Title: PSI 2019/2020 - Standard Model and Beyond part 2 - Lecture 6

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Collection: PSI 2019/2020 - Standard Model and Beyond - Part 2

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SM+B #24 $D(xy) = D(x)y + xD(y)$

3 main algebras:

i) $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (numbers) $D_{a,b} = [[a,b],\_] - 3[a,b,\_]$

$x \to D_{a,b}(x)$

ii) Lie (symmetry) $[\,,\,]$ $D_a = [a,\_]$

$so(n)$ $su(n)$ $sq(n)$ $\{g_2, f_4, e_6, e_7, e_8)$

iii) Jordan (QM observables) $D_{a,b} = \{a,\{b,\_\}\}$

$J\mathrm{spin}(n)$ $H_n(\mathbb{R})$ $H_n(\mathbb{C})$ $H_n(\mathbb{H})$ $H_3(\mathbb{O})$ $-\{b,\{a,\_\}\}$

SM+B #24

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3 main algebras:

i) $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (numbers) $\quad D_{a,b} = [[a,b], \_] - 3[a,b,\_]$

der: $0 \quad 0 \quad su(2) \quad g_2$ $\qquad x \to D_{a,b}(x)$

ii) Lie (symmetry) $[\ ,\ ]$ $\quad D_a = [a, \_]$

der: $so(n) \quad su(n) \quad sq(n) \quad \{g_2, f_4, e_6, e_7, e_8\}$

$A_n(\mathbb{R}) \quad A_n'(\mathbb{C}) \quad A_n(\mathbb{H}) \quad \mathbb{O}? \quad (\mathbb{O}\otimes\mathbb{K})P^2$

iii) Jordan (QM observables) $\quad D_{a,b} = \{a,\{b,\_\}\} - \{b,\{a,\_\}\}$

$J\mathrm{spin}(n) \hookrightarrow \mathrm{spin}(n)$, $H_n(\mathbb{R}) \hookrightarrow so(n)$, $H_n(\mathbb{C}) \hookrightarrow su(n)$, $H_n(\mathbb{H}) \hookrightarrow sq(n)$, $H_3(\mathbb{O}) \hookrightarrow f_4$

Key results for $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$ (arb. n) extend to $\mathbb{K}=\mathbb{O}$ ($n=3$)

Let $X,Y,Z\in H_n(\mathbb{K})$ $\quad A,B\in A_n(\mathbb{K})$

$$D_{a,b} \qquad D(M,N)=\sum_{i,j=1}^{n} \underline{D_{M_{ij},N_{ji}}} \qquad M_{ij}N_{ji}$$

$$\mathbb{K}\to\mathbb{K}$$

Jordan id: $(X^2\circ Y)\circ X = X^2\circ(Y\circ X)$ ← Jordan alg

$$[A,\{X,Y\}] = \{[A,X],Y\}+\{X,[A,Y]\} \leftarrow \underline{C_A} = [A,-]$$

$$[\underline{A},[\underline{B},\underline{X}]]-[\underline{B},[\underline{A},\underline{X}]] = [\underline{[A,B]',X}] + \underline{\tfrac{1}{n}D(A,B)}X \leftarrow$$

$$[C_A,C_B] \quad = \quad C_{[A,B]'} + \cdots$$

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$$[A,\{X,Y\}]=\{[A,X],Y\}+\{X,[A,Y]\} \leftarrow C_A=[A,-]$$

$$[A,[B,X]]-[B,[A,X]]=[[A,B]',X]+\frac{1}{n}D(A,B)X \leftarrow$$

$$\{X,\{Y,Z\}\}-\{Y,\{X,Z\}\}=[[X,Y]',Z]+\frac{1}{n}D(A,B)X$$

Key results for $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$ (arb. n) extend to $\mathbb{K}=\mathbb{O}$ $(n=3)$

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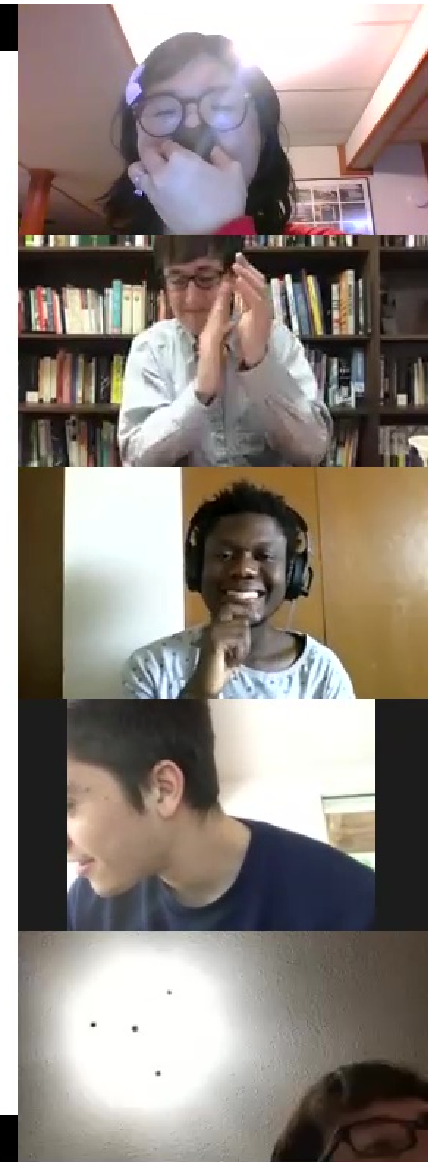
$\{X,\{Y,Z\}\}-\{Y,\{X,Z\}\}=[[X,Y]',Z]+\frac{1}{n}D(X,Y)Z$ ←

$$\mathrm{Der}(\underline{\underline{H_n(\mathbb{K})}}) = \underbrace{A_n'(\mathbb{K})}_{C_A} \oplus \underbrace{\mathrm{Der}(\mathbb{K})}_{D,\,D'}$$

$$[D, D'] = D''$$

$$[D, C_A] = C_{\underline{DA}}$$

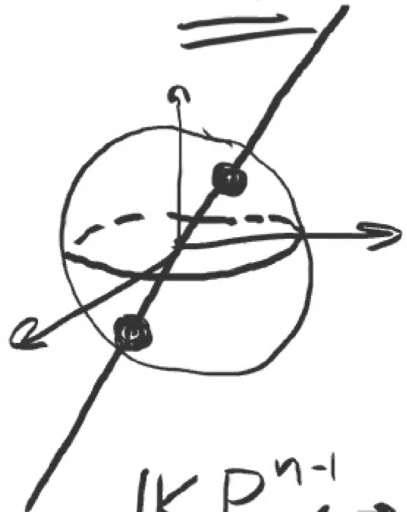
$$[C_A, C_B] = C_{[A,B]} + \frac{1}{n} D(A,B)$$



$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ $\qquad \underline{\underline{\mathbb{K}P^n}}$ $\qquad \underline{\underline{\mathbb{K}}}^{n+1} = (\underline{x_1}, \ldots, x_{n+1})$

$\uparrow_{k} \cong x_0 k \longrightarrow \frac{1}{x_0} k \cong k$

$\mathbb{R}^3 \hookrightarrow \mathbb{R}P^2$



$E_n, S_n, H_n, \mathbb{K}P^n$

$\mathbb{K}P^{n-1} \longleftrightarrow H_n(\mathbb{K}) \longleftarrow e^2 = e$

$\uparrow_{k-1} \qquad \uparrow_{k}$

$\mathbb{O}P^1 \quad \mathbb{O}P^2 \longleftarrow$ Moufang plane

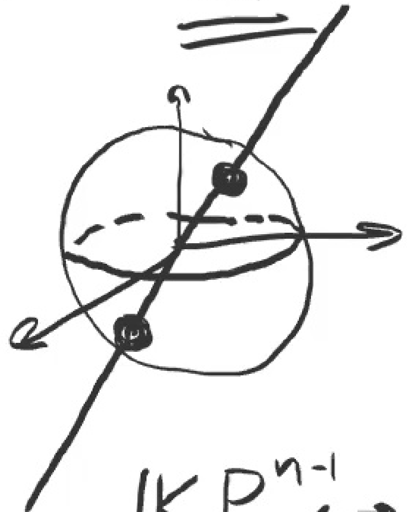
$f_4, \cancel{e} \, e_6, e_7, e_8$

Boris Rosenfeld: $\underline{\underline{(\mathbb{R}\otimes\mathbb{O})}}P^2, (\mathbb{R}\otimes\mathbb{C})P^2, \ldots$

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ $\qquad \underline{\underline{\mathbb{K}P^n}}$ $\qquad \underline{\underline{\mathbb{K}^{n+1}}} = (x_1, \dots, x_{n+1})$

$\uparrow_{k} \cong x_0 k \longrightarrow \frac{1}{x_0} k \cong k$

$\mathbb{R}^3 \hookleftarrow \mathbb{R}P^2$



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Boris Rosenfeld: $(\mathbb{R}\otimes\mathbb{O})P^2, (\mathbb{C}\otimes\mathbb{O})P^2, (\mathbb{H}\otimes\mathbb{O})P^2, (\mathbb{O}\otimes\mathbb{O})P^2$

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ $\qquad$ $\mathbb{K}P^n$ $\qquad$ $\mathbb{K}^{n+1} = (x_1, \dots, x_{n+1})$

$\uparrow_k \cong x_0 k \longrightarrow \frac{1}{x_0} k \cong k$

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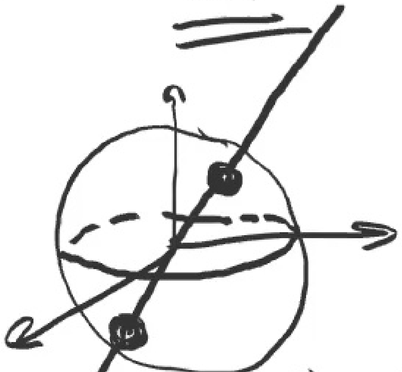

$E_n, S_n, H_n, \mathbb{K}P^n$

$\mathrm{Isom}(\mathbb{K}P^{n-1}) \longleftrightarrow \mathrm{Aut}(H_n(\mathbb{K}))$

$\underset{\uparrow_{k-1}}{\mathbb{K}P^{n-1}} \longleftrightarrow \underset{\uparrow_k}{H_n(\mathbb{K})} \leftarrow e^2 = e$

$\mathbb{O}P^1$ $\qquad$ $\mathbb{O}P^2 \leftarrow$ Moufang plane

$f_4, \quad e_6, e_7, e_8$

Boris Rosenfeld: $(\mathbb{R}\otimes\mathbb{O})P^2, (\mathbb{C}\otimes\mathbb{O})P^2, (\mathbb{H}\otimes\mathbb{O})P^2, (\mathbb{O}\otimes\mathbb{O})P^2$

$$\to \mathrm{Der}(\underline{\underline{H_n(\mathbb{K})}}) = A'_n(\underline{\underline{\mathbb{K}}}) \oplus \underline{\underline{\mathrm{Der}(\mathbb{K})}} \oplus$$

$$\underline{\underline{\mathbb{K}P^{n-1}}} \qquad \underline{\underline{C_A}} \qquad \underline{\underline{D, D'}}$$

$$\left\{ \begin{array}{l} [D, D'] = D'' \leftarrow \\ [D, C_A] = C_{\underline{DA}} \\ [C_A, C_B] = C_{[A,B]} + \frac{1}{n} D(A,B) \end{array} \right.$$

$$(\mathbb{K}_1 \otimes \mathbb{K}_2) P^2$$

$$L(\mathbb{K}_1, \mathbb{K}_2) = A_n'(\mathbb{K}_1 \otimes \mathbb{K}_2) \oplus \underline{Der(\mathbb{K}_1) \oplus Der(\mathbb{K}_2)}$$

$$C_A \qquad D, D', D''$$

$$[D, D'] = D'' \in Der(\mathbb{K}_1) \oplus Der(\mathbb{K}_2)$$

$$[D, C_A] = C_{DA}$$

$$[C_A, C_B] = C_{[A,B]'} + \frac{1}{n} D(A, B) \qquad \underline{\underline{D(A,B) = \sum_{i,j=1}^{n} \hat{D}_{A_{ij}, B_{ji}}}}$$

$$\rightarrow a = a_1 \otimes a_2 \qquad b = b_1 \otimes b_2$$

$$\hat{D}_{a,b} = \langle a_1 | b_1 \rangle D_{a_2, b_2} + \langle a_2 | b_2 \rangle D_{a_1, b_1}$$

$$L(\mathbb{K}_1,\mathbb{K}_2) = \overset{(\mathbb{K}_1\otimes\mathbb{K}_2)P^2}{A_n'(\mathbb{K}_1\otimes\mathbb{K}_2)} \oplus \underline{\mathrm{Der}(\mathbb{K}_1)\oplus\mathrm{Der}(\mathbb{K}_2)}$$

$$C_A \qquad\qquad D,\ D',\ D''$$

$$[D,D'] = D'' \in \mathrm{Der}(\mathbb{K}_1)\oplus\mathrm{Der}(\mathbb{K}_2)$$

$$[D,C_A] = C_{DA}$$

$$[C_A,C_B] = C_{[A,B]'} + \frac{1}{n}D(A,B) \qquad D(A,B) = \underline{\underline{\sum_{i,j=1}^{n}\hat{D}_{A_{ij},B_{ji}}}}$$

$$\rightarrow a = a_1\otimes a_2 \qquad b = b_1\otimes b_2$$

$$\hat{D}_{a,b} = \underline{\underline{\langle a_1|b_1\rangle D_{a_2,b_2}}} + \underline{\underline{\langle a_2|b_2\rangle D_{a_1,b_1}}}$$

$n=3$

$\mathbb{K}_2$

$\mathbb{K}_1$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$				f_4
$\mathbb{C}$				e_6
$\mathbb{H}$				e_7
$\mathbb{O}$	F_4	e_6	e_7	e_8

$\mathbb{O} \leftarrow g_2$

$f_4 \leftarrow 52 \quad (3\times 4\times 8) = 96$

$\overline{\mathbb{K}_1 \otimes \mathbb{K}_2} \qquad \dfrac{20}{116}$

$e_6 \leftarrow 78$

$= \overline{\mathbb{K}_1} \otimes \overline{\mathbb{K}_2}$

$\uparrow \qquad \uparrow$

$\cdots \qquad \cdots e_6$

$e_7 \leftarrow \underline{\underline{133}}$

$e_8 \leftarrow 248$

$$L(\mathbb{H},\mathbb{O}) = \overset{(116)}{A_3'(\mathbb{H}\otimes\mathbb{O})} \oplus \underline{\overset{(3)}{\mathrm{Der}(\mathbb{H})} \oplus \overset{(14)}{\mathrm{Der}(\mathbb{O})}}$$