

Title: PSI 2019/2020 - Standard Model and Beyond part 2 - Lecture 6

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Collection: PSI 2019/2020 - Standard Model and Beyond - Part 2

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SM+B #24

$$D(xy) = D(x)y + xD(y)$$

3 main algebras:

i) $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (numbers) $D_{a,b} = [[a,b], -] - 3[a,b, -]$

$$x \rightarrow D_{a,b}(x)$$

ii) Lie (symmetry) $[,]$

$so(n)$ $su(n)$ $sg(n)$

$$D_a = [a, -]$$

$\{g_2, f_4, e_6, e_7, e_8\}$

iii) Jordan (QM observables)

J spin(n) $H_n(\mathbb{R})$ $H_n(\mathbb{C})$ $H_n(\mathbb{H})$

$$D_{a,b} = \{a, \{b, -\}\}$$

$$H_3(\mathbb{O}) - \{b, \{a, -\}\}$$

SM+B #24

$$D(xy) = D(x)y + xD(y)$$

3 main algebras:

i) $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (numbers)
der: $\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$
 $0 \quad 0 \quad su(2) \quad g_2$

$$D_{a,b} = [[a,b], -] - 3[a,b, -]$$
$$x \rightarrow D_{a,b}(x)$$

ii) Lie (symmetry) $[,]$
der: $so(n) \quad su(n) \quad sg(n)$
 $A_n(\mathbb{R}) \quad A_n(\mathbb{C}) \quad A_n(\mathbb{H})$

$$D_a = [a, -]$$
$$\{g_2, f_4, e_6, e_7, e_8\}$$
$$\mathbb{O}^? \quad (\mathbb{O} \otimes \mathbb{K})P^2$$

iii) Jordan (QM observables)
 $\begin{matrix} \overline{JSpin(n)} \\ \hookrightarrow Spin(n) \end{matrix} \quad \begin{matrix} \overline{H_n(\mathbb{R})} \\ \hookrightarrow so(n) \end{matrix} \quad \begin{matrix} \overline{H_n(\mathbb{C})} \\ \hookrightarrow su(n) \end{matrix} \quad \begin{matrix} \overline{H_n(\mathbb{H})} \\ \hookrightarrow sg(n) \end{matrix}$

$$D_{a,b} = \{a, \{b, -\}\} \leftarrow$$
$$\begin{matrix} \hookrightarrow H_3(\mathbb{O}) \\ \hookrightarrow f_4 \end{matrix} \quad -\{b, \{a, -\}\}$$

Key results for $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (arb. n) extend to $K = \mathbb{O}$ ($n=3$)

Let $X, Y, Z \in M_n(K)$ $A, B \in A_n(K)$

$$D_{a,b} \quad D(M, N) = \sum_{i,j=1}^n \underline{D_{M_{ij}, N_{ji}}} \quad M_{ij} N_{ji}$$

$K \rightarrow K$

Jordan id: $(X^2 \circ Y) \circ X = X^2 \circ (Y \circ X) \leftarrow$ Jordan alg

$$[A, \{X, Y\}] = \{[A, X], Y\} + \{X, [A, Y]\} \leftarrow \underline{C_A} = [A, -]$$

$$\underline{[A, [B, X]]} - \underline{[B, [A, X]]} = \underline{[A, B]', X} + \underline{\frac{1}{n} D(A, B) X} \leftarrow$$

$$[C_A, C_B] = C_{[A, B]'} + \dots$$

Key results for $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (arb. n) extend to $K = \mathbb{O}$ ($n=3$)

Let $X, Y, Z \in M_n(K)$ $A, B \in A_n(K)$

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$$\{X, \{Y, Z\}\} - \{Y, \{X, Z\}\} = \underline{[X, Y]', Z} + \underline{\frac{1}{n} D(A, B) X}$$

Key results for $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (arb. n) extend to $K = \mathbb{O}$ ($n=3$)

Let $X, Y, Z \in H_n(K)$ $A, B \in A_n(K)$

$$D_{a,b} \quad D(M, N) = \sum_{i,j=1}^n \underline{D_{M_{ij}, N_{ji}}} \quad M_{ij} N_{ji}$$

$K \rightarrow K$

Jordan id: $(X^2 \circ Y) \circ X = X^2 \circ (Y \circ X) \leftarrow \text{Jordan alg}$

$$[A, \{X, Y\}] = \{[A, X], Y\} + \{X, [A, Y]\} \leftarrow \underline{C_A} = [A, -]$$

$$\underline{[A, [B, X]]} - \underline{[B, [A, X]]} = \underline{[A, B]', X} + \underline{\frac{1}{n} D(A, B) X} \leftarrow$$

$$\underline{\{X, \{Y, Z\}\}} - \underline{\{Y, \{X, Z\}\}} = \underline{[X, Y]', Z} + \underline{\frac{1}{n} D(X, Y) Z} \leftarrow$$

$$\text{Der}(\underline{H_n(K)}) = A'_n(K) \oplus \text{Der}(K) \leftarrow$$

\uparrow \uparrow
 C_A D, D'

$$[D, D'] = D'' \leftarrow$$

$$[D, C_A] = C_{DA} \leftarrow$$

$$[C_A, C_B] = C_{[A, B]} + \frac{1}{n} D(A, B)$$



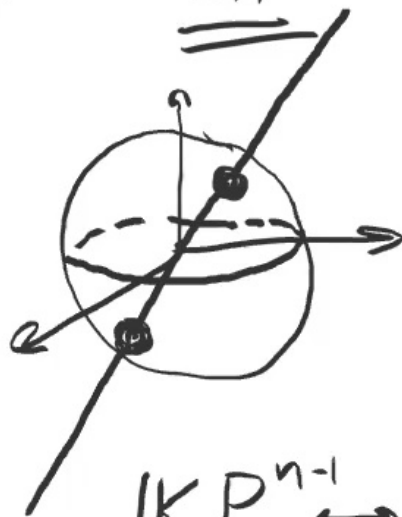
$$K = \mathbb{R}, \mathbb{C}, \mathbb{H}$$

$$\underline{\underline{K P^n}}$$

$$\underline{\underline{K^{n+1}}} = (x_1, \dots, x_{n+1})$$

$$\begin{array}{c} \uparrow \\ \underline{\underline{k}} \cong x_0 k \\ \underline{\underline{=}} \end{array} \quad \begin{array}{c} \leftarrow \\ \frac{1}{x_0} k \cong k \end{array}$$

$$\begin{array}{l} \mathbb{R}^3 \\ \hookrightarrow \mathbb{R} P^2 \end{array}$$



$$E_n, S_n, H_n, K P^n$$

$$\begin{array}{ccc} K P^{n-1} & \leftrightarrow & H_n(K) \leftarrow e^2 = e \\ \uparrow_{k-1} & & \uparrow_k \end{array}$$

$$\mathbb{O} P^1$$

$$\mathbb{O} P^2$$

← Montfarg plane

$$f_4, \neq e_6, e_2, e_8$$

Boris Rosenfeld:

$$\underline{\underline{(\mathbb{R} \otimes \mathbb{O}) P^2}}, (\mathbb{R} \otimes \mathbb{C}) P^2, \dots$$

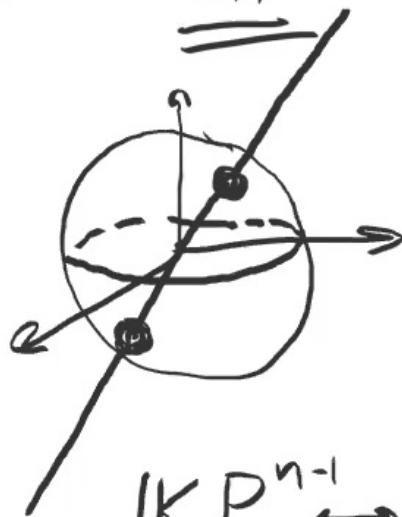
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$$\underline{\underline{K P^n}}$$

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$$E_n, S_n, H_n, \underline{\underline{K P^n}}$$

$$\underline{\underline{K P^{n-1}}} \leftrightarrow H_n(K) \leftarrow e^2 = e$$

$$\uparrow_{k-1}$$

$$\uparrow_k$$

$$\mathbb{O} P^1$$

$$\mathbb{O} P^2$$

← Montfong plane

Boris Rosenfeld: f_4, e_6, e_2, e_8

$$\underline{\underline{(\mathbb{R} \otimes \mathbb{O}) P^2}}, \underline{\underline{(\mathbb{C} \otimes \mathbb{O}) P^2}}, \underline{\underline{(\mathbb{H} \otimes \mathbb{O}) P^2}}, \underline{\underline{(\mathbb{O} \otimes \mathbb{O}) P^2}}$$

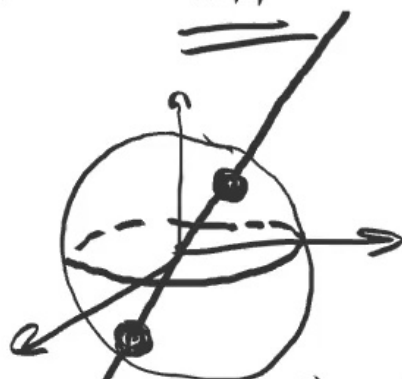
$$K = \mathbb{R}, \mathbb{C}, \mathbb{H}$$

$$\underline{\underline{K P^n}}$$

$$\underline{\underline{K^{n+1}}} = (x_1, \dots, x_{n+1})$$

$$\begin{aligned} \uparrow & k \cong x_0 k \\ \underline{\underline{=}} & \quad \quad \quad \frac{1}{x_0} k \cong k \end{aligned}$$

$$\begin{aligned} \hookrightarrow \mathbb{R}^3 \\ \hookrightarrow \mathbb{R} P^2 \end{aligned}$$



$$E_n, S_n, H_n, K P^n$$

$$\begin{aligned} \underline{\underline{Isom(K P^n)}} & \leftrightarrow \underline{\underline{Aut(H_n(K))}} \\ \underline{\underline{K P^{n-1}}} & \leftrightarrow \underline{\underline{H_n(K)}} \leftarrow e^2 = e \\ \uparrow_{k-1} & \quad \quad \quad \uparrow_k \end{aligned}$$

$$\mathbb{O} P^1 \quad \mathbb{O} P^2 \leftarrow \text{Moufang plane}$$

Boris Rosenfeld: f_4, e_6, e_2, e_8

$$\underline{\underline{(\mathbb{R} \otimes \mathbb{O}) P^2}}, \underline{\underline{(\mathbb{C} \otimes \mathbb{O}) P^2}}, \underline{\underline{(\mathbb{H} \otimes \mathbb{O}) P^2}}, \underline{\underline{(\mathbb{O} \otimes \mathbb{O}) P^2}}$$

$$\rightarrow \text{Der}(\underline{\underline{H_n(K)}}) = \underbrace{A'_n(\underline{\underline{K}})}_{\substack{\uparrow \\ C_A}} \oplus \underbrace{\text{Der}(\underline{\underline{K}})}_{\substack{\uparrow \\ D, D'}} \oplus \dots$$

$\nearrow \underline{\underline{K P^{n-1}}}$

$$\left\{ \begin{array}{l} [D, D'] = D'' \leftarrow \\ [D, C_A] = C_{DA} \leftarrow \\ [C_A, C_B] = C_{[A, B]} + \frac{1}{n} D(A, B) \end{array} \right.$$

$$(K_1 \otimes K_2)P^2$$

$$L(K_1, K_2) = A'_n(K_1 \otimes K_2) \oplus \underbrace{\text{Der}(K_1) \oplus \text{Der}(K_2)}_{D, D', D''} \oplus C_A$$

$$[D, D'] = D'' \in \text{Der}(K_1) \oplus \text{Der}(K_2)$$

$$[D, C_A] = C_{DA}$$

$$[C_A, C_B] = C_{[A, B]'} + \frac{1}{n} D(A, B)$$

$$D(A, B) = \sum_{i,j=1}^n \hat{D}_{\underline{A_{ij} B_{ji}}}$$

$$\rightarrow a = a_1 \otimes a_2 \quad b = b_1 \otimes b_2$$

$$\hat{D}_{a,b} = \langle a_1 | b_1 \rangle D_{a_2, b_2} + \langle a_2 | b_2 \rangle D_{a_1, b_1}$$

$$(K_1 \otimes K_2)P^2$$

$$L(K_1, K_2) = A'_n(K_1 \otimes K_2) \oplus \underbrace{\text{Der}(K_1) \oplus \text{Der}(K_2)}_{D, D', D''}$$

$$C_A$$

$$[D, D'] = D'' \in \text{Der}(K_1) \oplus \text{Der}(K_2)$$

$$[D, C_A] = C_{DA}$$

$$[C_A, C_B] = C_{[A, B]'} + \frac{1}{n} D(A, B)$$

$$D(A, B) = \sum_{i,j=1}^n \hat{D}_{\underline{A_{ij}}, \underline{B_{ji}}}$$

$$\rightarrow a = a_1 \otimes a_2 \quad b = b_1 \otimes b_2$$

$$\hat{D}_{a,b} = \underline{\langle a_1 | b_1 \rangle} \underline{D_{a_2, b_2}} + \underline{\langle a_2 | b_2 \rangle} \underline{D_{a_1, b_1}}$$

$n=3$

\mathbb{K}_2

	\mathbb{R}	\mathbb{C}	\mathbb{H}	$\mathbb{O} = \mathfrak{g}_2$	$\begin{pmatrix} \odot & \odot & \odot \\ \vdots & \odot & \odot \\ \vdots & \vdots & \odot \end{pmatrix}$
\mathbb{R}				\mathfrak{f}_4	$\leftarrow 52 \quad (3 \times 4 \times 8) = 96$
\mathbb{C}				\mathfrak{e}_6	$\leftarrow 78 \quad \overline{\mathbb{K}_1 \otimes \mathbb{K}_2} \quad \frac{20}{116}$
\mathbb{H}				\mathfrak{e}_7	$\leftarrow \underline{133} \quad \begin{matrix} \uparrow & \uparrow \\ \vdots & -\mathfrak{e}_6 \end{matrix}$
\mathbb{O}	\mathfrak{f}_4	\mathfrak{e}_6	\mathfrak{e}_7	\mathfrak{e}_8	$\leftarrow 248$

$$L(\mathbb{H}, \mathbb{O}) = A'_3 \begin{matrix} (116) \\ \uparrow \quad \uparrow \\ (\mathbb{H} \otimes \mathbb{O}) \end{matrix} \oplus \text{Der}(\mathbb{H}) \oplus \text{Der}(\mathbb{O})$$

(3)
 (14)