

Title: Coulomb branches for quaternionic representations

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Abstract: I will review the construction of Coulomb branches in 3D gauge theory for a compact Lie group G and a quaternionic representation E . In the case when E is polarized, these branches are determined by topological boundary conditions built from the gauged A-model of the two polar halves of E . No analogue of this is apparent in the absence of a polarization, nonetheless the Coulomb branch can be defined by the use of a $\hat{\epsilon}^{\text{TM}}$ -square root of E (related to the Spin representation). These branches ought to be part of a 3D topological field theory, but that is only apparent in special cases.



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Coulomb branches for quaternionic representations

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Background

- ① Pure topological gauge theory for a compact group G in 3D can be defined naïvely as the *sphere topology* of the classifying space BG .
- ② It has a refinement in terms of the holomorphic Lagrangian geometry on an algebraic completely integrable system \mathcal{C} (*Toda system*), with base the adjoint orbits $B := \mathfrak{g}_{\mathbb{C}}/G_{\mathbb{C}} = \text{Spec } H^*(BG)$.
- ③ The space \mathcal{C} is called the *Coulomb branch* for pure 3D gauge theory.
- ④ Some boundary conditions for this theory are 2D gauged A -models. *Compact symplectic manifolds* give (coherent sheaves supported on) holomorphic Lagrangians in \mathcal{C} , finite over the base.
- ⑤ The sections of these sheaves are their equivariant cohomologies; more generally, the spaces of states for the circle, made G -equivariant.
- ⑥ For instance, \mathcal{C} itself is foliated by the flag varieties of G (ranging over all the small quantum parameters).
- ⑦ This can be viewed as a *character theory* for topological G -actions on linear categories, with \mathcal{C} playing the role of conjugacy classes, in the sense of semiclassical calculus (irreducibles \leftrightarrow Lagrangians).



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Example

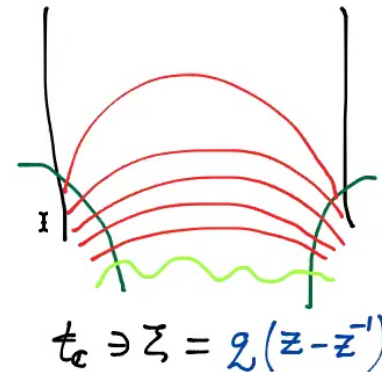
When $G = U_1$, $\mathcal{C} = T^\vee \mathbb{C}^\times = \text{Spec } \mathbb{C}[\tau, z^\pm] = \text{Spec } H_*^{U_1}(\Omega U_1)$,
with the Pontryagin product on ΩU_1 .

(In general: $H_*^G(\Omega G)$, with G conjugating ΩG ; a space first described by
Bezrukavnikov-Finkelberg-Mirkovic.)

Then, \mathbb{P}^1 corresponds to the Lagrangians $\tau = q(z - z^{-1})$,
the graphs of the differentials of $W(z) = q(z + z^{-1})$ in the toric mirror.

For SO_3 , $\mathcal{C}(SO_3)$ arises from $\mathcal{C}(U_1)$ by
dividing out the ± 1 Weyl action, blowing
up $(\tau = 0, z = \pm 1)$, and deleting the
proper transform of $\tau = 0$.

The \mathbb{P}^1 -Lagrangians foliate the space,
missing only the cotangent fibers over
 $z = \pm 1$ (the point flag variety).



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Gauge theory “with matter”

For non-compact manifolds, the Lagrangians (4) need not be finite over B .

For example, the standard representation \mathbb{C} of U_1 gives the Lagrangian $\{z = \tau\}$ in $T^*\mathbb{C}^\times$, with ring of functions $\mathbb{C}[\tau^\pm]$.

This is the equivariant *symplectic cohomology* of \mathbb{C} .

For a general complex representation V of G , one regularizes the symplectic cohomology by adding equivariance under scaling, with an equivariant *mass parameter* μ . (Here, μ just shifts τ , so is dispensable.)

Formally near $\mu = \infty$ (but not for finite μ) the Lagrangian is a section of B ; the formula in terms of the weights $\{\nu\}$ of V is (in rank one)

$$z = \prod_{\nu} (\mu + \nu\tau)^{\nu} \quad (\text{Mirror formula})$$

This infiniteness of the GLSM V , as boundary condition of gauge theory, seems to be remedied by introducing *matter fields* in the quaternionic representation $E := V \oplus \bar{V}$ in the 3D gauge theory.



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Coulomb branches with matter

There is a Coulomb branch $\mathcal{C}(E)$ for more \mathbb{H} -representations E of G . Gauge theory with matter assigns a vector space to S^2 , which is an E_3 algebra for pictorial reasons (*sphere topology*), and \mathcal{C} is Spec of that. Based on substantial ideas from physics, Braverman, Finkelberg and Nakajima defined $\mathcal{C}(E)$ for the polarized case $V \oplus \bar{V}$.

In this lecture, we will

- 1 Recall the definition of $\mathcal{C}(V \oplus \bar{V})$
- 2 Give an explicit description in terms of \mathcal{C} and the Mirror formula
- 3 Extend the definition to non-polarized representations
- 4 Give the Abelianization theorem for \mathcal{C} , allowing all computations.

Remark

The gauge theory should be recoverable from the Coulomb branch as its *Rozansky-Witten theory*. However, while this seems to work for \mathcal{C} , most Coulomb branches are singular, so the RW construction not clearly defined.



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Acknowledgements

This is only a partial list, given my poor knowledge of the literature.

- The Coulomb branch for SU_2 was described by Seiberg and Witten
- Follow-up physics work for SU_n (Argyres et al.)
- The zero Coulomb branch was introduced by Bezrukavnikov, Finkelberg, Mirkovic in connection with the Toda system and Langlands duality (but not recognized as such at the time)
- Early computations on Coulomb branches by Hanany, Witten, et al.
- Braverman, Finkelberg, Nakajima first defined $\mathcal{C}(V \oplus \bar{V})$
- Bullimore, Dimofte, Gaiotto gave complete computations in the abelian case and conjectural constructions in general
- Recent work of Dimofte, Bullimore and others on the Physics side
- Sam Raskin explained to me a chiral homology construction of Coulomb branches and the mod 2 obstructions¹(due to Witten). This motivated the present construction.



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Properties of Coulomb branches

- 1 The zero branch \mathcal{C} is an abelian group scheme over the base B .
 - 2 Each $\mathcal{C}(E)$ carries a Poisson structure (of degree (-2))
 - 3 Each $\mathcal{C}(E)$ maps to B with Lagrangian fibers, and is free as a module.
 - 4 Each $\mathcal{C}(E)$ carries a fiber-wise action of \mathcal{C} , and is birational to \mathcal{C} .
 - 5 There are compatible multiplications $\mathcal{C}(E) \times_B \mathcal{C}(F) \rightarrow \mathcal{C}(E \oplus F)$.
 - 6 $\mathcal{C}(V \oplus \bar{V})$ has two regular Lagrangian sections, corresponding to V, \bar{V} , whose ratio is given by the earlier Mirror formula.
 - 7 *With the mass parameter included*, $\mathcal{C}(V \oplus \bar{V})$ is the quotient in affine schemes of two copies of \mathcal{C} relatively translated by the mirror section.
 - 8 Abelianization: $\mathcal{C}(G; E) = \mathcal{C}(T; E \ominus \mathfrak{g}_{\mathbb{C}}^{\oplus 2}) / W$, when the latter makes sense (e.g. the roots are weights of E but less is needed).
- (3) and (8) \Rightarrow all Coulomb branches from \mathcal{C} and from SU_2 , $V = \mathbb{C}^2$.



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Pure topological gauge theory

Two naïve versions of gauge theory in $3D$ (related by Koszul duality) can be described homotopically. They fit in the framework of fully extended TQFT, but lose information.

Much like in $2D$, string topology for a compact manifold M can be generated by $C^*(M)$ or $C_*(\Omega M)$, we can use $C^*(M)$ and the E_2 algebra $C_*(\Omega^2 M)$ to define *sphere topology*, a partially defined $3D$ TQFT.

It is fully defined up to dimension 2; some $3D$ operations also exist (e.g. the E_3 structure on the space assigned to the sphere).

Here, $M = BG$ (so we can dispense with (co)chains). The space of states for S^2 is computed, in the two cases, as the E_2 Hochschild cohomology of $H^*(BG)$, resp. $H_*(\Omega G)$, giving formal completions of \mathcal{C} along Lagrangians (the cotangent fiber at 1, or the zero-section, respectively, for the torus).

From the two naïve answers, the true answer can be guessed as $H_G^*(\Omega G)$. More generally, a closed surface Σ leads to the analogous equivariant homology of the stack of G -bundles. This gives a partial TQFT.



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Wild speculation

This topological construction makes it seem that adding a representation E is irrelevant, because of its contractibility.

What shows up instead is a categorified version of the *virtual fundamental class* for an obstructed deformation problem. This does in fact carry a virtual fundamental class from the boundary theory, the GLSM of V .

It appears, on the level of surfaces, in the guise of a *constructible sheaf* on the stack Bun_G of G -bundles, which is (conjecturally) multiplicative for the TQFT gluing operations. A key contribution of [BFN] is a coefficient system on ΩG in which the sphere multiplication is strictly defined, bypassing the deformation and coherence checks.

The theory with matter should be generated by a conjectural *gauged holomorphic Fukaya 2-category* of E . For $V \oplus \bar{V}$, a simplified model could be the gauged endofunctor category of the Fukaya category of V . This seems difficult to define precisely.



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At a loop $\gamma \in G_{\mathbb{C}}((z))$, the fiber $L_V(\gamma)$ is the kernel of

$$V[[z]] \oplus V[[z]] \xrightarrow{\text{Id}-\gamma} V((z))$$

which has finite codimension in $V[[z]]$.

(This is H^0 of the V -bundle over the double-centered disk.)

We can dispense with the infinite-dimensional bundles by using $\text{Bun}_G(\mathbb{P}^1)$ in guise of $G \backslash \Omega G$ and replace L_V with the total space of the *index sheaf*

$$\text{Spec Sym} \left[R\Gamma(\mathbb{P}^1; \mathcal{V}(-1))^{\vee} \right],$$

where \mathcal{V} is the universal bundle with fiber \mathcal{V} on $\mathbb{P}^1 \times \text{Bun}_G$.

The sheaf $\mathcal{S}_V \rightarrow \text{Bun}_G$ is the fiber-wise compactly supported cohomology.

The “square root” of \mathcal{S} that we will need in the non-polarized case is more transparent in this interpretation.

Remark

This last construction can be applied to any closed surface, but only yields a topological answer *after stabilization* (perturbation of $\bar{\partial}$, or high order poles to avoid Brill-Noether phenomena).



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Quaternionic representations

When E has no invariant Lagrangian, the 3D theory has no obvious topological boundary conditions that would allow its reconstruction by a Mirror formula. (It may well have *conformal* boundary conditions.)

Instead, we will modify of the BFN construction of the constructible sheaf.

Note that using E in lieu of V produces a sheaf \mathcal{S}_E for the doubled representation $E \oplus \bar{E}$. So what is needed is a square root of $\mathfrak{f}_! \mathcal{S}_E$, in the following sense.

Each stratum of ΩG is shifted cohomologically by capping with the Euler class of H^0 of the universal bundle. Now, in K -theory, the Euler class of a vector bundle is its exterior algebra. A square root would be the Spin bundle, if one exists. This requires a reduction along $\text{Spin} \rightarrow \text{SO} \rightarrow \text{U}$ of the structure group.

We will find something close enough to this for $H^0(E)$.



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A Bott sequence (Wood's theorem)

This is the sequence, classified by $\eta : \Sigma^2 KO \rightarrow \Sigma^1 KO$,

$$KO \xrightarrow{\mathbb{C} \otimes_{\mathbb{R}} \cdot} KU \xrightarrow{\Omega^2(\mathbb{H} \otimes_{\mathbb{C}} \cdot)} \Omega^2 BSp = \Sigma^2 KO \quad (P)$$

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whose relevance is seen when integrating E over \mathbb{P}^1 ,

$$E : BG \rightarrow BSp \rightsquigarrow R\Gamma(E(-1)) : \Omega G \rightarrow \Omega^2 BSp = \Sigma^2 KO :$$

it refines the complex structure of $R\Gamma$ to $\Sigma^2 KO$.

A polarization V of E would lift $R\Gamma(E)$ to $R\Gamma(V) \in KU$ in (P), and we could build L_V, \mathcal{S}_V as before.

Now, an obstruction calculation shows that a lift in (P) *always* exists, just not an *additive* one. So the constructible sheaf \mathcal{S}_V would seem not (E_2) multiplicative.

But that is not quite right.



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Mod 2 obstruction to multiplicativity

By virtue of (P), additivity holds up to suspension by a real virtual vector bundle. Should this be oriented, the Thom isomorphism would remove the obstruction. (Similarly, a Spin structure would remove it in K -theory.)

The obstruction to lifting BO to $BSpin$ consists of w_1, w_2 , in the bottom of BO ; this is a spectrum W with $\pi_1 = \pi_2 = \mathbb{Z}/2$ connected by Sq^2 . The obstruction to Spin reduction in (P) is then read from $\eta : \Sigma^2 KO \rightarrow \Sigma^1 W$. Since we need E_2 additivity in the reduction, the obstruction is

$$E : BG \rightarrow BSp \rightarrow \Sigma^3 W$$

and consists of a class in $H^4(BG; \mathbb{Z}/2)$ and, subject to its vanishing, a second one in $H^5(BG; \mathbb{Z}/2)$.

Remark

The argument needs fleshing out: one uses a lifting of ΩG in (P) to suspend the index sheaf and produce a linear space with a real structure.



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Questions and Conjectures

- 1 The construction should work on an arbitrary curve and produce a vector space independent of the complex structure. TQFT would need a factorization rule for these, which must be spelt out. There will be no rigid construction as in [BFN].
- 2 Abelianization in higher genus should be a strong version of the Atiyah-Bott construction, splitting the homology. This may allow effective computation of the TQFT structure.
- 3 In the polarized case, the TQFT should admit the GLSM of V as a topological boundary theory.
- 4 Without polarization, conformal boundary conditions should exist. Can one construct the 3D TQFT from those, as one can from the topological boundary theory in the polarized case?

THANK YOU FOR LISTENING