

Title: Quantum-reduced loop gravity from the perspective of full LQG

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Abstract: Quantum-reduced loop gravity is a model of loop quantum gravity, whose characteristic feature is the considerable simplicity of its kinematical structure in comparison with that of full loop quantum gravity. The model therefore provides an accessible testing ground for probing the physical implications of loop quantum gravity. In my talk I will give a brief introduction to quantum-reduced loop gravity, and examine the relation between the quantum-reduced model and full loop quantum gravity. In particular, I will focus on clarifying how the operators of the quantum reduced model are related to those of the full theory. I will show that despite their simplicity, the operators of the quantum-reduced model are simply the operators of the full theory acting on states in the Hilbert space of the quantum-reduced model. In order to pass from the full theory operators to the "reduced" operators, one only has to keep in mind that the states of the quantum-reduced model are labeled with large spins, and discard terms which are of lower than leading order in j .

Quantum-reduced loop gravity
from the perspective of full loop quantum gravity

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30 April 2020

Introduction: Models of loop quantum gravity

The foundations of loop quantum gravity are very solid from the mathematical, and (in my opinion) the conceptual point of view.

From the practical point of view, however, it tends to be technically very challenging to perform any concrete calculations about physics.

This has motivated the development of various simplified approaches in order to probe the physical implications of the theory. For example:

- Models based on a classical symmetry reduction
(e.g. loop quantum cosmology, spherically symmetric models)
- Quantum-reduced loop gravity
- Effective dynamics



The kinematical structure of loop quantum gravity

Kinematical Hilbert space of LQG: The space of cylindrical functions

$$\Psi_{\Gamma}(h_{e_1}, \dots, h_{e_N})$$

Γ : Graph, with edges e_1, \dots, e_N

h_{e_1}, \dots, h_{e_N} : $SU(2)$ group elements (holonomies)

A basis on \mathcal{H}_{kin} is given by the functions
$$\prod_{e \in \Gamma} D_{m_e n_e}^{(j_e)}(h_e)$$

Elementary operators of the theory: Holonomy and flux operators.

The holonomy operator acts on cylindrical functions by multiplication:

$$\widehat{D_{mn}^{(j)}}(h_e) \Psi_{\Gamma}(h_{e_1}, \dots, h_{e_N}) = D_{mn}^{(j)}(h_e) \Psi_{\Gamma}(h_{e_1}, \dots, h_{e_N})$$

Its action is computed using the $SU(2)$ Clebsch–Gordan series

$$D_{m_1 n_1}^{(j_1)}(h_e) D_{m_2 n_2}^{(j_2)}(h_e) = \sum_j C_{m_1 m_2 m_1+m_2}^{(j_1 j_2 j)} C_{n_1 n_2 n_1+n_2}^{(j_1 j_2 j)} D_{m_1+m_2 n_1+n_2}^{(j)}(h_e)$$

Flux operator

Define an auxiliary operator $J_i^{(v,e)}$ as

$$J_i^{(v,e)} \Psi_\Gamma = \begin{cases} L_i^{(k)} \Psi_\Gamma(h_{e_1}, \dots, h_{e_k}, \dots, h_{e_N}) & \text{if } e = e_k \text{ and } e \text{ begins at } v \\ R_i^{(k)} \Psi_\Gamma(h_{e_1}, \dots, h_{e_k}, \dots, h_{e_N}) & \text{if } e = e_k \text{ and } e \text{ ends at } v \end{cases}$$

$L_i^{(k)}, R_i^{(k)}$: Left- and right-invariant vector fields of $SU(2)$ acting on h_{e_k}

The operator acts on holonomies as

$$\begin{aligned} J_i^{(v,e)} D^{(j)}(h_e) &= iD^{(j)}(h_e) \tau_i^{(j)} & (e \text{ begins at } v) \\ J_i^{(v,e)} D^{(j)}(h_e) &= -i\tau_i^{(j)} D^{(j)}(h_e) & (e \text{ ends at } v) \end{aligned}$$

$\tau_i^{(j)}$: anti-Hermitian generators of $SU(2)$ in the spin- j representation

Using the operator $J_i^{(v,e)}$, the flux operator can be expressed as

$$E_i(S) \Psi_\Gamma(h_{e_1}, \dots, h_{e_N}) = 8\pi\beta G \sum_{x \in S} \sum_{e \text{ at } x} \frac{1}{2} \kappa(S, e) J_i^{(x,e)} \Psi_\Gamma(h_{e_1}, \dots, h_{e_N})$$

where $\kappa(S, e) = +1, 0$ or -1

The reduced Hilbert space

The Hilbert space of the quantum-reduced model is obtained by looking for (weak) solutions of certain reduction constraints, which are designed to implement a gauge fixing to a diagonal triad field in the quantum theory.

The reduced Hilbert space is a subspace of the kinematical Hilbert space of full LQG. The basis states spanning the reduced Hilbert space are characterized by the following requirements:

- The edges of the graph are aligned along the x -, y - and z -axes of a background coordinate system.
- The spin quantum number associated to each edge is large: $j_e \gg 1$.
- Each edge carries a representation matrix, both of whose magnetic indices take either the maximal or the minimal value (i.e. j_e or $-j_e$) with respect to the direction of the edge.


Reduced spin network states

Notation:

- $|jm\rangle_i$ ($i = x, y, z$): Eigenstates of J^2 and J_i with eigenvalues $j(j+1)$ and m
- $D_{mn}^{(j)}(h)_i \equiv {}_i\langle jm|D^{(j)}(h)|jn\rangle_i$: Wigner matrices in the basis $|jm\rangle_i$

Then a "reduced spin network state" is defined by a wave function of the form

$$\prod_{e \in \Gamma} D_{\sigma_e j_e \sigma_e j_e}^{(j_e)}(h_e)_{i_e}$$

where $\sigma_e = +1$ or -1 , and $i_e = x, y$ or z (according to the direction of e). 

Note: No intertwiners are involved in the reduced spin network state.

The relation $D_{jj}^{(j)}(h_e^{-1}) = D_{-j-j}^{(j)}(h_e)$ implies that one can either

- Take a fixed orientation of the graph, and have both values of σ_e ; or
- Take $\sigma_e = +1$ for every e , but keep track of the orientation of the graph.

Reduced operators

Elementary operators of the quantum-reduced model introduced as projections of the corresponding operators of full LQG down to the reduced Hilbert space.

The reduced holonomy operator acts according to the "reduced recoupling rule"

$$\begin{aligned} {}^R D_{ss}^{(s)}(h_e) D_{jj}^{(j)}(h_e) &= D_{j+s, j+s}^{(j+s)}(h_e) \\ {}^R D_{-s, -s}^{(s)}(h_e) D_{jj}^{(j)}(h_e) &= D_{j-s, j-s}^{(j-s)}(h_e) \end{aligned}$$

which is essentially the multiplication law of $U(1)$.

The reduced flux operator is associated to surfaces S_i such that the background coordinate $x^i = \text{const.}$ on S_i . The action of the operator is diagonal:

$$\begin{aligned} {}^R E_i(S_i) D_{\sigma_j \sigma_j}^{(j)}(h_e)_i &= (8\pi\beta G) \sigma_j D_{\sigma_j \sigma_j}^{(j)}(h_e)_i \\ {}^R E_i(S_k) D_{\sigma_j \sigma_j}^{(j)}(h_e)_l &= 0 \quad \text{if } i \neq k \text{ or } k \neq l \end{aligned}$$

Consequently, operators constructed out of the flux operator (e.g. volume) are extremely simple in the quantum-reduced model.

Relation between reduced operators and full theory operators

The main result of this talk: (arxiv:2004.00309)

The "reduced" operators of quantum-reduced loop gravity are simply the operators of full LQG acting on states in the Hilbert space of the quantum-reduced model.

(... even though the reduced operators are very simple, and were initially introduced as projections of the full operators onto the reduced Hilbert space.)

More precisely, when an operator of the full theory acts on a state in the reduced Hilbert space, the term of leading order in j reproduces the action of the corresponding reduced operator.

Thus the passage from the full theory operators to the "reduced" operators simply consists of recalling that the states of the quantum-reduced model carry large spins, and discarding terms of lower order in j .

In some cases this approach does not exactly reproduce the reduced operator in its standard form. In particular, our calculations lead to a modified version of the reduced holonomy operator.

Holonomy operator: Example 1 ($s = 1/2$)

$$D_{AB}^{(1/2)}(h_e) D_{jj}^{(j)}(h_e) = \sum_k C_{jA j+A}^{(j \ 1/2 \ k)} C_{jB j+B}^{(j \ 1/2 \ k)} D_{j+A j+B}^{(k)}(h_e)$$

$$\text{Case } A = B = +: \quad D_{++}^{(1/2)}(h_e) D_{jj}^{(j)}(h_e) = D_{j+1/2 j+1/2}^{(j+1/2)}(h_e)$$

$$\begin{aligned} \text{Case } A = B = -: \quad & D_{--}^{(1/2)}(h_e) D_{jj}^{(j)}(h_e) \\ &= \left(C_{j \ 1/2 \ j-1/2}^{(j \ 1/2 \ j+1/2)} \right)^2 D_{j-1/2 j-1/2}^{(j+1/2)}(h_e) + \left(C_{j \ 1/2 \ j-1/2}^{(j \ 1/2 \ j-1/2)} \right)^2 D_{j-1/2 j-1/2}^{(j-1/2)}(h_e) \\ &= \frac{1}{2j+1} D_{j-1/2 j-1/2}^{(j+1/2)}(h_e) + \frac{2j}{2j+1} D_{j-1/2 j-1/2}^{(j-1/2)}(h_e) \quad \star \end{aligned}$$

$$\therefore D_{--}^{(1/2)}(h_e) D_{jj}^{(j)}(h_e) = D_{j-1/2 j-1/2}^{(j-1/2)}(h_e) + \mathcal{O}\left(\frac{1}{j}\right)$$

Case $A = +, B = -$:

$$D_{+-}^{(1/2)}(h_e) D_{jj}^{(j)}(h_e) = C_{j \ 1/2 \ j+1/2}^{(j \ 1/2 \ j+1/2)} C_{j \ 1/2 \ j-1/2}^{(j \ 1/2 \ j+1/2)} D_{j+1/2 j-1/2}^{(j+1/2)}(h_e) = \mathcal{O}\left(\frac{1}{\sqrt{j}}\right)$$

At leading order in j , we recover the action of the reduced holonomy operator.

Holonomy operator: Example 2 ($s = 1$)

This example will reveal a new feature, which was not encountered in the spin-1/2 example. As in the first example, one finds

$$D_{11}^{(1)}(h_e)D_{jj}^{(j)}(h_e) = D_{j+1\ j+1}^{(j+1)}(h_e)$$

$$D_{-1\ -1}^{(1)}(h_e)D_{jj}^{(j)}(h_e) = D_{j-1\ j-1}^{(j-1)}(h_e) + \mathcal{O}\left(\frac{1}{j}\right)$$

$$D_{mn}^{(1)}(h_e)D_{jj}^{(j)}(h_e) = \mathcal{O}\left(\frac{1}{\sqrt{j}}\right) \quad (m \neq n)$$

The new feature arises in the case $m = n = 0$. We have

$$D_{00}^{(1)}(h_e)D_{jj}^{(j)}(h_e) = \left(C_{j\ 0\ j}^{(j\ 1\ j+1)}\right)^2 D_{jj}^{(j+1)}(h_e) + \left(C_{j\ 0\ j}^{(j\ 1\ j)}\right)^2 D_{jj}^{(j)}(h_e)$$

$$= \frac{1}{j+1} D_{jj}^{(j+1)}(h_e) + \frac{j}{j+1} D_{jj}^{(j)}(h_e)$$

That is: $D_{00}^{(1)}(h_e)D_{jj}^{(j)}(h_e) = D_{jj}^{(j)}(h_e) + \mathcal{O}\left(\frac{1}{j}\right)$

Hence also $D_{00}^{(1)}(h_e)$ acts appropriately as a "reduced" operator, in contrast to the standard formulation of the quantum-reduced model.

Holonomy operator: The general case

Consider the operator $D_{mn}^{(s)}(h_e)$ acting on the state $D_{jj}^{(j)}(h_e)$ (assuming $s \ll j$). For the diagonal components we expect to find

$$D_{mm}^{(s)}(h_e)D_{jj}^{(j)}(h_e) = D_{j+m, j+m}^{(j+m)}(h_e) + \mathcal{O}\left(\frac{1}{j}\right)$$

We have

$$\begin{aligned} & D_{mm}^{(s)}(h_e)D_{jj}^{(j)}(h_e) \\ &= \left(C_{j, m, j+m}^{(j, s, j+m)}\right)^2 D_{j+m, j+m}^{(j+m)}(h_e) + \sum_{k \neq j+m} \left(C_{j, m, j+m}^{(j, s, k)}\right)^2 D_{j+m, j+m}^{(k)}(h_e) \end{aligned}$$

The result now follows from

$$C_{j, m, j+m}^{(j, s, j+m)} = \sqrt{\frac{(2j)!(2j+2m+1)!}{(2j-s+m)!(2j+s+m+1)!}} = 1 + \mathcal{O}\left(\frac{1}{j}\right)$$

and

$$1 = \sum_k \left(C_{j, m, j+m}^{(j, s, k)}\right)^2 = \left(C_{j, m, j+m}^{(j, s, j+m)}\right)^2 + \sum_{k \neq j+m} \left(C_{j, m, j+m}^{(j, s, k)}\right)^2$$

For the off-diagonal components this implies $D_{mn}^{(s)}(h_e)D_{jj}^{(j)}(h_e) = \mathcal{O}\left(\frac{1}{\sqrt{j}}\right)$

Holonomy operator: Summary

In conclusion, we have shown that

$$D_{mm}^{(s)}(h_e)D_{jj}^{(j)}(h_e) = D_{j+m, j+m}^{(j+m)}(h_e) + \mathcal{O}\left(\frac{1}{j}\right) \quad \blackstar$$
$$D_{mn}^{(s)}(h_e)D_{jj}^{(j)}(h_e) = \mathcal{O}\left(\frac{1}{\sqrt{j}}\right) \quad (m \neq n)$$

We have therefore reproduced a modified version of the multiplication law of the reduced holonomy operator. Under the modified multiplication law, every diagonal component of the holonomy operator acts appropriately as a reduced operator.

For $s > 1/2$, this is a departure from the standard formulation of the quantum-reduced model, in which only the components labeled by the maximal or the minimal value of the magnetic indices are considered as valid operators.

In the modified version of the multiplication law, the magnetic index of the operator $D_{mm}^{(s)}(h_e)$ plays the same role as the spin of the operator $D_{ss}^{(s)}(h_e)$ does in the original version.

The operator $J_i^{(v,e)}$

Suppose, for concreteness, that e is oriented along the z -axis
 v is the beginning point of e

Then

$$J_i^{(v,e)} D_{jj}^{(j)}(h_e) = i D_{jm}^{(j)}(h_e) (\tau_i^{(j)})_{mj}$$

Inserting the matrix elements of the generators

$$(\tau_x^{(j)})_{mj} = -i \sqrt{\frac{j}{2}} \delta_{m,j-1} \quad (\tau_y^{(j)})_{mj} = \sqrt{\frac{j}{2}} \delta_{m,j-1} \quad (\tau_z^{(j)})_{mj} = -ij \delta_{mj}$$

we obtain

$$J_z^{(v,e)} D_{jj}^{(j)}(h_e) = j D_{jj}^{(j)}(h_e)$$

and

$$J_x^{(v,e)} D_{jj}^{(j)}(h_e) = J_y^{(v,e)} D_{jj}^{(j)}(h_e) = \mathcal{O}(\sqrt{j})$$

For a reduced holonomy oriented along the i -axis, we find

$$\begin{aligned} J_i^{(v,e)} D_{jj}^{(j)}(h_e)_i &= \pm j D_{jj}^{(j)}(h_e)_i \\ J_k^{(v,e)} D_{jj}^{(j)}(h_e)_i &= \mathcal{O}(\sqrt{j}) \quad (k \neq i) \end{aligned}$$

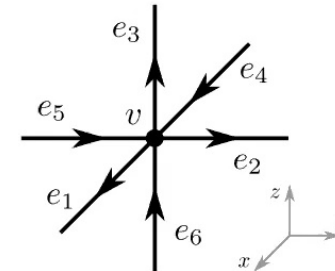
Volume operator

Acting on a 6-valent "cubical" node v , the Ashtekar–Lewandowski volume operator takes the form

$$V_v = \sqrt{|q_v|}$$

where

$$q_v = \frac{1}{8} \epsilon^{ijk} (J_i^{(v,e_1)} - J_i^{(v,e_4)}) (J_j^{(v,e_2)} - J_j^{(v,e_5)}) (J_k^{(v,e_3)} - J_k^{(v,e_6)})$$



When we apply this operator to the state

$$|\Psi_0\rangle = D_{j_1 j_1}^{(j_1)}(h_{e_1})_x D_{j_2 j_2}^{(j_2)}(h_{e_2})_y D_{j_3 j_3}^{(j_3)}(h_{e_3})_z D_{j_4 j_4}^{(j_4)}(h_{e_4})_x D_{j_5 j_5}^{(j_5)}(h_{e_5})_y D_{j_6 j_6}^{(j_6)}(h_{e_6})_z$$

we find that

- The term of leading order in j arises when $(i, j, k) = (x, y, z)$
- The terms with $(i, j, k) \neq (x, y, z)$ are suppressed by at least one power of j

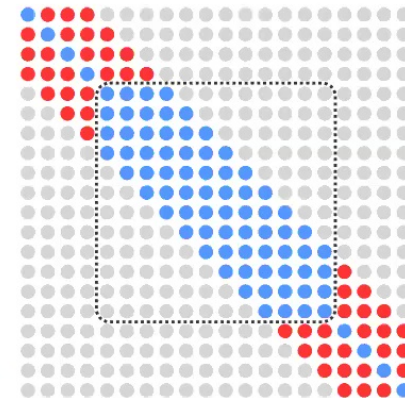
Therefore
$$q_v |\Psi_0\rangle = \frac{1}{8} (j_1 + j_4)(j_2 + j_5)(j_3 + j_6) |\Psi_0\rangle + \mathcal{O}(j^2)$$

Volume operator

How to take the square root to extract V_v from q_v ?

Consider q_v as a matrix on $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_6}$ (the "generalized intertwiner space" of the node).

- $|\Psi_0\rangle$ corresponds to $(1, 0, \dots, 0)$
- The red matrix elements are (at most) of order j^2
- Outside of the central block, the blue matrix elements are of order j^3



Taking the red matrix elements as a perturbation over the blue matrix elements, standard perturbation theory gives

$$\sqrt{|q_v|} |\Psi_0\rangle = \sqrt{\lambda_0} |\Psi_0\rangle + \sum_i \frac{W_{i0}}{\sqrt{\lambda_i} + \sqrt{\lambda_0}} |\Psi_i\rangle + \text{higher order}$$

λ_i : Eigenvalues of the unperturbed operator

W_{ik} : Matrix elements of the perturbation

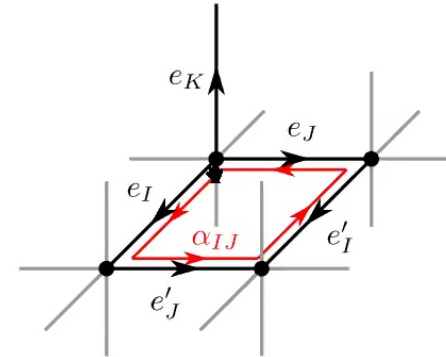
Conclusion:
$$V_v |\Psi_0\rangle = \sqrt{\frac{1}{8}(j_1 + j_4)(j_2 + j_5)(j_3 + j_6)} |\Psi_0\rangle + \mathcal{O}(\sqrt{j})$$

Thiemann's Hamiltonian

In the regularization used in the quantum-reduced model, the Euclidean part of Thiemann's Hamiltonian consists of terms of the form

$$\text{Tr} \left(D^{(s)}(h_{\alpha_{IJ}}) D^{(s)}(h_{e_K}^{-1}) V_v D^{(s)}(h_{e_K}) \right)$$

The loop α_{IJ} is defined according to the graph-preserving prescription shown in the drawing.



To extract the term of leading order in j in the action of the Hamiltonian on a reduced spin network state:

- Expand the trace in the standard basis (in which J_z is diagonal)
- Write the holonomy around the loop as $h_{\alpha_{IJ}} = h_{e_J}^{-1} h_{e'_I}^{-1} h_{e'_J} h_{e_I}$
- Express the holonomy of each edge in the appropriate basis
- Discard all terms involving off-diagonal components of holonomies

For $s = 1/2$ the leading term reproduces the reduced Hamiltonian constructed previously in the literature of the quantum-reduced model.

Conclusions

We have shown that, despite their considerable simplicity, the operators of the quantum-reduced model are simply the operators of full LQG acting on states in the reduced Hilbert space.

More precisely, when an operator of the full theory acts on a reduced state, the leading term in j reproduces the action of the corresponding reduced operator.

If one still wishes to think of the reduced operators as projections of the full theory operators down to the reduced Hilbert space, our result shows that the terms projected out are small compared with the terms preserved by the projection.

At the technical level, the only genuine assumption of the quantum-reduced model is therefore the structure of the reduced Hilbert space. If the reduced Hilbert space is considered to be given, the very simple "reduced" operators are obtained without introducing any additional assumptions.

Alternative interpretation of the result: There exists a subspace of the kinematical Hilbert space of LQG, which is preserved by the basic operators of LQG, and in which the basic operators act in a very simple way.

Conclusions

In the case of holonomy, we discovered a modified version of the reduced holonomy operator. Every diagonal component of the modified operator, and not only those labeled with $m = n = \pm j$, acts as a valid reduced operator. (For $j = 1/2$, the modified operator is of course identical with the standard one.)

In order to determine which version of the holonomy operator is physically correct, one could repeat some calculation in the quantum-reduced model (e.g. the semiclassical analysis of the Hamiltonian) but letting the Hamiltonian carry a spin higher than $1/2$.

Our findings suggest that the quantum-reduced model could be seen as the leading term in a large- j expansion of a particular sector of full LQG. How to turn this speculation into a precise statement, and in particular understand the physical meaning of such an expansion?

The results presented clarify the relation between the quantum-reduced model and full LQG as far as operators are concerned. Obtaining a more satisfactory and systematic understanding of the reduced Hilbert space from the point of view of the full theory still remains a question to be investigated.