Title: On Gauss-Bonnet gravity in four dimensions

Speakers: David Kubiznak

Series: Strong Gravity

Date: April 30, 2020 - 1:00 PM

URL: http://pirsa.org/20040094

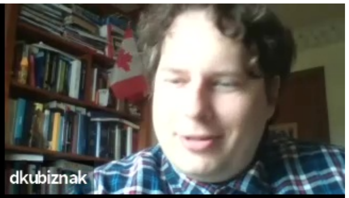
Abstract: We comment on the recently introduced Gauss-Bonnet gravity in four dimensions. We argue that it does not make sense to consider this theory to be defined by a set of D->4 solutions of the higher-dimensional Gauss-Bonnet gravity. We show that a well-defined D->4 limit of Gauss-Bonnet Gravity is obtained generalizing a method employed by Mann and Ross to obtain a limit of the Einstein gravity in D=2 dimensions. This is a scalar-tensor theory of the Horndeski type obtained by dimensional reduction methods. By considering simple spacetimes beyond spherical symmetry (Taub-NUT spaces) we show that the naive limit of the higher-dimensional theory to four dimensions is not well defined and contrast the resultant metrics with the actual solutions of the new theory. Theory and solutions in lower dimensions will also be briefly discussed.

On Gauss-Bonnet gravity in four dimensions

David Kubizňák
(Perimeter Institute)

$$\boxed{\mathcal{G} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2}$$

Strong Gravity Seminar
Perimeter Institute, Waterloo, Canada
April 30, 2020



Lovelock Theorem

D. Lovelock, *The Einstein Tensor and Its Generalizations", Journal of Mathematical Physics.* ***12*** *(3): 498–501 (1971).*

In **four dimensions**, the Einstein-Hilbert action is the only local action (apart from the cosmological constant and topological terms) that leads to the **second order** differential equations for the metric.

Einstein's theory is the unique theory!

Gauss-Bonnet gravity in 4 dimensions

D. Glavan and C. Lin, Einstein-Gauss-Bonnet gravity in 4-dimensional space-time, Phys. Rev. Lett. 124 (2020) 081301, [1905.03601].

Plan of the talk

I. What is Gauss-Bonnet gravity?

II. 4-dimensional GB gravity?
   a) Glavan-Lin proposal
   b) Problems with the “naïve” D->4 limit

III. Horndeski type Gauss-Bonnet gravity in 4D
   a) Kaluza-Klein approach
   b) Conformal trick derivation
   c) Solutions

IV. Lower-dimensional theory and solutions

V. Summary

Based on

Paper 1: R. A. Hennigar, D. Kubiznak, R. B. Mann and C. Pollack, *On Taking the D->4 limit of Gauss-Bonnet Gravity: Theory and Solutions*, 2004.09472.

Paper 2: R. A. Hennigar, D. Kubiznak, R. B. Mann and C. Pollack, *Lower-dimensional Gauss-bonnet gravity and BTZ black holes*, 2004.12995.

Paper 1 has some overlap with

P. G. Fernandes, P. Carrilho, T. Clifton and D. J. Mulryne, *Derivation of Regularized Field Equations for the Einstein-Gauss-Bonnet Theory in Four Dimensions*, 2004.08362.

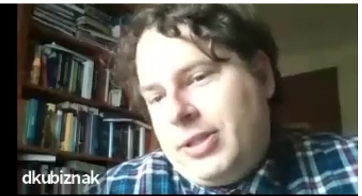
Einstein Theory 1

To write the **Gravitational Action** we want

- **Scalar Lagrangian** -- diffeomorphism invariance
- **Second-order (E-L)** equations for the metric
- Due to the **Equivalence principle** there is no scalar with only first derivatives: $$I = I(g, \partial g)$$
- Best one can do is to write

$$S_{\mathrm{EH}}[g] = \frac{1}{16\pi G}\int \sqrt{-g}R(g, \partial g, \partial^2 g)$$

How come the Einstein equations are 2[nd] order?



Einstein Theory 2

$$\delta S_{\rm EH} = \frac{1}{16\pi G}\int \delta(\sqrt{-g}R_{\alpha\beta}g^{\alpha\beta})$$

$$= \frac{1}{16\pi G}\int (\underbrace{R\delta\sqrt{-g}+\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}}_{\text{easy to calculate: } \sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu}} + \underbrace{\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}}_{\text{difficult: throw away}})$$

So we recover $G_{\mu\nu}(g, \partial g, \partial^2 g) = 0$

Luckily we were (almost) right throwing away the last term:

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu V^\mu\,, \quad V_\mu = \nabla^\beta(\delta g_{\mu\beta}) - g^{\alpha\beta}\nabla_\mu(\delta g_{\alpha\beta})$$

$$\boxed{\sqrt{-g}R(g, \partial g, \partial^2 g) = \sqrt{-g}\tilde{R}(g, \partial g) + \partial_\mu \hat{R}^\mu(g, \partial g)}$$

Einstein Theory 3

$$\mathcal{L} = R$$

- Is this just the simplest choice or can we add other scalars?

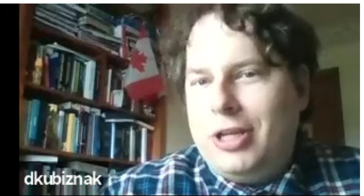
$$R^2\,,\ R_{\mu\nu}R^{\mu\nu}\,,\ R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}$$

$$R^3\,, \nabla_\mu R \nabla^\mu R\,, \ldots$$

- **Lovelock Theorem (1971):** In **4D**, the Einstein-Hilbert action is the only local action, apart from the cosmological constant and **topological terms (total derivatives)**, that leads to the **second-order** PDEs for the metric.

Example of topological term (in 4D)

$$\mathcal{G} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$$



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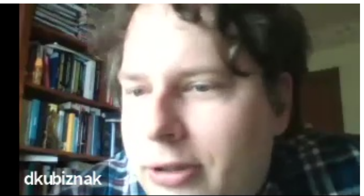
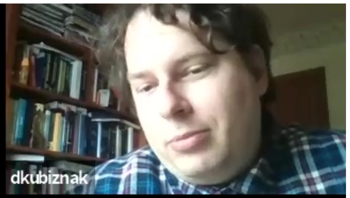




Gauss-Bonnet gravity

$$\boxed{\mathcal{G} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2}$$

- Interestingly, the **Gauss-Bonnet term** is **topological** (total derivative) **only in 4D.**
- In **D<4** it **identically vanishes**!
- In **D=5 and higher** dimensions it yields non-trivial EOMs:

$$\begin{aligned} H_{\alpha\beta} \;&=\; -\frac{1}{2}g_{\alpha\beta}\mathcal{G} + 2RR_{\alpha\beta} - 4R_{\alpha\gamma}R_{\beta}{}^{\gamma} \\ &\quad +4R_{\gamma\alpha\beta\delta}R^{\gamma\delta} + 2R_{\alpha}{}^{\gamma\delta\kappa}R_{\beta\gamma\delta\kappa} = 0\,. \end{aligned}$$

These are 2nd-order PDEs !!!

$$\boxed{\sqrt{-g}\,\mathcal{G}(g,\partial g,\partial^2 g) = \sqrt{-g}\,\tilde{\mathcal{G}}(g,\partial g) + \partial_\mu \hat{\mathcal{G}}^\mu(g,\partial g)}$$

Lovelock gravity

= Unique higher-curvature (with local action) gravity that yields **2nd-order PDEs** for the metric

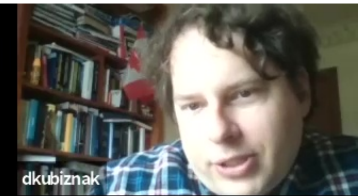
$$\mathcal{L} = \frac{1}{16\pi G_N} \sum_{k=0}^{K} \alpha_k \mathcal{L}^{(k)} \qquad K = \lfloor \tfrac{d-1}{2} \rfloor$$

where $\mathcal{L}^{(k)}$ are the 2k-dimensional Euler densities

$$\mathcal{L}^{(k)} = \frac{1}{2^k} \delta^{a_1 b_1 \dots a_k b_k}_{c_1 d_1 \dots c_k d_k} R_{a_1 b_1}{}^{c_1 d_1} \dots R_{a_k b_k}{}^{c_k d_k}$$

- k=0: cosmological term $\Lambda = -\alpha_0/2$
- k=1: Einstein-Hilbert term R **(topological in 2D)**
- k=2: Gauss-Bonnet term $\mathcal{G}$ **(topological in 4D)**
- k=3: 3rd-order Lovelock **(topological in 6D)**

"Natural generalization of Einstein's theory in higher dimensions"



Beyond Lovelock

Less restricted theories with **higher-order EOM** and some nice properties – "*desperate times require desperate measures*"

- **Quasi-topological gravity** (EOM on SSS are 2nd-order)

  R. C. Myers and B. Robinson, Black Holes in Quasi-topological Gravity, JHEP 1008 (2010) 067, [1003.5357].

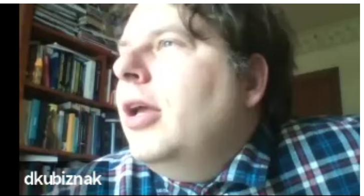
- **Einsteinian Cubic Gravity** (active already in 4D)

  P. Bueno and P. A. Cano, Einsteinian cubic gravity, Phys. Rev. D94 (2016) 104005,

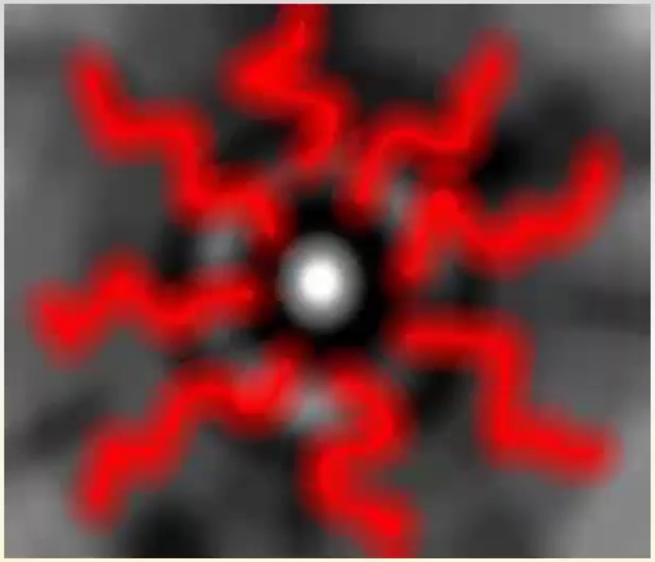
- **SMF VSSS Gravity** (coincides with Einsteinian in 4D)

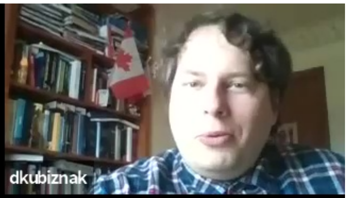
  R. A. Hennigar, D. Kubiznak and R. B. Mann,Generalized quasitopological gravity, Phys.Rev. D95(2017) 104042, [1703.01631]

In what follows we are not interested in these theories and focus on 1st-order and 2nd-order Lovelock gravity!



II) 4-dimensional Gauss-Bonnet gravity?





Glavan-Lin proposal 2

Observation: Gauss-Bonnet equations proportional to (D-4):

$$\sum_{p=0}^{\frac{D}{2}-1} \alpha_p \,(D-2p)\, \epsilon_{a_1 \dots a_D} R^{a_1,a_2} \wedge \dots \wedge R^{a_{2p-1},a_{2p}} \wedge e^{a_{2p+1}} \wedge \dots \wedge e^{a_{D-1}} = 0.$$

Proposal: “dimensional regularization”

$$\boxed{\alpha \to \alpha/(D-4)} \qquad \boxed{D \to 4?}$$

Does not effect gravitational dof???

Glavan-Lin proposal 3

Theory: EHGB action in D dimensions

$$S_D = \int d^D x \sqrt{-g}\left(R - 2\Lambda + \hat{\alpha}\mathcal{G}\right)$$

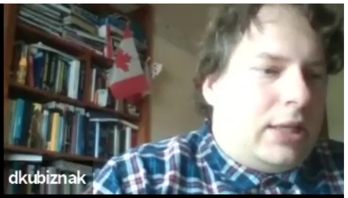
Consider: Enhanced symmetry solutions (maximally symmetric spaces, FLRW, SSS) by looking at the **few relevant equations** and taking the D->4 limit.

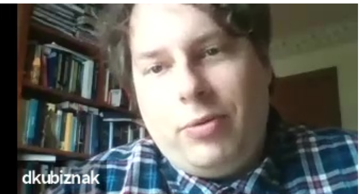
- 2 branches of (A)dS solutions
- Perturbing around $$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}$$

  (unable to distinguish from GR)
- Cosmology

$$3M_P^2 H^2 + 6\alpha H^4 = \frac{1}{2}\dot{\phi}^2 + V(\phi),$$
$$-M_P^2 \Gamma \dot{H} = \frac{1}{2}\dot{\phi}^2,$$

$$\Gamma \equiv 1 + 4\alpha H^2/M_P^2.$$





Glavan-Lin proposal 4

Black hole solutions:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2_{D-2}\,,$$

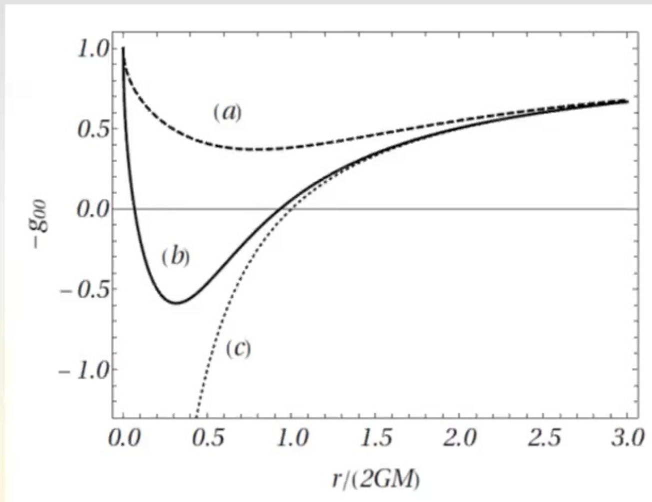
$$f_\pm(r) = 1 + \frac{r^2}{2(D-3)(D-4)\alpha}\left[1 \pm \sqrt{1 + \frac{64\pi G_D (D-3)(D-4)\alpha M}{(D-2)\Omega_{D-2} r^{D-1}} + \frac{8(D-3)(D-4)\alpha\Lambda_0}{(D-1)(D-2)}}\right]$$

- D->4 limit straightforward
- Minus branch is Schwarzschild-like

$$-\,g_{00} \overset{r\to\infty}{\sim} 1 - \frac{2GM}{r}$$

- “Singularity free”

$$R \propto r^{-3/2}$$



Problems

1) Is the D->4 limit well defined?

- M. Gurses, T. C. Sisman and B. Tekin, *Is there a novel Einstein-Gauss-Bonnet theory in four dimensions?*, 2004.03390.

  Abstract: **No!**

- W.-Y. Ai, *A note on the novel 4D Einstein-Gauss-Bonnet gravity*, 2004.02858.

Moral: Theory **always remains higher-dimensional** (extra equations in higher dimensions)

Explicitly: Can be seen on Taub-NUT spacetimes:

- R. A. Hennigar, D. Kubiznak, R. B. Mann and C. Pollack, *On Taking the D->4 limit of Gauss-Bonnet Gravity: Theory and Solutions*, 2004.09472.

Different higher-dimensional bases (CP2, S2, T2) yield **different 4D Taub-NUTs** in the limit D->4.

Problems

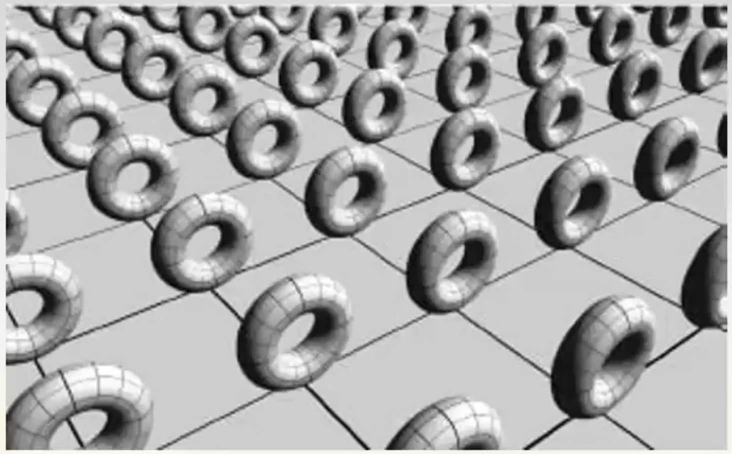
2) Graviton scattering amplitudes:

J. Bonifacio, K. Hinterbichler and L. A. Johnson, *Amplitudes and 4D Gauss-Bonnet Theory*, 2004.10716.

- There are **no** other than GR tree level graviton scattering amplitudes in 4D
- By taking the limit of higher-dimensional Gauss-Bonnet scattering amplitudes, one obtains amplitudes of a certain **scalar-tensor theory**

So, is there Gauss-Bonnet gravity in 4D?

III) Horndeski type Gauss-Bonnet gravity in 4D



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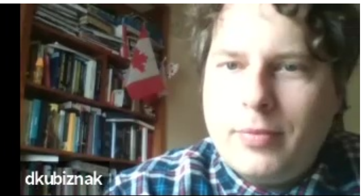


Kaluza-Klein approach

H. Lu and Y. Pang, *Horndeski Gravity as D->4 Limit of Gauss-Bonnet*, 2003.11552. (also also T. Kobayashi, 2003.12771)

- Start with EHGB $$S_D = \int d^D x \sqrt{-g}\big(R - 2\Lambda + \hat{\alpha}\mathcal{G}\big)$$
- **Compactify** on

$$ds_D^2 = ds_p^2 + e^{2\phi} d\Sigma^2_{D-p,\,\lambda} \qquad R_{abcd} = \lambda(g_{ac}g_{bd} - g_{ad}g_{bc})$$

- Resultant **effective p-dimensional** action is

$$\begin{aligned}S_p = \frac{1}{16\pi G_p}\int d^p x\sqrt{-g}e^{(D-p)\phi}\Big\{&R - 2\Lambda_0 + (D-p)(D-p-1)\big((\partial\phi)^2 + \lambda e^{-2\phi}\big)\\
&+\alpha\Big(\mathrm{GB} - 2(D-p)(D-p-1)\left[2G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \lambda R e^{-2\phi}\right]\\
&-(D-p)(D-p-1)(D-p-2)\left[2(\partial\phi)^2\Box\phi + (D-p-1)((\partial\phi)^2)^2\right]\\
&+(D-p)(D-p-1)(D-p-2)(D-p-3)\left[2\lambda(\partial\phi)^2 e^{-2\phi} + \lambda^2 e^{-4\phi}\right]\Big)\Big\},\end{aligned}$$

Kaluza-Klein approach

- In $p \leq 4$ one can substract topological (zero) term

$$-\frac{\alpha}{16\pi G_p}\int d^p x\sqrt{-g}\,\mathrm{GB}$$

- Rescale the coupling alpha and take the limit:

$$\alpha \to \frac{\alpha}{D-p} \qquad D \to p$$

(Limit of **0-dim.** internal space)

Gauss-Bonnet in $p \leq 4$

$$S = \int d^p x\sqrt{-g}\Big[R - 2\Lambda + \alpha\Big(\phi\mathcal{G} + 4G^{ab}\partial_a\phi\partial_b\phi - 4(\partial\phi)^2\Box\phi + 2((\nabla\phi)^2)^2\Big)\Big]\,,$$

$$S_\lambda = \int d^p x\sqrt{-g}\Big(-2\lambda R e^{-2\phi} - 12\lambda(\partial\phi)^2 e^{-2\phi} - 6\lambda^2 e^{-4\phi}\Big)$$

The new scalar-tensor theory

$$S = \int d^p x \sqrt{-g}\Big[R - 2\Lambda + \alpha\Big(\phi\mathcal{G} + 4G^{ab}\partial_a\phi\partial_b\phi$$
$$-4(\partial\phi)^2\Box\phi + 2((\nabla\phi)^2)^2\Big)\Big]\,,$$

- It is a **Horndeski-type Theory**

  G. W. Horndeski, *Second-order scalar-tensor field equations in a four-dimensional space*, International Journal of Theoretical Physics 10 (1974) 363-384.

- It can also be derived by generalizing a **conformal trick** used many years ago by Mann and Ross to derive D->2 limit of GR

- The advantage is that this trick does not “require an assumption of extra dimensions”

Conformal trick: D->2 limit of GR

R. B. Mann and S. F. Ross, The D->2 limit of general relativity, Class. Quant. Grav. 10 (1993) 1405{1408, [gr-qc/9208004].

- Start with EH $S_D^{\mathrm{EH}} = \kappa \int d^D x \sqrt{-g} R$ (topological in D=2)
- Evaluate it for the **conformally rescaled** metric $\tilde{g} = e^{\psi} g$
- Expand around $\epsilon = D - 2$ and add a (in D=2 topological) **counter term** to have a finite limit D->2.

$$
\begin{aligned}
S_D^{\mathrm{EH}} &= \kappa \left(\int d^D x \sqrt{-\tilde{g}} \tilde{R} - \int d^D x \sqrt{-g} R \right) \\
&= \kappa \int d^D x \left[e^{\epsilon\psi/2} \left[\left(R - (\epsilon+1) \Box \psi \right) - \frac{1}{4} \epsilon (\epsilon+1) (\partial\psi)^2 \right] - R \right]
\end{aligned}
$$

- **Rescale the coupling** according to $\boxed{\frac{1}{2}(D-2)\kappa \to \kappa}$

Conformal trick: D->2 limit of GR

R. B. Mann and S. F. Ross, The D->2 limit of general relativity, Class. Quant. Grav. 10 (1993) 1405{1408, [gr-qc/9208004].

- Take the limit D->2, and throw away boundary terms, to obtain the following **scalar-tensor theory**

$$S_2^{\mathrm{EH}} \equiv \lim_{\epsilon\to 0} S_D^{\mathrm{EH}} = \kappa \int d^2x\sqrt{-g}\Big[\psi R + \frac{1}{2}(\partial\psi)^2\Big]$$

  - This theory has interesting **BH solutions**, whose metric is in some sense D->2 limit of D-dimensional Kerr solution, e.g.

    A. M. Frassino, R. B. Mann and J. R. Mureika, Lower-Dimensional Black Hole Chemistry, Phys. Rev. D 92 (2015) 124069, [1509.05481].

Conformal trick: D->4 limit of GB

P. G. Fernandes, P. Carrilho, T. Clifton and D. J. Mulryne, *Derivation of Regularized Field Equations for the Einstein-Gauss-Bonnet Theory in Four Dimensions*, 2004.08362.
R. A. Hennigar, D. Kubiznak, R. B. Mann and C. Pollack, *On Taking the D->4 limit of Gauss-Bonnet Gravity: Theory and Solutions*, 2004.09472.

- Replace EH part with the GB part

$$S_D^{GB} = \alpha\left(\int d^D x\sqrt{-\tilde{g}}\tilde{\mathcal{G}} - \int d^D x\sqrt{-g}\mathcal{G}\right)$$

- Expand around $\epsilon = (D-4)$, rescale the coupling according to $(D-4)\alpha \to \alpha$, and take the limit D->4.

- After field redefinition: $\psi \to -2\phi\,,\quad g_{ab} \to -\frac{1}{2}g_{ab}\,,\quad \alpha \to \alpha/2$

$$S = \int d^p x\sqrt{-g}\Big[R - 2\Lambda + \alpha\Big(\phi\mathcal{G} + 4G^{ab}\partial_a\phi\partial_b\phi - 4(\partial\phi)^2\Box\phi + 2((\nabla\phi)^2)^2\Big)\Big]\,,$$

Solutions: GB Black Hole

H. Lu and Y. Pang, *Horndeski Gravity as D->4 Limit of Gauss-Bonnet*, 2003.11552.

- Constructed the SSS solutions of the **Horndeski-GB theory**

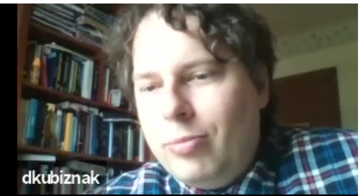
$$ds^2 = -f dt^2 + \frac{dr^2}{fh} + r^2 d\Omega^2$$

- **Special class** of solutions has $h = 1$

$$f_\pm = 1 + \frac{r^2}{2\alpha}\left(1 \pm \sqrt{1 + \frac{4}{3}\alpha\Lambda + \frac{8\alpha M}{r^3}}\right)$$

$$\phi_\pm = \int \frac{1 \pm \sqrt{f}}{\sqrt{f} r} dr$$

Metric **coincides** with the naïve D->4 limit. True also for non-trivial internal space curvature lambda.



Solutions: GB Black Hole

- The same spacetime considered as "**quantum gravity** corrected metric"
  - Y. Tomozawa, Quantum corrections to gravity, 1107.1424.
  - G. Cognola, R. Myrzakulov, L. Sebastiani and S. Zerbini, Einstein gravity with Gauss-Bonnet entropic corrections, Phys. Rev. D 88 024006, [1304.1878].
- **Have a theory** – so can use Wald's formalism to calculate **entropy**, which picks up *logarithmic corrections*:

$$S = \frac{1}{G_4}\left(\pi r_+^2 + 4\alpha\pi \log\frac{r_+}{L}\right)$$

- Many papers have studied observational features
  - Light bending
  - Black hole shadow
  - Thin accretion disc

Conformal trick: D->4 limit of GB

P. G. Fernandes, P. Carrilho, T. Clifton and D. J. Mulryne, *Derivation of Regularized Field Equations for the Einstein-Gauss-Bonnet Theory in Four Dimensions*, 2004.08362.
R. A. Hennigar, D. Kubiznak, R. B. Mann and C. Pollack, *On Taking the D->4 limit of Gauss-Bonnet Gravity: Theory and Solutions*, 2004.09472.

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$$S = \int d^p x\sqrt{-g}\Big[R - 2\Lambda + \alpha\Big(\phi\mathcal{G} + 4G^{ab}\partial_a\phi\partial_b\phi - 4(\partial\phi)^2\Box\phi + 2((\nabla\phi)^2)^2\Big)\Big]\,,$$

Solutions: GB Taub-NUT?

R. A. Hennigar, D. Kubiznak, R. B. Mann and C. Pollack, *On Taking the D->4 limit of Gauss-Bonnet Gravity: Theory and Solutions*, 2004.09472.

$$ds^2 = -f(dt + 2n\cos\theta d\varphi)^2 + \frac{dr^2}{fh} + (r^2 + n^2)d\Omega^2$$

- Concentrating on $h = 1$ (works for higher-dim GB)

δf $\quad \phi_\pm = \int \frac{\sqrt{f(3fn^2 + n^2 + r^2)} \pm rf}{f(r^2 + n^2)}$

δh Gives hope for D->4 limit of higher-dimensional GB Taub-NUTs, but!

Solutions: GB Taub-NUT?

R. A. Hennigar, D. Kubiznak, R. B. Mann and C. Pollack, *On Taking the D->4 limit of Gauss-Bonnet Gravity: Theory and Solutions*, 2004.09472.

$$ds^2 = -f(dt + 2n\cos\theta d\varphi)^2 + \frac{dr^2}{fh} + (r^2+n^2)d\Omega^2$$

- Concentrating on $h = 1$ (works for higher-dim GB)

δf

$$\phi_\pm = \int \frac{\sqrt{f(3fn^2+n^2+r^2)} \pm rf}{f(r^2+n^2)}$$

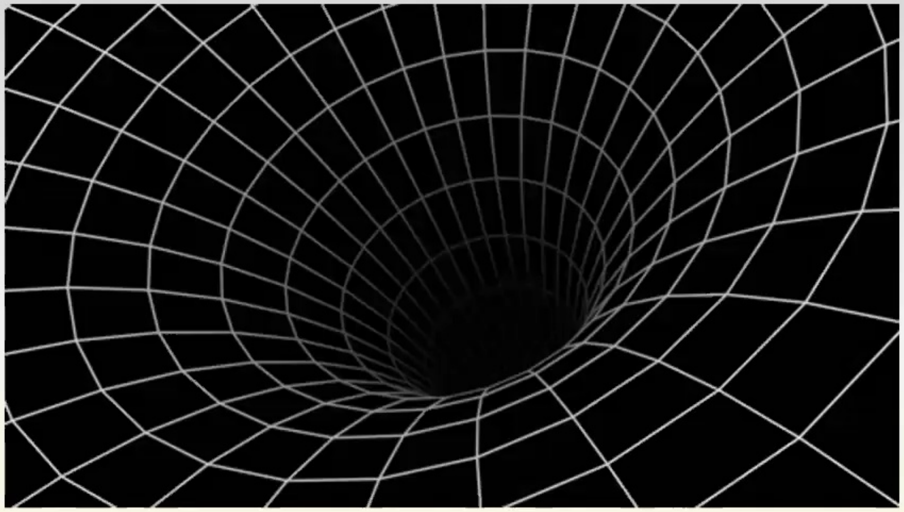
δh Gives hope for D->4 limit of higher-dimensional GB Taub-NUTs, but!

$\delta\phi$

$$f = -\frac{1}{4}\frac{r^2+n^2}{l^2} \qquad \Lambda = \frac{3}{16}\frac{4n^2+\alpha}{n^4}, \quad l = \pm n$$

There is NO Lorentzian Taub-NUT solution! (with h=1)

IV) Lower-dimensional Theory and Solutions



https://www.quantamagazine.org/tag/quantum-gravity/

New Horndeski 3D GB Gravity

$$S = \int d^p x \sqrt{-g}\Big[R - 2\Lambda + \alpha\Big(\phi\mathcal{G} + 4G^{ab}\partial_a\phi\partial_b\phi - 4(\partial\phi)^2\Box\phi + 2((\nabla\phi)^2)^2\Big)\Big],$$

- **The theory is applicable in p=3 dimensions**

- Obtained by D->3 **KK procedure** with 0-dimensional internal space

  H. Lu and Y. Pang, *Horndeski Gravity as D->4 Limit of Gauss-Bonnet*, 2003.11552.

- Can equally be obtained via the **conformal trick** in D=3 dimensions

  R. A. Hennigar, D. Kubiznak, R. B. Mann and C. Pollack, *Lower-dimensional Gauss-bonnet gravity and BTZ black holes*, 2004.12995.

Gauss-Bonnet BTZ Black Holes

R. A. Hennigar, D. Kubiznak, R. B. Mann and C. Pollack, *Lower-dimensional Gauss-bonnet gravity and BTZ black holes*, 2004.12995.

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\varphi^2$$

$$\phi = \ln(r/l)$$

$$f_\pm = -\frac{r^2}{2\alpha}\left(1 \pm \sqrt{1 - \frac{4\alpha m}{r^2} + \frac{4\alpha}{\ell^2}}\right)$$

- f- branch describes a BH which is asymptotically BTZ (with modified Lambda)
- It has singularity in the origin (repulsive potential)
- It has identical thermodynamics to standard BTZ black hole (entropy calculated through Wald formalism)

Gauss-Bonnet BTZ Black Holes

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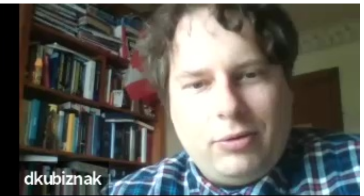
$$f_\pm = -\frac{r^2}{2\alpha}\left(1 \pm \sqrt{1 - \frac{4\alpha m}{r^2} + \frac{4\alpha}{\ell^2}}\right)$$

- Metric first constructed by naïve **D->3 limit** of D-dimensional GB black hole

$$f_\epsilon = \kappa - \frac{r^2}{2\alpha}\left(1 \pm \sqrt{1 - 4\alpha\Lambda - \frac{4\alpha(\kappa - \Lambda r_+^2)}{r^2} + \frac{4\alpha^2\kappa^2}{r^2 r_+^2}}\right)$$

R. A. Konoplya and A. Zhidenko, *BTZ black holes with higher curvature corrections in the 3D Einstein-Lovelock theory*, 2003.12171.



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Only kappa=0 metric is a solution!

Alternative 3D Horndeski-GB Gravity

$$S_3^{(2)} = \int d^3x \sqrt{-g} e^{\phi} (R - 2\Lambda) + S_3^{\mathcal{G}}$$

$$S_3^{\mathcal{G}} = \alpha \int d^3x \sqrt{-g} e^{\phi} \left[-4G^{ab} \partial_a \phi \partial_b \phi + 2(\partial \phi)^2 \Box \phi \right]$$

- Obtained by **D->4 KK procedure** with **1-dimensional** internal space

  H. Lu and Y. Pang, *Horndeski Gravity as D->4 Limit of Gauss-Bonnet*, 2003.11552.

- Not clear how to obtain using the conformal trick

- Admits **BH solutions** which seem quite interesting

  R. A. Hennigar, D. Kubiznak, R. B. Mann and C. Pollack, *Lower-dimensional Gauss-bonnet gravity and BTZ black holes*, 2004.12995.

Summary

1) In past 2 months, papers on **4D Gauss-Bonnet Gravity** plagued Arxiv.

2) The original proposal by Glavan and Lin of taking the **naïve D->4 limit** of solutions/EOMs while rescaling the coupling constant
$$\boxed{\alpha \to \alpha/(D-4)}$$
is interesting but **does not work** – one does not obtain a 4D metric theory.

3) However, one can make sense of the limit in two alternative scenarios – by **KK compactification** and by **conformal trick**. The resultant theory is a **scalar-tensor theory of Horndeski type**.
$$\boxed{\begin{aligned} S = \int d^p x \sqrt{-g}\Big[R - 2\Lambda + \alpha\Big(\phi\mathcal{G} + 4G^{ab}\partial_a\phi\partial_b\phi \\ -4(\partial\phi)^2\Box\phi + 2((\nabla\phi)^2)^2\Big)\Big]\,, \end{aligned}}$$

4) This theory is **applicable in p=4,3,2 dimensions**.

Summary

5) **The theory is "nice"** as

a) It is derived from a more fundamental theory (c.f. EdGB, or other Horndeski theories)

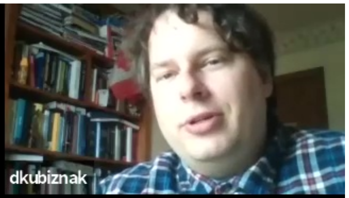
b) Admits **interesting analytic solutions**, some of which coincide with the naïve D->4 limit.

c) For example: **BH solutions**

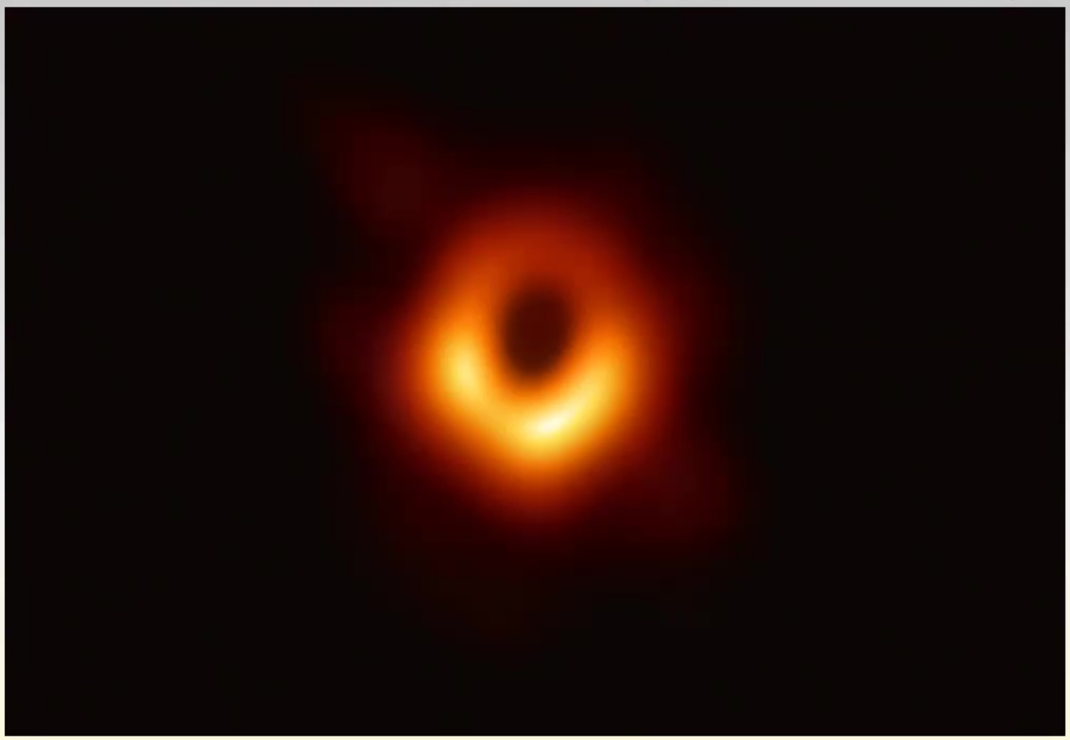
$$f_{\pm} = 1 + \frac{r^2}{2\alpha}\Big(1 \pm \sqrt{1 + \frac{4}{3}\alpha\Lambda + \frac{8\alpha M}{r^3}}\Big)$$

Seem interesting from both **theoretical** and **observational** aspects (logarithmic corrections to *entropy*, "removal of *singularities*", observational predictions)

6) Is there **Gauss-Bonnet gravity in the sky**?



Gauss-Bonnet gravity in the sky?



https://medium.com/predict/black-holes-shadow-seen-for-the-first-time-46578a7a2787

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