGRAPHIC THEORIES - DEFORMATIONS OF BOUNDARY SPHERE

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- Example: global AdS – $\mathcal{M}$ is a hemisphere.

- The optical Ricci scalar obeys an elegant equation;

$$\nabla^2 R = -3 \tilde{R}_{ab} \tilde{R}^{ab} \leq 0$$

where the traceless optical Ricci tensor is; $\tilde{R}_{ab} = R_{ab} - \frac{1}{3} R \, g_{ab}$.

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HOLOGRAPHIC THEORIES - DEFORMATIONS OF BOUNDARY SPHERE

- Integrating over $\mathcal{M}$ and using divergence thm;

$$0 \geq \int_{\partial M} dA^a \partial_a R = \frac{6}{\tilde{c}_T} \int_\Sigma \sqrt{g} \, \langle T_{tt} \rangle = \frac{6}{\tilde{c}_T} E \,, \quad \tilde{c}_T = \frac{\ell^2}{16 \pi G}$$

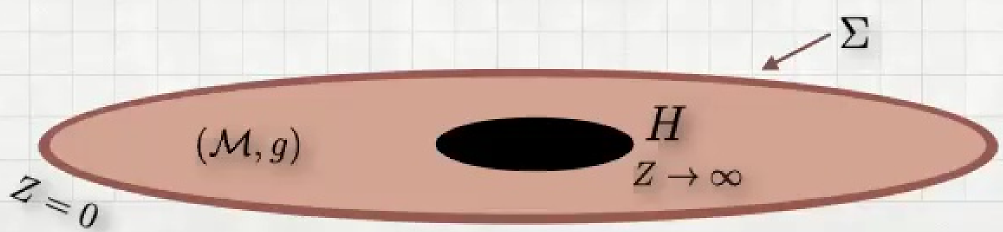
  Related to [Galloway, Woolgar] and also Gauss-Bonnet them of [Anderson]

- We don't expect bulk horizons, but for $T \to 0$ limit horizon contributions vanish.

- Global AdS has energy $E = 0$; hence we see $\Delta E \leq 0$ for deformations Σ of round sphere $\bar{\Sigma}$.

- Equality iff $\tilde{R}_{ab} = 0 \implies R = \text{const} \implies \bar{R} =$const, so a sphere. Hence the sphere is the global maximizer. Remarkable given we can't prove this for free field theory!

- Note; for confining bulk these results trivially extend to low temperature as $\Delta F = \Delta E$.



HOLOGRAPHIC THEORIES - DEFORMATIONS OF BOUNDARY TORUS

- We would like a proof for ΔF also for the deconfined phase. This is more complicated as then the reference bulk geometry is AdS-Schwarzschild which itself has a non-trivial free energy.
- What can we do? We have a result for ΔE – the energy, not free energy – for a toroidal boundary at finite temperature, fixed area.
- Now the reference geometry is the flat torus, and bulk is toroidally compactified planar AdS-Schwarzschild.
- It is considerably more complicated! Use the technique of Robinson – spatial geometry of AdS-Schwarzschild is conformally flat and cohomogeneity one.

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HOLOGRAPHIC THEORIES - DEFORMATIONS OF BOUNDARY TORUS

- Start by considering Cotton tensor of optical geometry;

$$R_{abc} = \nabla_c R_{ab} - \nabla_b R_{ac} + \frac{1}{4}\left(g_{ac}\partial_b R - g_{ab}\partial_c R\right)$$

which vanishes for AdS-Schwarzschild.

- Then define function;

$$\Phi(x) = \left(R - \frac{6}{Z^2}\right) - H(Z)$$

where $H(Z)$ is chosen such that $\Phi = 0$ for planar AdS-Schwarzschild.

From this we define a covector field;

$$V_a = \partial_a \Phi + \frac{1}{H(Z)}\left(\frac{12}{Z^3} - H'(Z)\right)\Phi\,\partial_a Z$$

which also vanishes for Schwarzschild.

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HOLOGRAPHIC THEORIES - DEFORMATIONS OF BOUNDARY TORUS

- We take solutions A, B, C to the following linear ode's;

$$A'(Z) = \frac{3}{2}\frac{A(Z)}{H(Z)}\left(\frac{12}{Z^3} - H'(Z)\right) - 9B(Z)$$

$$B'(Z) = \frac{1}{12}\frac{A(Z)}{H(Z)^2}\left(\frac{12}{Z^3} - H'(Z)\right)^2 - \frac{3}{Z}B(Z) + \frac{18}{Z^3}\frac{C(Z)}{H(Z)^2}$$

$$C'(Z) = -\frac{3}{Z^3 H(Z)}\left(10B(Z)H(Z) + 12C(Z) + Z^2 C(Z)H(Z)\right)$$

HOLOGRAPHIC THEORIES - DEFORMATIONS OF BOUNDARY TORUS

- Using these we construct a vector field as;

$$J_a(x) = \frac{A(Z)}{9}\partial_a R - 6B(Z)\frac{(\partial Z)^2}{Z^2}\,\partial_a Z + C(Z)\partial_a Z$$

- Obtain generalization of an identity Robinson used to prove uniqueness of Schwarzschild;

$$\nabla^a J_a = -A(Z)\frac{Z^2}{(\partial Z)^2}\left(\frac{1}{6}R_{abc}R^{abc} + \frac{1}{72}V_a V^a\right) - C(Z)\frac{3}{ZH(Z)^2}\Phi^2\,.$$

- Hence integrating we find; c.f. $\nabla^2 R = -3\tilde{R}_{ab}\tilde{R}^{ab} \leq 0$

$$\int_{\partial\mathcal{M}} dA^a J_a \leq 0 \quad \text{if} \quad A(Z), C(Z) \geq 0 \text{ for } Z \geq 0$$

with equality if the bulk is planar Schwarzschild.

- Now we have a horizon and boundary to $\mathcal{M}$ – so what are these boundary terms? And can we find positive A, C?

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HOLOGRAPHIC THEORIES - DEFORMATIONS OF BOUNDARY TORUS

- In Schwarzschild coordinates;

$$ds^2_{(4)} = \frac{\ell^2}{z^2}\left(-\left(1-\left(\frac{z}{z_0}\right)^3\right)dt^2 + \delta_{ij}dx^i dx^j + \left(1-\left(\frac{z}{z_0}\right)^3\right)^{-1} dz^2\right)$$

then $Z(z) = \frac{z}{\sqrt{1-\left(\frac{z}{z_0}\right)^3}}$.

- Taking $\mu = z/z_0$ one can show the general solution for A, B, C is;

$$A(Z(\mu)) = \frac{1}{5}\frac{1}{(2+\mu^3)^2}\left(20a - 45b\mu\left(4-\mu^3\right) - 6c\mu^3\right)$$

$$B(Z(\mu)) = \frac{4}{5z_0}\frac{(1-\mu^3)^{3/2}}{(2+\mu^3)^3}\left(10b\left(1-\mu^3\right) + c\mu^2\right)$$

$$C(Z(\mu)) = \frac{2}{15z_0^3}\frac{(1-\mu^3)^{3/2}}{(2+\mu^3)^2}\left(-20a\mu^3 + 90b\left(\frac{2}{\mu^2} - 5\mu\right) + 3c\left(10+\mu^3\right)\right)$$

for constants a, b, c; w.l.o.g. we take $a = 1$; we set $b = 0$ and redefine $k = \frac{2}{3} - \frac{c}{5}$ finding

$$A, C \geq 0 \quad \text{for} \quad 0 \leq k \leq \frac{6}{11}$$

HOLOGRAPHIC THEORIES - DEFORMATIONS OF BOUNDARY TORUS

- Then the boundary terms at the 'conformal boundary' and any horizons are given as;

$$\int_{\partial\mathcal{M}_\infty} dA^a J_a = \frac{2}{3}\frac{E}{\tilde{c}_T} - 2\left(\frac{4\pi T}{3}\right)^3\left(\frac{2}{3} - k\right)A(\Sigma)$$
$$\int_{\partial\mathcal{M}_H} dA^a J_a = -\frac{2}{3}k\frac{TS_H}{\tilde{c}_T} - \frac{2(4\pi)^2}{9}kT\,\chi(H)$$

- Let us assume the bulk horizon remains toroidal as for planar Schwarzschild. Then $\chi(H) = 0$ and so,

$$0 \geq \frac{3}{2}\int_{\partial\mathcal{M}} dA^a J_a = \frac{1}{\tilde{c}_T}\left(E - kTS\right) - \left(\frac{4\pi T}{3}\right)^3 (2 - 3k)\, A(\Sigma)$$

with equality for Schwarzschild.

- Since we fix temperature T and area $A(\Sigma)$ then we see,

$$0 \geq \frac{1}{\tilde{c}_T}\left(\Delta E - kT\Delta S\right)$$

For $k = 0$ this implies $\Delta E \leq 0$ – in fact we may take $0 \leq k \leq \frac{6}{11}$ but we get to $k = 1$.

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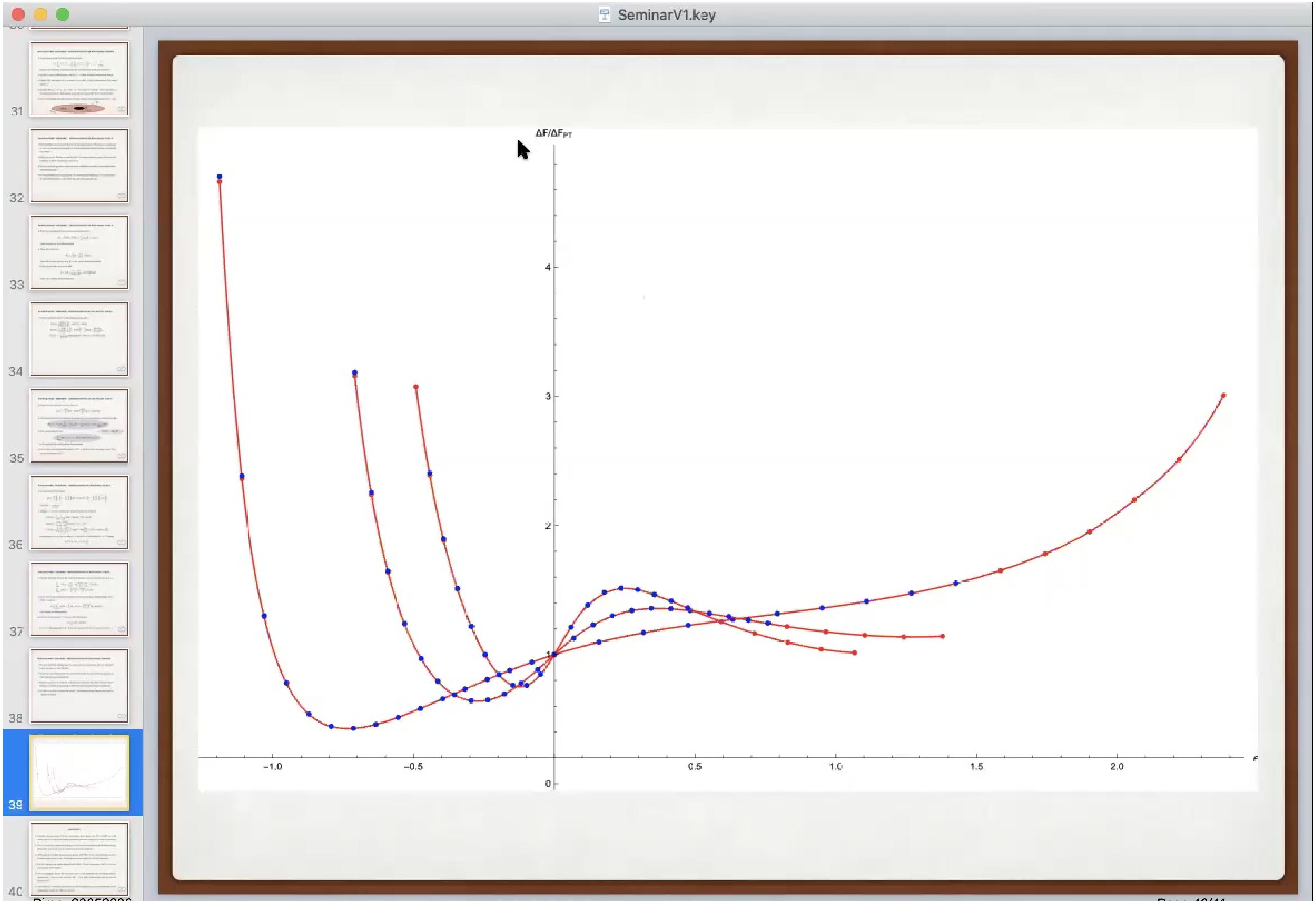
HOLOGRAPHIC THEORIES - DEFORMATIONS OF BOUNDARY SPHERE

- We have shown for deformations of a sphere at zero temperature $\Delta E \leq 0$. But how does it compare to other theories?
- Recall for small deformations the result is universal. So the interesting question is what happens non-perturbatively.
- Work in progress; we solve the bulk Einstein equations with the deformed sphere boundary condition using numerical GR techniques (harmonic Einstein approach).
- We then use a trick to extract the energy – the boundary stress tensor is very hard to compute accurately.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ΔF/ΔF_PT
4
3
2
1
0
−1.0
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0.5
1.0
1.5
2.0
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SUMMARY

- Casimir vacuum energy or finite temperature free energy of a $(2+1)$-QFT on a 2-d curved space Σ at fixed area and temperature gives an energetic measure on geometry.
- This is an intrinsic quantum analog to classical measures such as the bending energy. Physically relevant for the dynamics of monolayer materials.
- QFTs appear to dislike homogeneous spaces. For CFTs at zero temperature, and free fermions and scalars at any temperature a round sphere is a local maximum.
- For free theories our results suggest that $\sigma \Delta K < 0$ and consequently $\Delta F < 0$ for any deformation of the sphere.
- For a holographic theory we can prove $\Delta E < 0$ for deformations of a sphere at zero temperature. A proof may exist for $\Delta F < 0$ at finite temperature, but we haven't found it yet!
- And finally, in detail the dependence of ΔE on geometry at zero temperature is very surprisingly similar for different theories.