Title: Quantum group in 3d quantum gravity

Speakers: Florian Girelli

Series: Quantum Gravity

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Abstract: It is well-known that quantum groups are relevant to describe the quantum regime of 3d gravity. They encode a deformation of the gauge symmetries (Lorentz symmetries) parametrized by the value of the cosmological constant. They appear as some kind of regularization either through the quantization of the Chern-Simons formulation (Fock-Rosly formulation/combinatorial quantization, path integral quantization) or the state sum approach (Turaev-Viro model). Such deformation might be perplexing from a classical picture since the action is defined in terms of plain/undeformed gauge symmetry. I would like to present here a novel way to derive/justify such quantum group deformation, starting from the classical action.

3d gravity and quantum groups

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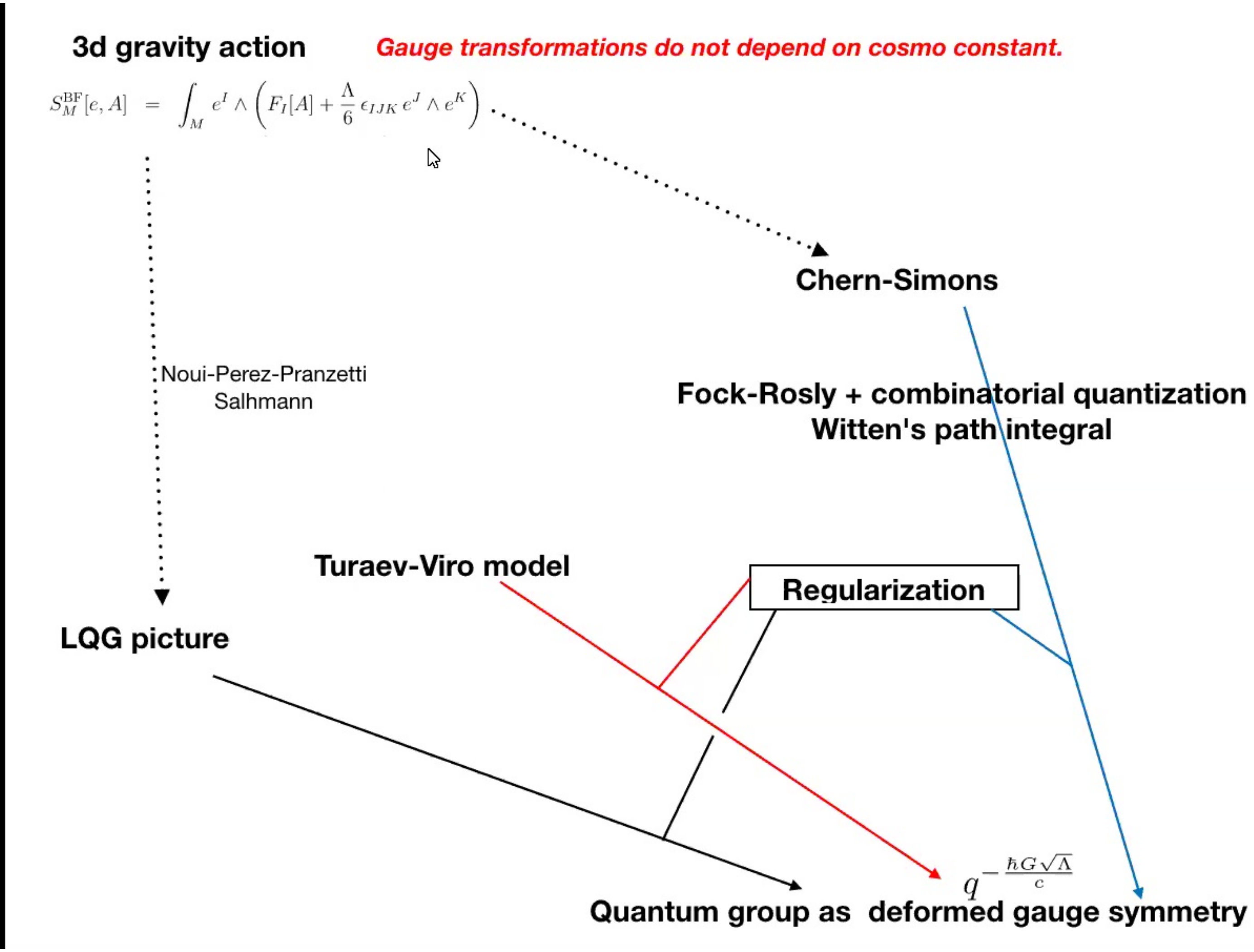
in collaboration with M. Dupuis, L. Freidel, A. Osumanu and J. Rennert

also based on some old works with V. Bonzom, M. Dupuis and E. Livine


3d gravity action
Gauge transformations do not depend on cosmo constant.
$S_M^{\mathrm{BF}}[e,A] = \int_M e^I \wedge \left(F_I[A] + \frac{\Lambda}{6}\epsilon_{IJK}\, e^J \wedge e^K \right)$
Chern-Simons
Noui-Perez-Pranzetti
Salhmann
Fock-Rosly + combinatorial quantization
Witten's path integral
Turaev-Viro model
Regularization
LQG picture
$q^{-\frac{\hbar G\sqrt{\Lambda}}{c}}$
Quantum group as deformed gauge symmetry


Florian Girelli

3d gravity action

$$S_M^{\rm BF}[e,A] \;=\; \int_M e^I\wedge\left(F_I[A]+\frac{\Lambda}{6}\epsilon_{IJK}\,e^J\wedge e^K\right)$$

Chern-Simons

Turaev-Viro model

Regularization

"The journey is more important than the destination"

Quantum group as deformed gauge symmetry

The expedition

- Preparing for the trip:
  - Symmetry from boundary 
  - Divide to truncate 
- The Journey…
- We made it, 

then what? 4d?

3d gravity with a cosmological constant

The action: **Space-time** $M \sim \mathbb{R} \times \Sigma$ **Metric** $\eta_{IJ} = \mathrm{diag}(+,+,\sigma)$ $\alpha = \frac{1}{16\pi G}$

$$S_{BF}[A,e] = \alpha \int_M \left(\eta_{IJ} e^I \wedge F^J[A] + \sigma \frac{\Lambda}{6} \epsilon_{IJK}\, e^I \wedge e^J \wedge e^K \right)$$

Notation: we'll often use $(A \times B)^I = \epsilon^I_{JK} A^J \wedge B^K$ between 1-forms

EOM

$$\mathrm{d}A^I + \frac{1}{2}(A \times A)^I + \frac{\Lambda}{2}(e \times e)^I = 0$$

$$\mathrm{d}e^I + (A \times e)^I = 0$$

Symmetries

Gauge transf $\delta_\alpha e^I = (e \times \alpha)^I, \quad \delta_\alpha A^I = \mathrm{d}_A \alpha^I,$ **does not depend on the cosmo const**

"Translations" $\delta_\phi e^I = \mathrm{d}_A \phi^I, \quad \delta_\phi A^I = \Lambda\,(e \times \phi)^I$

Charge algebra $J_\alpha \approx \oint_{\partial\Sigma} \alpha_I e^I, \quad P_\phi \approx \oint_{\partial\Sigma} \phi^I A_I$

$$\{J_\alpha, J_\beta\} = J_{(\alpha\times\beta)}, \quad \{P_\phi, P_\psi\} = -\sigma\Lambda J_{(\phi\times\psi')}, \quad \{J_\alpha, P_\phi\} = P_{(\alpha\times\phi)} + \oint_{\partial\Sigma} \phi^I \mathrm{d}\alpha_I.$$

Usual central extension
cancelled if parameters are constant on boundary — which we take.

3d gravity with a cosmological constant

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$\mathfrak{so}(2,2),\ \mathfrak{so}(3,1),\ \mathfrak{so}(4).$

$A \equiv A^I \mathbf{J}_I \qquad e \equiv e^I \mathbf{P}_I$ Frame field is with value in the "boosts". Difficult to discretize…

3d gravity with a cosmological constant
new variables — boundary term changes the shape of the symmetries

We add to the action a boundary term, parametrized by a vector *n*

$$\int_M e^I \wedge \left(F_I[A] + \sigma \frac{\Lambda}{6} \epsilon_{IJK}\, e^J \wedge e^K \right) + \frac{1}{2} \int_{\partial M} (e \times e)_I n^I .$$

We demand that it is constant under variations

$$\delta n = 0$$

New pre-symplectic potential

$$\Theta_{QG} = \int_\Sigma e_I \wedge \delta A^I - \frac{1}{2} \delta \int_\Sigma (e \times e)_I n^I = \int_\Sigma e_I \wedge \delta \omega^I - \frac{1}{2} \int_\Sigma (e \times e)_I \cdot \delta n^I$$

($\delta n^I \to 0$)

New connection, depending also on frame field! $\omega^I \equiv A^I + (n \times e)^I .$

*Can also be seen as a **canonical transformation**.*

(Remember Holst term vs Ashtekar-Barbero canonical map. Cf general discussion in Laurent's recent talk)

3d gravity with a cosmological constant

new variables — boundary term changes the shape of the symmetries

We add to the action a boundary term, parametrized by a vector *n* $\delta n = 0$

$$\int_M e^I \wedge \left(F_I[A] + \sigma \frac{\Lambda}{6} \epsilon_{IJK} \, e^J \wedge e^K \right) + \frac{1}{2} \int_{\partial M} (e \times e)_I n^I .$$

Let us rewrite the action with the new connection. $\omega^I \equiv A^I + (n \times e)^I .$

$$F[A] = F[\omega + e \times n] = F[\omega] + \mathrm{d}_\omega (e \times n) + \frac{1}{2} (e \times n) \times (e \times n)$$

$$\mathrm{d}_\omega \alpha = \mathrm{d}\alpha + \omega \times \alpha .$$

3d gravity with a cosmological constant
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$\mathrm{d}_\omega \alpha = \mathrm{d}\alpha + \omega \times \alpha.$

But $\frac{1}{2} e \cdot ((e \times n) \times (e \times n)) = \frac{\sigma n^2}{6} e \cdot (e \times e)$ so taking $n^2 = -\Lambda$ **will cancel the volume term**

$$\int_M e^I \wedge \left(F_I[A] + \frac{\Lambda}{6} \epsilon_{IJK}\, e^J \wedge e^K \right) + \frac{1}{2} \int_{\partial M} (e \times e)_I n^I = \int_M \left(e \cdot F[\omega] - \frac{1}{2} (e \times e) \cdot \mathrm{d}_\omega n \right).$$

3d gravity with a cosmological constant

new variables — boundary term changes the shape of the symmetries

We take as a starting point this action.

$n^2 = -\Lambda \qquad \delta n = 0$

$$\int_M \left(e \cdot F[\omega] - \frac{1}{2}(e \times e) \cdot \mathrm{d}_\omega n \right).$$

$$\omega^I \equiv A^I + (n \times e)^I.$$

	Lorentzian $\sigma = -1$	Euclidian $\sigma = +1$
$\Lambda > 0$	n is time-like	n is pure imaginary
$\Lambda < 0$	n is space-like	n is space-like
$\Lambda = 0$	n is light-like or n=0	n is Grassmanian or n=0

3d gravity with a cosmological constant

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$$\int_M \left(e \cdot F[\omega] - \frac{1}{2}(e \times e) \cdot \mathrm{d}_\omega n \right).$$

EOM: $F_I[\omega] - (e \times \mathrm{d}_\omega n)_I \ = 0$

$$\mathrm{d}_\omega e^I + \tfrac{1}{2}[(e \times e) \times n]^I = 0$$

Torsion equation does depend on the cosmo. const!

3d gravity with a cosmological constant
new variables — boundary term changes the shape of the symmetries

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$$\int_M \left(e \cdot F[\omega] - \frac{1}{2}(e \times e) \cdot \mathrm{d}_\omega n \right).$$

Action symmetries:

Gauge transformations

$$\begin{aligned} \delta'_\alpha n^I &= 0, \\ \delta'_\alpha e^I &= (e \times \alpha)^I, \\ \delta'_\alpha \omega^I &= \mathrm{d}_\omega \alpha^I + (e \times (n \times \alpha))^I \equiv D\alpha^I. \end{aligned}$$

Gauge transformations now depend on cosmo. const!

Deformed/new notion of covariant derivative

"Translations"

$$\begin{aligned} \delta'_\phi n^I &= 0, \\ \delta'_\phi \omega^I &= (\phi \times \mathrm{d}_\omega n)^I \\ \delta'_\phi e^I &= \mathrm{d}_\omega \phi^I + ((e \times \phi) \times n)^I \equiv \tilde{D}\phi^I. \end{aligned}$$

Deformed/new notion of covariant derivative

3d gravity with a cosmological constant

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Charge algebra: $J'_\alpha = \oint_{\partial\Sigma} \alpha_I e^I = J_\alpha, \qquad P'_\phi = \oint_{\partial\Sigma} \phi^I \omega_I = P_\phi + J_{\phi \times n}.$

$$\{J_\alpha, J_\beta\} = J_{\alpha\times\beta}, \qquad \{P'_\alpha, P'_\beta\} = P'_{(\alpha\times\beta)\times n} + \oint_{\partial\Sigma} (\alpha \times \beta) \cdot \mathrm{d}n.$$

← **Demanding $\mathrm{d}n = 0$ leads to a closed algebra**

$$\{J_\alpha, P'_\phi\} = P'_{\alpha\times\phi} - J_{(\alpha\times n)\times\phi} + \oint_{\partial\Sigma} \phi \cdot \mathrm{d}\alpha.$$

↑ **Usual central extension**

3d gravity with a cosmological constant

new variables — boundary term changes the shape of the symmetries

$$n^2 = -\Lambda \qquad \delta n = 0 \qquad \mathrm{d}n = 0$$

The vector *n* parameterizes the boundary term: direction and norm.

We can take *n* defining the third direction

$$n^I = (0, 0, n^3)$$

3d gravity with a cosmological constant

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so might need to choose the metric so that it is consistent.

3d gravity with a cosmological constant

new variables — boundary term changes the shape of the symmetries

$$n^2 = -\Lambda \qquad \delta n = 0 \qquad \mathrm{d}n = 0$$

The vector *n* parameterizes the boundary term: direction and norm.

We can take n in the third direction

$$n^I = (0, 0, \theta)$$

might need to change the shape of metric

	Lorentzian $\sigma = -1$	Euclidian $\sigma = +1$
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no metric change (n is pure imaginary)

no metric change (n is light-like or n=0; n is Grassmanian or n=0)

$\Lambda < 0$ → $$\eta_{IJ} = \mathrm{diag}(+, -s\sigma, -s)$$

s **is sign of cosmo const**

$$\Lambda = s|\Lambda| = s\theta^2$$

3d gravity with a cosmological constant

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We take as a starting point this action. $n^2 = -\Lambda \qquad \delta n = 0 \qquad \mathrm{d}n = 0$

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$$\{J_\alpha, J_\beta\} = J_{\alpha\times\beta}, \qquad \{P'_\alpha, P'_\beta\} = P'_{(\alpha\times\beta)\times n}$$

What is this algebra?

$$\{J_\alpha, P'_\phi\} = P'_{\alpha\times\phi} - J_{(\alpha\times n)\times\phi}$$

Usual central extension
cancelled if parameters are constant on *boundary* — which we take.

3d gravity with a cosmological constant

new variables — boundary term changes the shape of the symmetries

$$n^2 = -\Lambda \qquad \delta n = 0 \qquad dn = 0$$

$$n^I = (0, 0, \theta)$$

Relevant charge Lie algebra

$$J \to \mathbf{J}, \quad P' \to \tau \qquad \tau_I \equiv \mathbf{P}_I + n^J \epsilon_{IJK} \mathbf{J}^K = \mathbf{P}_I + (n \times \mathbf{J})_I$$

$$[\mathbf{J}^I, \mathbf{J}^J] = \epsilon^{IJ}{}_K \mathbf{J}^K \qquad [\tau_I, \tau_J] = C_{IJ}{}^K \tau_K \quad \text{with} \quad C_{IJ}{}^K = \sigma(n_I \delta_J^K - n_J \delta_I^K).$$

an₂ **Lie algebra**

Cross-term: $[\mathbf{J}^I, \tau_J] = C_{JK}{}^I \mathbf{J}^K + \epsilon^I{}_J{}^K \tau_K$

DNE for so(4) **Will not consider it in the following**

This amounts to the Iwasawa decomposition of the relevant Lorentz Lie algebra.

$$\mathfrak{d}_{\sigma s} = \mathfrak{su}_{\sigma s} \bowtie \mathfrak{an}_2. \qquad (\sigma, s) \neq (+,+)$$

$$\mathbf{J}_I \triangleright \tau_J \equiv [\mathbf{J}_I, \tau_J]_{\mathfrak{an}} = \epsilon_{IJ}{}^K \tau_K, \qquad \mathbf{J}_I \triangleleft \tau_J \equiv [\tau_J, \mathbf{J}_I]_{\mathfrak{su}} = C_{IKJ} \mathbf{J}^K.$$

Killing form $\langle \tau_J, \mathbf{J}^I \rangle = \delta_J^I = \langle \mathbf{J}^I, \tau_J \rangle, \qquad \langle \mathbf{J}^I, \mathbf{J}^J \rangle = 0 = \langle \tau_I, \tau_J \rangle.$

Summary up to now

Starting from the usual action, we performed a change of variables, parametrized by a vector.

$$e^I P_I \to e^I \tau_I \qquad \text{frame field with value in } an \qquad n^2 = -\Lambda \qquad \delta n = 0 \qquad \mathrm{d}n = 0$$

$$A_I J^I \to \omega_I J^I = (A^I + (n \times e)^I) J_I \qquad n^I = (0, 0, \theta)$$

$$\int_M e^I \wedge \left(F_I[A] + \sigma \frac{\Lambda}{6} \epsilon_{IJK}\, e^J \wedge e^K \right) \longrightarrow \int_M \left(e \cdot F[\omega] - \frac{1}{2}(e \times e) \cdot \mathrm{d}_\omega n \right).$$

EOM:

$$\mathrm{d}A + \frac{1}{2}[A, A] + \frac{\Lambda}{2}[e, e] = 0 \qquad \text{New curvature} \quad \mathrm{d}\omega + \frac{1}{2}[\omega, \omega] + e \triangleright \omega = 0$$

$$\mathrm{d}e + A \triangleright e = 0 \qquad \text{New torsion} \quad \mathrm{d}e + \frac{1}{2}[e, e] + \omega \triangleright e = 0$$

New covariant derivatives —> **all symmetry transformations depend on the cosmo const.**

$$\alpha \in \mathfrak{su} \qquad D\alpha = \mathrm{d}\alpha + [\omega, \alpha] + e \triangleright \alpha \qquad \text{in direction } su$$

$$\phi \in \mathfrak{an} \qquad \tilde{D}\phi = \mathrm{d}\phi + [e, \phi] + \omega \triangleright \phi \qquad \text{in direction } an$$

$$\mathrm{d}(\alpha \cdot \phi) = D\alpha \cdot \phi + \alpha \cdot \tilde{D}\phi.$$

$$\delta_\alpha \omega = D\alpha, \qquad \delta_\alpha e = e \triangleleft \alpha$$

$$\delta_\phi \omega = \omega \triangleleft \phi, \qquad \delta_\phi e = \tilde{D}\phi.$$

Summary up to now

Starting from the usual action, we performed a change of variables, parametrized by a vector.

$$e^I P_I \to e^I \tau_I$$
$$A_I J^I \to \omega_I J^I = (A^I + (n \times e)^I) J_I$$

$$n^2 = -\Lambda$$
$$n^I = (0, 0, \theta)$$

$\delta n = 0$ $\quad$ $\mathrm{d}n = 0$

Removing either assumption will allow to go beyond the quantum group picture.

EOM:

$$\mathrm{d}A + \frac{1}{2}[A, A] + \frac{\Lambda}{2}[e, e] = 0$$
$$\mathrm{d}e + A \triangleright e = 0$$

New curvature $\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] + e \triangleright \omega = 0$

New torsion $\mathrm{d}e + \frac{1}{2}[e, e] + \omega \triangleright e = 0$

New covariant derivatives —> **all symmetry transformations depend on the cosmo const.**

$\alpha \in \mathfrak{su}$ $\quad D\alpha = \mathrm{d}\alpha + [\omega, \alpha] + e \triangleright \alpha$ $\quad$ in direction *su*

$\phi \in \mathfrak{an}$ $\quad \tilde{D}\phi = \mathrm{d}\phi + [e, \phi] + \omega \triangleright \phi$ $\quad$ in direction *an*

$$\mathrm{d}(\alpha \cdot \phi) = D\alpha \cdot \phi + \alpha \cdot \tilde{D}\phi.$$

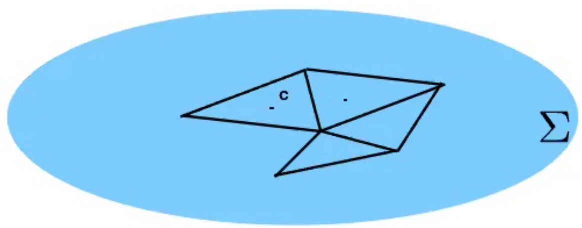
$$\delta_\alpha \omega = D\alpha, \qquad \delta_\alpha e = e \triangleleft \alpha$$
$$\delta_\phi \omega = \omega \triangleleft \phi, \qquad \delta_\phi e = \tilde{D}\phi.$$

Divide to truncate

We intend to determine the symplectic form in order to find the discrete variables.

$$\Omega = \int_\Sigma \langle \delta e \curlywedge \delta A \rangle = \int_\Sigma \langle \delta e \curlywedge \delta \omega \rangle = \sum_i \int_{c_i^*} \langle \delta e \curlywedge \delta \omega \rangle \, .$$

Divide Let us decompose the (space) manifold in cells — triangles.



Divide to truncate

We intend to determine the symplectic form in order to find the discrete variables.

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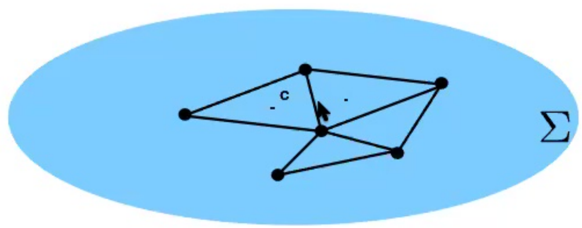


Truncate In the cells, we solve the constraints

$$\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] + e \triangleright \omega = 0$$

$$\mathrm{d}e + \frac{1}{2}[e, e] + \omega \triangleright e = 0$$

Any source of these equations is pushed to the vertices of the cellular decomposition.
Proper treatment is deferred to later or see work with B. Shoshany for the flat case.

Need to find the constraint solutions and plug them back into symplectic form.

Divide to truncate

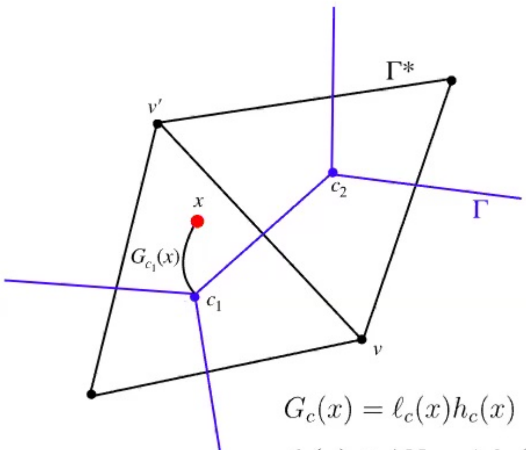
For simplicity, we will restrict ourselves to the Euclidian case with $\Lambda < 0$

Iwasawa decomposition of Lie group is simpler in this case $\mathrm{SL}(2,\mathbb{C}) \sim \mathrm{SU}(2) \bowtie \mathrm{AN}$

$$\mathrm{d}\omega + \frac{1}{2}[\omega,\omega] + e \triangleright \omega = 0$$
$$\mathrm{d}e + \frac{1}{2}[e,e] + \omega \triangleright e = 0$$

Solution:

$$\omega_{|c^*} = h_c^{-1}\mathrm{d}h_c + \left(h_c^{-1}(\ell^{-1}\mathrm{d}\ell_c)h_c\right)_{|\mathfrak{su}}$$
$$e_{|c^*} = \left(h_c^{-1}(\ell_c^{-1}\mathrm{d}\ell_c)h_c\right)_{|\mathfrak{an}} .$$



$$G_c(x) = \ell_c(x) h_c(x)$$
$$\ell_c(x) \in \mathrm{AN} \text{ and } h_c(x) \in \mathrm{SU}.$$

$$\downarrow \quad \Lambda \to 0$$

$$\omega_{|c^*} = h_c^{-1}\mathrm{d}h_c, \quad e_{|c^*} = h_c^{-1}\mathrm{d}X\, h_c,$$

with $\ell = (X,1) \in \mathbb{R}^3$.

Recover usual formula in the flat case

Divide to truncate

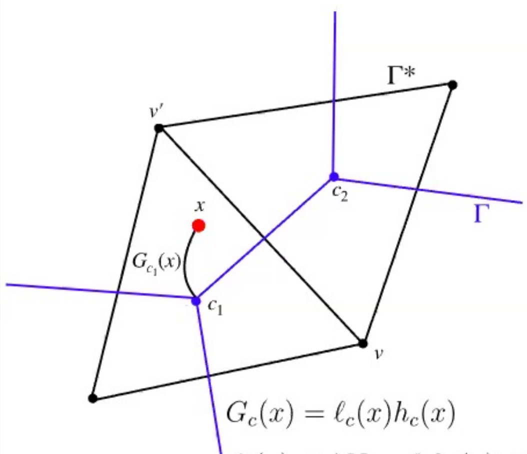
For simplicity, we will restrict ourselves to the Euclidian case with $\Lambda > 0$

Iwasawa decomposition of Lie group is simpler in this case $\mathrm{SL}(2,\mathbb{C}) \sim \mathrm{SU}(2) \bowtie \mathrm{AN}$

$$\mathrm{d}\omega + \frac{1}{2}[\omega,\omega] + e \triangleright \omega = 0$$
$$\mathrm{d}e + \frac{1}{2}[e,e] + \omega \triangleright e = 0$$

Solution:

$$\omega_{|c^*} = h_c^{-1}\mathrm{d}h_c + \left(h_c^{-1}(\ell^{-1}\mathrm{d}\ell_c)h_c\right)_{|\mathfrak{su}}$$
$$e_{|c^*} = \left(h_c^{-1}(\ell_c^{-1}\mathrm{d}\ell_c)h_c\right)_{|\mathfrak{an}} .$$



$G_c(x) = \ell_c(x)h_c(x)$

$\ell_c(x) \in \mathrm{AN}$ and $h_c(x) \in \mathrm{SU}$.

Lemma: *The truncated symplectic form is*

$$\Omega_{|c} = \int_{c^*} \langle \delta e \curlywedge \delta\omega \rangle \approx \Omega_c$$

with $\Omega_c = -\int_{c^*} \mathrm{d}\delta\langle \ell_c^{-1}\mathrm{d}\ell_c, \delta h_c h_c^{-1}\rangle$

$$= -\int_{\partial c^*} \delta\langle \ell_c^{-1}\mathrm{d}\ell_c, \delta h_c h_c^{-1}\rangle$$

Divide to truncate

For simplicity, we will restrict ourselves to the Euclidian case with $\Lambda > 0$

Iwasawa decomposition of Lie group is simpler in this case $\mathrm{SL}(2,\mathbb{C}) \sim \mathrm{SU}(2) \bowtie \mathrm{AN}$

$$\mathrm{d}\omega + \frac{1}{2}[\omega,\omega] + e \triangleright \omega = 0$$
$$\mathrm{d}e + \frac{1}{2}[e,e] + \omega \triangleright e = 0$$

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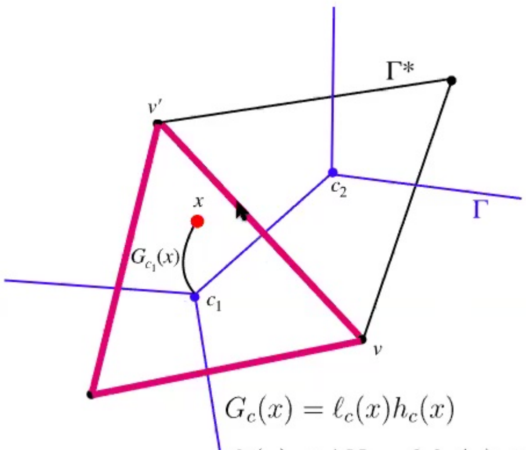

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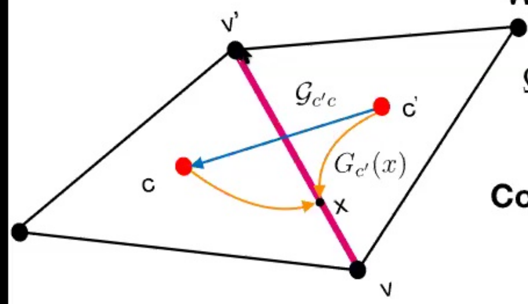
The Journey



Two adjacent cells share an edge.
We can determine the contribution for the given edge.

$$\Omega_{cc'} \equiv \Omega_c^{[vv']} + \Omega_{c'}^{[v'v]} = \Omega_c^{[vv']} - \Omega_{c'}^{[vv']}$$

Continuity equation <-> triangles in different frames

$$G_{c'}(x) = \mathcal{G}_{c'c} G_c(x) \qquad x \in [vv'].$$

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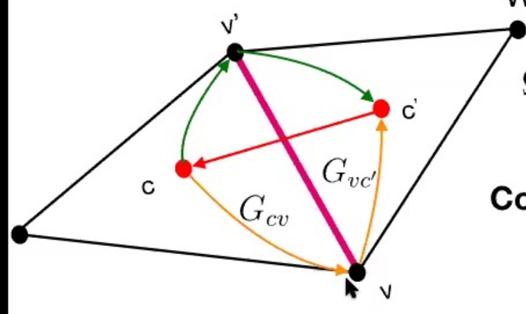

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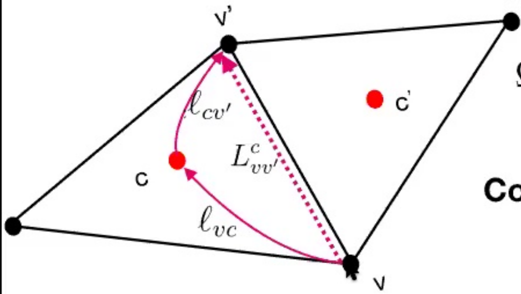

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in SU

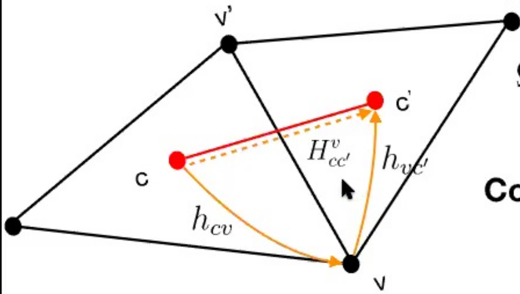
$$\Leftrightarrow \ L^c_{vv'}H^{v'}_{cc'} = H^v_{cc'}L^{c'}_{vv'}.$$

in AN or R^3

with $L^c_{vv'} \equiv \ell_{vc}\ell_{cv'}, \qquad H^v_{cc'} \equiv h_{cv}h_{vc'}.$

which are called the *triangular holonomies*
aka **triangular operators in Kitaev's model**.

The Journey



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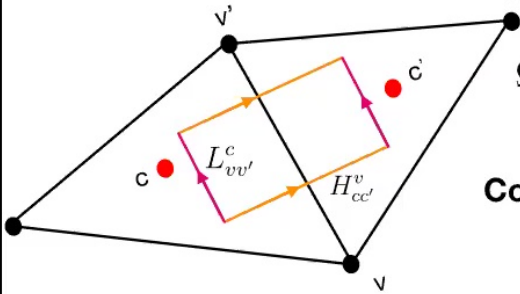

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$$\Leftrightarrow \quad L^c_{vv'} H^{v'}_{cc'} = H^v_{cc'} L^{c'}_{vv'}.$$ **ribbon structure**

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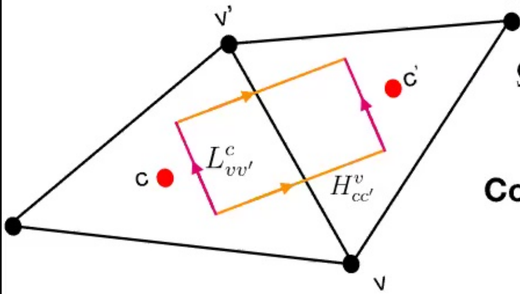

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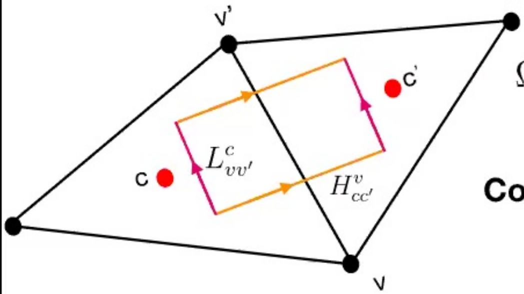

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ribbon structure $L^c_{vv'} H^{v'}_{cc'} = H^v_{cc'} L^{c'}_{vv'}.$ **with** $L^c_{vv'} \equiv \ell_{vc}\ell_{cv'}, \qquad H^v_{cc'} \equiv h_{cv}h_{vc'}.$

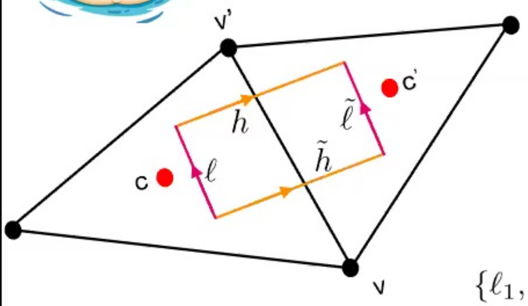
We can evaluate the symplectic form:

Theorem 1 *The symplectic form associated to a link* $[cc']$ *is given by*

$$\Omega_{cc'} = \Omega_c^{[vv']} - \Omega_{c'}^{[vv']} = \frac{1}{2}\left(\langle \Delta H^v_{cc'} \wedge \Delta L^c_{vv'} \rangle + \left\langle \underline{\Delta} H^{v'}_{cc'} \wedge \underline{\Delta} L^{c'}_{vv'} \right\rangle\right)$$

$$\Delta u = \delta u u^{-1} \qquad \underline{\Delta} u = u^{-1}\delta u$$

This is a symplectic form— hence we have a *phase space*. It is called a Heisenberg double: it is the generalization of the usual cotangent space.



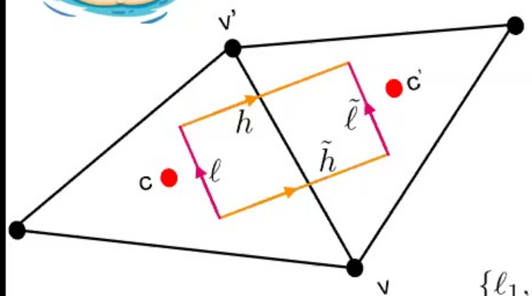
We made it!

We have derived the ribbon model introduced by Bonzom, Dupuis, FG, Livine (also by Freidel-Zapata).

$$\Omega = \frac{1}{2}\left(\left\langle \Delta\tilde{h} \wedge \Delta\ell \right\rangle + \left\langle \underline{\Delta} h \wedge \underline{\Delta}\tilde{\ell} \right\rangle\right)$$

Alekseev-Malkin ↓ with $\ell\, h = \tilde{h}\;\tilde{\ell}.$

$$\{\ell_1, \ell_2\} = -[r, \ell_1\ell_2], \quad \{h_1, h_2\} = -[r^t, h_1 h_2]$$ + crossed terms...



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if cosmo const is zero, we recover T*SU(2) as ISO(3)

$$\ell \to X \in \mathbb{R}^3, \quad \tilde{h} = h, \quad \tilde{X} = hXh^{-1}$$

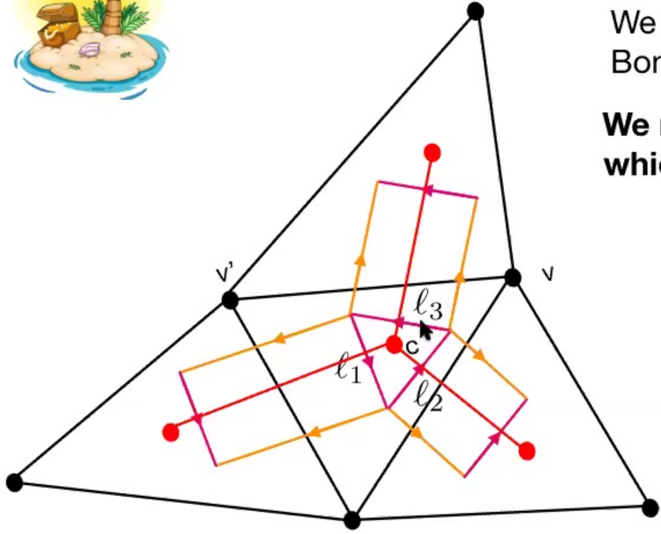
$$\{X_i, X_j\} = \epsilon_{ij}^k X_k \qquad \{h, h\} = 0$$

if cosmo const is not zero (<0), SL(2,C) is the phase space

$$h = \begin{pmatrix} \alpha & -\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}(2), \quad \det h = 1$$

$$\{\alpha, \bar{\alpha}\} = i\kappa\gamma\bar{\gamma}, \quad \{\alpha, \gamma\} = -i\frac{\kappa}{2}\alpha\gamma, \quad \{\alpha, \bar{\gamma}\} = -i\frac{\kappa}{2}\alpha\bar{\gamma}, \quad \{\gamma, \bar{\gamma}\} = 0$$

$$\kappa = G\sqrt{|\Lambda|}$$



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We replace the dual graph by a ribbon graph, which carries the info about all the variables.

Gauss law can be seen as an holonomy constraint in the AN (or R^3) sector.

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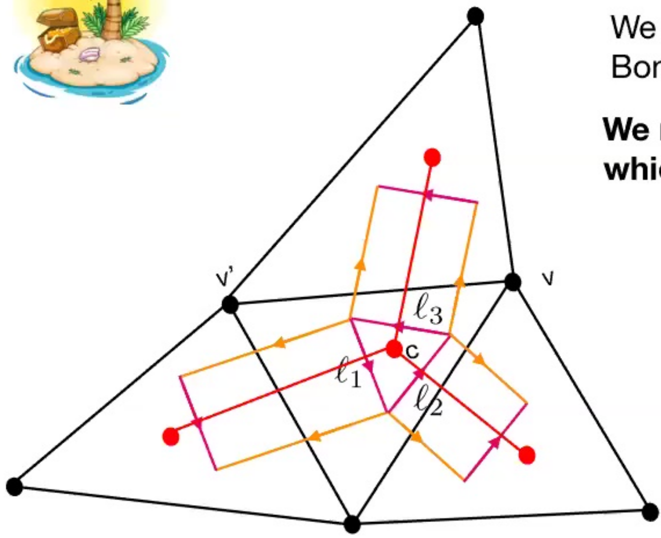

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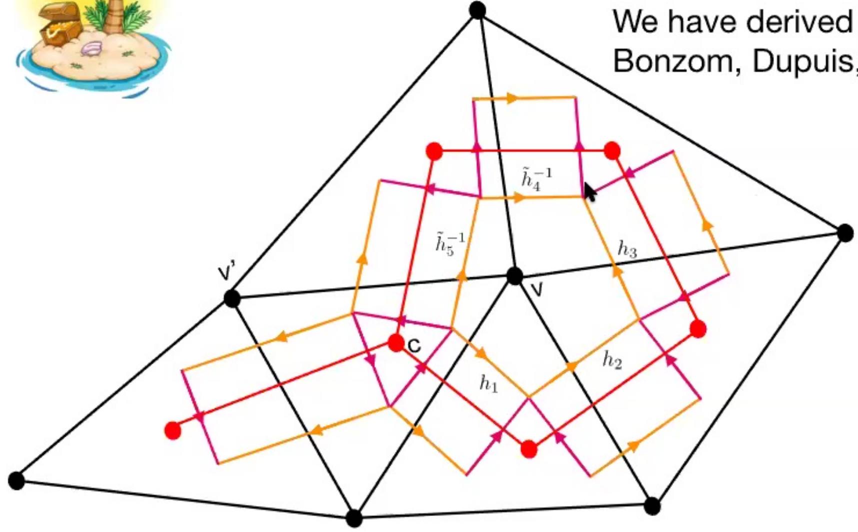

Flatness constraint is now depending on the different holonomies [h transports the flux, but the flux also transports the h].

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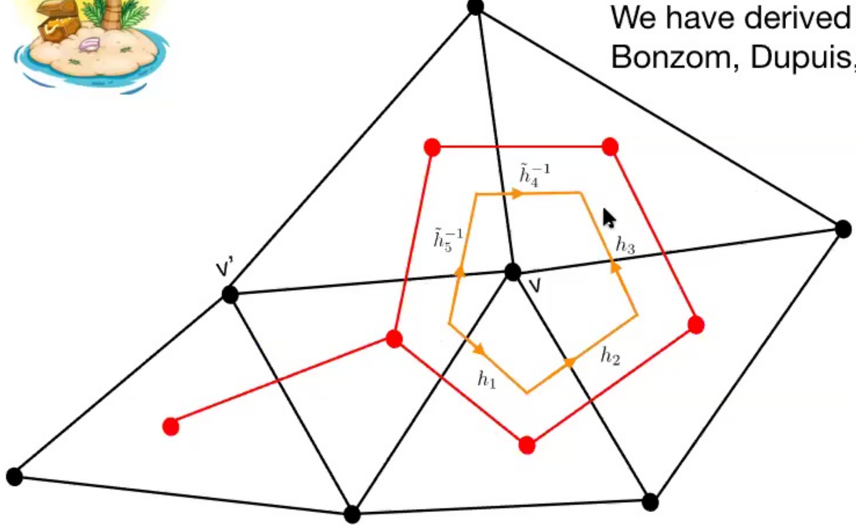

$$\tilde{h}_5^{-1}\tilde{h}_4^{-1}h_3h_2h_1 = 1$$

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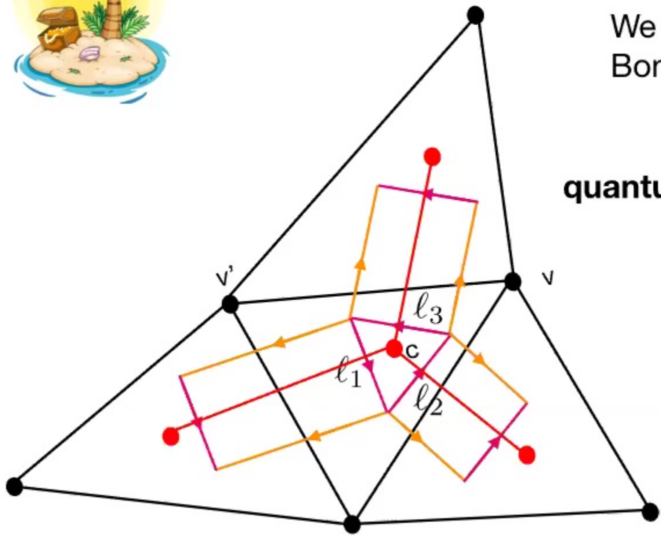

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Quantization of the model gives rise to quantum group spin networks and the TV amplitude as discussed by Bonzom, Dupuis, FG.

$$r = \frac{i\kappa}{4}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 4 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \to R = \begin{pmatrix}q^{\frac{1}{4}} & 0 & 0 & 0\\ 0 & q^{-\frac{1}{4}} & 0 & 0\\ 0 & q^{-\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) & q^{-\frac{1}{4}} & 0\\ 0 & 0 & 0 & q^{\frac{1}{4}}\end{pmatrix},$$

$$\lambda \in \mathbb{R},\ z \in \mathbb{C}$$

$$q = e^{\hbar\kappa}$$

$$\ell = \begin{pmatrix}\lambda & 0\\ z & \lambda^{-1}\end{pmatrix} \to \hat{\ell} = \begin{pmatrix}K & 0\\ (q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_+, & K^{-1}\end{pmatrix}$$

$$\{\ell_1, \ell_2\} = -[r, \ell_1\ell_2] \longrightarrow R\,\widehat{\ell}_1\,\widehat{\ell}_2 = \widehat{\ell}_2\,\widehat{\ell}_1\,R,$$

$$\downarrow$$

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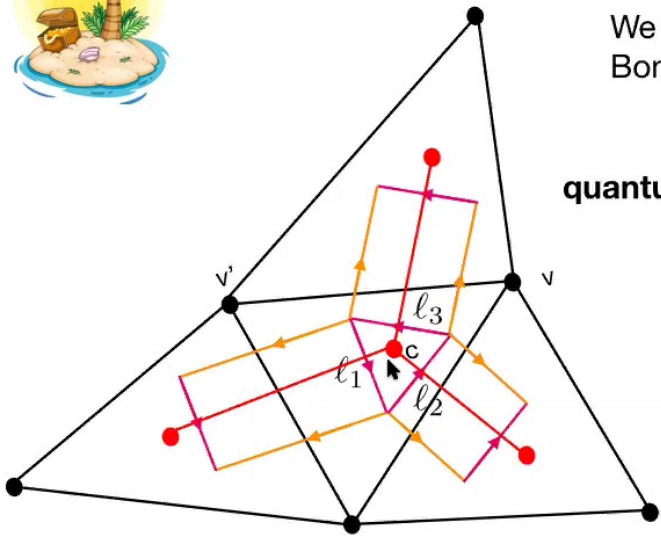

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$$(\ell_1)_{ij}(\ell_2)_{jl}(\ell_3)_{lm} = 1_{im} \to \widehat{\ell}_{ij} \otimes \widehat{\ell}_{jl} \otimes \widehat{\ell}_{lm} = \widehat{1}_{im}$$

→ **coproduct of** $\mathcal{U}_q(\mathrm{SU}(2))$

→ **Solution is quantum group intertwiner**

$$\kappa = G\sqrt{|\Lambda|}$$

Summary

Technical comments:

- We recovered the quantum group symmetries using 2 steps:
  - modifying the gauge symmetries by adding a boundary term/performing a canonical transformation
  - dividing and truncating the degrees of freedom by going on-shell.

- *The Euclidian case with positive cosmological constant has to be treated separately due to the reality conditions.*

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    - dividing and truncating the degrees of freedom by going on-shell.

- *The Euclidian case with positive cosmological constant has to be treated separately due to the reality conditions.*

- There is some room to go beyond the quantum group case, by removing some conditions on the vector *n.*

- *Different vectors n can be related by unitary transformations, can we find some relations between different quantum deformations parametrized by different n?*

- Can we define spin networks with different *n*? (domain walls? cf Livine)

Summary

More general comments:

- The truncation encodes the principle of decomposing the space into blocks solving the constraints. In the present case we have some deformed flatness which encodes homogeneously curved geometries (cf geometric structures in Carlip's book).

- *Construction illustrates again the power of putting terms that do not change EOM (boundary/topological term) but still renders the theory either* ***more manageable*** *or* ***with different symmetry*** *structure (cf teleparallel vs GR)*

- The vector *n* is analogue to the Immirzi parameter or the theta term in YM, except that we restricted it here to a specific value, the cosmological constant.
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- We recovered the analogue of the Kitaev model for Lie groups. We should use this model to explore some gravity questions!! "*Analogue quantum gravity*"?
- Is 3d useful to 4d to determine whether we should use a quantum group? (Yes probably)

Preliminary comments on 4d

with A. Osumanu

$$\int_{\mathcal{M}} \star(e\wedge e)_{KL} \wedge \left(R[\omega]^{KL} \; - \frac{\Lambda}{6}\, e^K \wedge e^L\right) + \frac{1}{3}\int_{\partial\mathcal{M}} \star(e\wedge e)_{KL} \wedge e^K\, n^L$$

Add a boundary term, to implement canonical map

$$\Theta = \int_\Sigma \star(e\wedge e)_{KL} \wedge \delta\omega^{KL} + \frac{1}{3}\delta \int_\Sigma \star(e\wedge e)_{KL}\wedge e^K n^L = \int_\Sigma \star(e\wedge e)_{KL}\wedge \delta\Omega^{KL}$$

$$\Omega^{KL} = \omega^{KL} + \frac{1}{2} e^{[K} n^{L]} \Leftrightarrow \omega^{IJ} = \Omega^{IJ} + \mathcal{I}^{IJ},$$ *same change of coordinates as in 3d case*

$$\mathcal{I}^{IJ} \equiv \frac{1}{2} n^{[I} e^{J]} = \frac{1}{2} C^{IJ}{}_K e^K, \quad C^{IJ}{}_K = n^I \delta^J{}_K - n^J \delta^I{}_K.$$

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$$\begin{aligned} R[\omega]_{KL} &= R[\Omega]_{KL} + \mathrm{d}_\Omega \mathcal{I}_{KL} + \mathcal{I}_K{}^M \wedge \mathcal{I}_{ML} \\ &= R[\Omega]_{KL} + \mathrm{d}_\Omega \mathcal{I}_{KL} + \frac{1}{4}\left(n\cdot e \wedge (n_K e_L - n_L e_K) - n^2 e_K \wedge e_L\right) \end{aligned}$$

if $n^2 = -4\frac{\Lambda}{3}$ → **we can cancel the volume term**

Preliminary comments on 4d

with A. Osumanu

$$\int_{\mathcal{M}} \star(e\wedge e)_{KL}\wedge\left(R[\omega]^{KL} - \frac{\Lambda}{6}\, e^K\wedge e^L\right) + \frac{1}{3}\int_{\partial\mathcal{M}} \star(e\wedge e)_{KL}\wedge e^K\, n^L$$

Add a boundary term, to implement canonical map

$$\Theta = \int_\Sigma \star(e\wedge e)_{KL}\wedge\delta\omega^{KL} + \frac{1}{3}\delta\int_\Sigma \star(e\wedge e)_{KL}\wedge e^K n^L = \int_\Sigma \star(e\wedge e)_{KL}\wedge\delta\Omega^{KL}$$

$$\Omega^{KL} = \omega^{KL} + \frac{1}{2}e^{[K}n^{L]} \Leftrightarrow \omega^{IJ} = \Omega^{IJ} + \mathcal{I}^{IJ},$$ *same change of coordinates as in 3d case*

$$n^2 = -4\frac{\Lambda}{3}$$

	Euclidian	Lorentzian
Flat: $\Lambda = 0$	$n = 0$ or n is Grassmanian	$n = 0$ or n is light-like
AdS:$\Lambda < 0$	n is space-like	n is space-like or *imaginary* time-like
dS:$\Lambda > 0$	n is *imaginary*	n is time-like or *imaginary* space-like

"Guidance" on which deformation we could get!

Preliminary comments on 4d

with A. Osumanu

$$\int_{\mathcal{M}} \star(e\wedge e)_{KL} \wedge \left(R[\omega]^{KL} \; - \frac{\Lambda}{6}\, e^K \wedge e^L \right)$$

Add a boundary term, to implement canonical map

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Performing the Hamiltonian analysis, do we get a deformed Gauss constraint?
Should we have the discretized flux as a non-abelian AN holonomy?

Do we get de Sitter spin networks as a generalization of Freidel—Livine—Pranzetti?