Title: Holomorphic Floer theory and deformation quantization

Speakers: Yan Soibelman

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Abstract: Geometry of a pair of complex Lagrangian submanifolds of a complex symplectic manifold appears in many areas of mathematics and physics, including exponential integrals in finite and infinite dimensions, wall-crossing formulas in 2d and 4d, representation theory, resurgence of WKB series and so on.

In 2014 we started a joint project with Maxim Kontsevich which we named "Holomorphic Floer Theory" (HFT for short) in order to study all these (and other) phenomena as a part of a bigger picture.

Aim of my talk is to discuss aspects of HFT related to deformation quantization of complex symplectic manifolds, including the conjectural Riemann-Hilbert correspondence. Although some parts of this story have been already reported elsewhere, the topic has many ramifications which have not been discussed earlier.

Zoom Link: https://pitp.zoom.us/j/496558472

Holomorphic Floer theory and deformation quantization

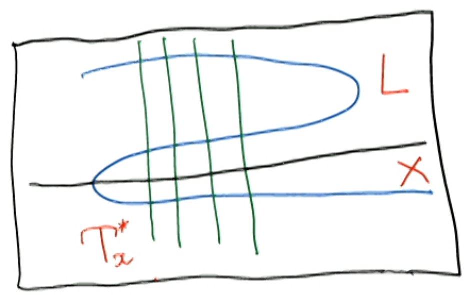
Yan Soibelman

Perimeter Institute, March 19,2020

D-modules on a curve

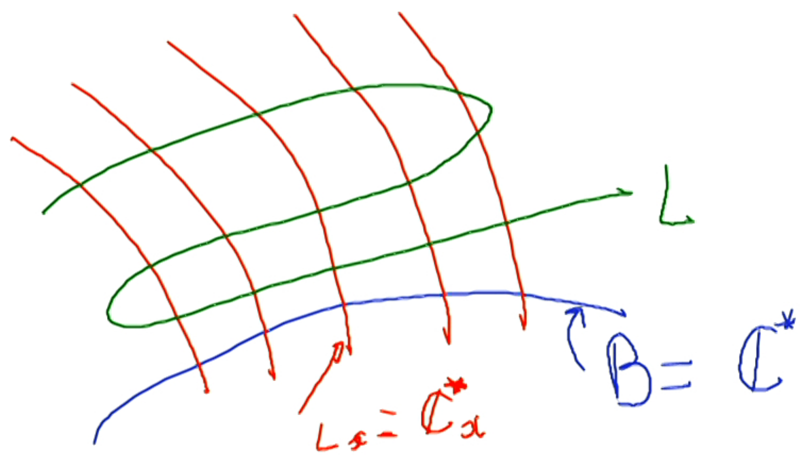
Transversals to SC

Sunday, November 01, 2015
11:46 AM



Spectral curve and family of transversal Lagrangians

D_q-modules on $\mathbb{C}^*$



$$(L, \mathrm{triv}) \longmapsto \mathrm{Hom}\big((L, \mathrm{triv}), (L_x, \rho_x)\big)$$

After Kapustin, Orlov, Witten and Gukov: boundary conditions $\mathcal{B}$ in the A-model can be coistorpic. In particular one can consider A-branes $\mathcal{B}_{cc}$ which are supported on the whole symplectic manifold M. Such coisotropic branes do not exist for general symplectic manifolds, but they can exist for holomorphic symplectic ones (e.g. HK manifolds).

In mathematics: coisotropic branes and their interactions are not well-understood in terms of Floer theory (e.g. Fukaya categories). The case of a canonical coisotropic brane supported on the whole M can be treated algebro-geometrically using the paper by Kontsevich (2000) on deformation quantization of algebraic varieties. Set-up: one has to choose a "quantizable" compactification of an affine Poisson algebraic variety and then apply deformation quantization techniques to the Poisson algebra of functions, considered as filtered (by order of pole at infinity) algebra. We will modify this idea later.

Brane quantization in physics: $M \simeq T^*X$, then the algebra $A_M := Hom(\mathcal{B}_{cc}, \mathcal{B}_{cc})$ acts on the vector space $V_L := Hom(\mathcal{B}_{cc}, \mathcal{B}_L)$ and gives a D-module on X corresponding to L.
In general, for an affine symplectic M the corresponding algebra $A_M = \mathcal{O}_\hbar(M)$ is a deformation quantization of the algebra $\mathcal{O}(M)$ of regular functions on M. To a complex Lagrangian $L \subset M$ one should be able to assign a module of finite type over quantized $\mathcal{O}(M)$.
This gives the "brane description" of the category of "holonomic DQ-modules", generalizing the category of holonomic D_X-modules from the case $M = T^*X$.
Informally holonomic DQ-modules can be thought of as "coherent sheaves with holomorphic Lagrangian support on the quantized M".

Conclusion: The functor $\mathcal{B}_L \to$ the A_M-module V_L should give an equivalence of the category of Lagrangian A-branes with the category of holonomic modules over the quantum algebra A_M.
Mathematically, let $(M, \omega^{2,0})$ be a holomorphic (e.g. complex algebraic) symplectic manifold, $\omega := Re(\omega^{2,0}), B = Im(\omega^{2,0})$ (here B is the B-field). Then:

Conjectural Riemann-Hilbert correspondence

In the derived sense the category of holonomic modules over the quantized sheaf of regular functions $\mathcal{O}_{M,\hbar}$ is equivalent to the Fukaya category $\mathcal{F}(M, \omega)$ with the B-field B.

Terminology of RH: The de Rham side is equivalent to the Betti side.
In order to make a more precise conjecture we will need to use the language of analytic 2-stacks. Next seven slides can be ignored if you are interested in examples only.

Conjecture: The Betti and de Rham sides depend analytically on the natural parameters, e.g. $[\omega]$.

Note: A priori the dependence on parameters is formal - in the case of formal deformation quantization the category is linear over $\mathbb{C}[[\hbar]]$ and the Fukaya category is linear over the Novikov field $\mathbb{C}((e^{\frac{\mathbb{R}}{\hbar}}))$.

Expectation: Under some conditions the series converge. The Betti and de Rham sides of the RH-correspondence are global sections of analytic sheaves of A_∞-categories over an analytic 2-stack (formalism of analytic dg-stacks: M. Porta and others).

The analytic 2-stack on the Betti side

Data for the Fukaya category: $(M, \omega, J, vol^{\otimes 2}, (U_i)_{i \in I}, B)$, where

- (M, ω) is a smooth connected symplectic manifold;
- J is an almost complex structure compactible with ω;
- $vol^{\otimes 2}$ is a trivialization of the square of the corresponding canonical bundle;
- $(U_i)_{i \in I}$ is an open covering of M;
- $B = (B_{ijk})$ is a Čech 2-cocycle on M with values in $U(1)$.

These data are parametrized by an analytic 2-stack $\mathcal{X}$ (for a $\mathbf{g}$=DGLA of symmetries one constructs Deligne's deformation 2-groupoid for $\mathbf{g} = \oplus_{n \geq -1} \mathbf{g}^n$). This does not require M to be a complex manifold.

Conjecture

There is an analytic sheaf $\mathcal{F}$ of triangulated A_∞-categories over a 2-substack $\mathcal{X}_{<1} \subset \mathcal{X}$ with global sections being the "universal" Fukaya category.

What is the origin of this substack?

Conjecture

There is a function $\rho : \mathcal{X} \to [0, +\infty)$ which satisfies the property $\rho(M, \lambda\omega, vol^{\otimes 2}, J, (U_i)_{i \in I}, B) = \lambda^{-1}\rho(M, \omega, vol^{\otimes 2}, J, (U_i)_{i \in I}, B)$, and ρ induces a continuous function with respect to the class $[\omega] \in H^2(M, \mathbb{R})$.

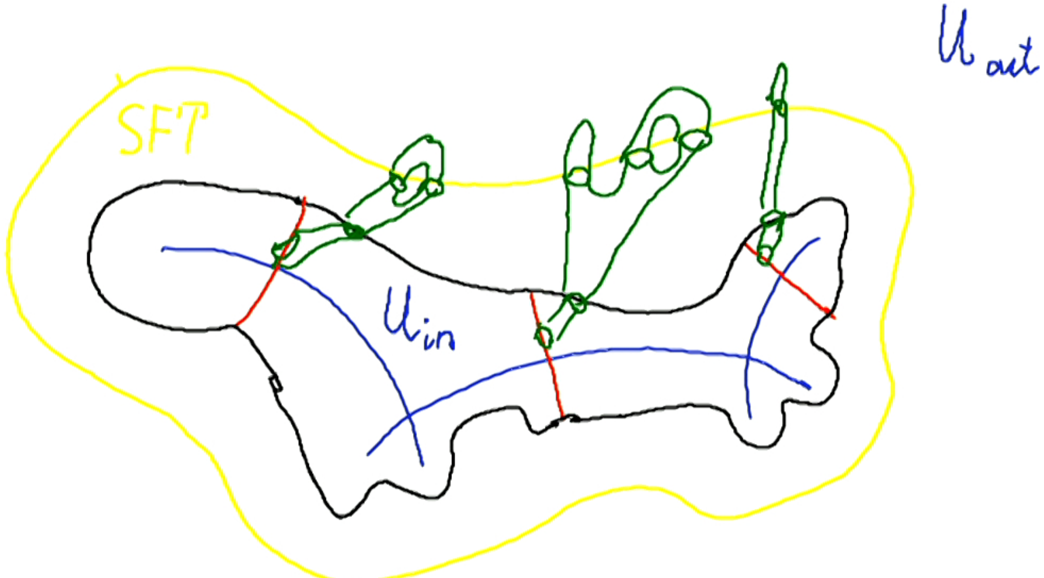
Morally, the number of stable J-holomorphic maps $\phi : \mathbb{CP}^1 \to (M, \omega, J)$ such that $\int_{\mathbb{CP}^1} \phi^*(\omega) \leq C$ is bounded from above by $e^{C(\rho + o(1))}$ for any $C > 0$. The substack $\mathcal{X}_{<1}$ is defined by the condition $\rho < 1$. The sheaf of categories $\mathcal{F}$ is enriched over $Perf(\mathcal{X}_{<1})$.

Comment on the function ρ

On can try to find more explicit bound on the value of ρ in the case of Calabi-Yau manifolds. It is expected that for hyper-Kähler manifolds and abelian varieties $\rho = 0$. The non-trivial answer is expected for the famous quintic. In general ρ should be given by the logarithm of the minimal radius of convergence of the periods of the mirror dual Calabi-Yau manifold.

- The 2-stack $\mathcal{X}$ admits a covering by simpler analytic 2-stacks $\mathcal{U}$ corresponding to tube domains of the form $Cone \times H^2(M, U(1))$, where $Cone \subset H^2(M, \mathbb{R})$ is an open convex cone (symplectic form +B-field).
- For each $\mathcal{U}$ there is a 2-substack $\mathcal{U}_{<1} \subset \mathcal{U}$ and the 2-stacks $\mathcal{U}_{<1}$ give rise to a covering of $\mathcal{X}_{<1}$.
- The pull-back of $\mathcal{F}$ to $\mathcal{U}_{<1}$ gives for each tube domain $Z = Cone \times H^2(M, U(1))$ a sheaf of categories enriched over $Perf(\mathcal{O}_Z^{an})$, where a $\mathcal{O}_Z^{an}$ is an appropriately defined sheaf of analytic functions on the tube domain.
- Can use this to formalize the analytic dependence on $[\omega]$. Then formal expansion of the analytic A_∞-data along an analytic path $\hbar \mapsto \omega/\hbar + O(\hbar)$ in $H^2(M, \mathbb{R})$ gives rise to the Novikov ring.
- Also need singular Lagrangian subvarieties as supports of objects This is done in C^∞ setting as a part of our formalism of 5 equations (separate story).

Figure illustrating formalism of 5 equations



Formalism of 5 equations

The analytic 2-stack on the de Rham side

The de Rham counterpart:

- For a holomorphic symplectic manifold $(M, \omega^{2,0})$ there is an analytic 2-stack $\mathcal{Y}$ parametrizing the quantization data for M (to define it one uses seminorms on the DGLA controlling the deformation quantization).
- $\mathcal{Y}$ can be mapped to the 2-stacks $\mathcal{U}_{<1}$ arising from tube domains as above.
- The triangulated A_∞-categories of holonomic modules over quantizations of M fit together into an analytic sheaf of A_∞-categories Hol over $\mathcal{Y}$.
- The global section category of the pull-back from $\mathcal{U}_{<1}$ to $\mathcal{Y}$ of the above-discussed sheaf $\mathcal{F}$ of Fukaya categories now can be compared with the global section category of Hol.
- The “universal" RH-correspondence claims an equivalence of the global section categories.

Let us discuss the RH-correspondence when M is a complex surface.

There are three types of surfaces which correspond to three types of equations:

- **a)** For differential equations: $M = T^*X$ endowed with the standard $\omega^{2,0}$. Here X is a curve (rational case).
- **b)** For q-difference equations: $M = (\mathbb{C}^*)^2, \omega^{2,0} = \frac{dx}{x} \wedge \frac{dy}{y}$ (trigonometric case);
- **c)** For elliptic difference equations: $M = E_\tau \times \mathbb{C}^*$, $\omega^{2,0} = d\alpha \wedge \frac{dx}{x}$. Here E_τ is a smooth elliptic curve and α is the nowhere vanishing holomorphic 1-form on E_τ (elliptic case).

RH-correspondence: equivalence of two categories
("de Rham side=Betti side"), both depending on a partial compactification
$M_{log} \supset M$ called the log extension of M.

The log extension of M is a Poisson variety obtained from a full Poisson compactification $\overline{M} \supset M$ such that $\overline{M} - M = \cup_{i \in I} D_i$ is a normal crossing divisor. In terms of $\overline{M}$ we have

$$M_{log} = \cup_{i \in I_{log} \subset I} D_i^0,$$

where $D_i^0 \subset D_i$ are open parts of those divisors where $\omega^{2,0}$ has pole of order 1.

Remark: This definition works in higher dimensions as well. In that case one needs the more complicated structure of the "symplectic variety with corners", which is unnecessary when $dim_{\mathbb{C}} M = 2$. At the categorical level it leads to the notion of Calabi-Yau category with corners (separate story).

What is M_{log} in the cases a)-c)?

In case **b)** take $\overline{M}$ to be any compact toric surface. Then M_{log} is obtained from $\overline{M}$ by deleting some toric divisors. Alternatively M_{log} is obtained from M by adding a union of $D_i^0 \simeq \mathbb{C}^*$.

The geometry **b)** is related to the Riemann-Hilbert correspondence for q-difference equations $f(qx) = A(x)f(x), x \in \mathbb{C}^*$, where $q \in \mathbb{C}^*$ is fixed.

In case **c)** take $\overline{M} = M_{log} = E_\tau \times \mathbb{CP}^1$. Then we have two log divisors, each isomorphic to an elliptic curve.

The geometry **c)** is related to the RH-correspondence for elliptic difference equations $f(x+h) = A(x)f(x), x \in E_\tau$, where $h \in E_\tau$ is fixed.

Note: There is another version of elliptic case which is related to Sklyanin algebras. Then $M = \mathbb{CP}^2 - \{\text{smooth cubic}\}$.

Log-extension of the cotangent bundle of a curve

The geometry of **a)** is related to the RH-correspondence for holonomic D-modules on X, e.g. bundles with possibly irregular meromoprphic flat connections. M_{log} depends on the type of irregular singularities for the holonomic D-modules being considered.

Let X be a complex projective curve, $S = \{x_1, ..., x_n\} \subset X$ a finite subset. A singular term at a point $x_i \in S$ is a Puiseux polynomial in negative powers with respect to a local parameter: $c_\alpha(x) = \sum_{\lambda \in \mathbb{Q}_{\leq 0}} c_{\alpha,\lambda}(x - x_i)^\lambda$ (modulo the action of Galois group). Each polynomial c_α has a multiplicity $m_\alpha \geq 1$.

Claim

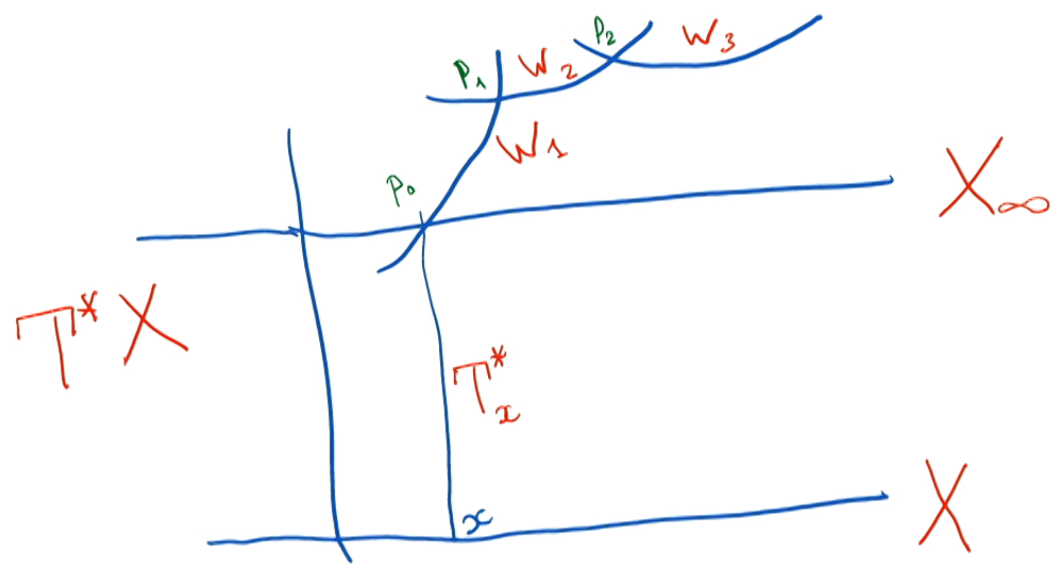
For a given choice of singular terms with given multiplicities (+some numerical conditions) there is a canonical embedding of the symplectic manifold $M = T^*X$ as an open symplectic leaf in a Poisson manifold M_{log} (log-extension of M).

Construction of the log-extension

- Consider the fiberwise projectivization $\overline{T^*X}$. The standard holomorphic symplectic form $\omega^{2,0} = \omega^{2,0}_{T^*X}$ has poles of order 2 at the divisor at infinity $X_\infty \simeq X$.
- Starting with the points $S \subset X = X_\infty \subset \overline{T^*X}$ perform a sequence of blow-ups $W_i = Bl_{p_i}(W_{i-1})$, $W_0 = \overline{T^*X}$ such that p_i is either a smooth point of a divisor in W_{i-1} at which the pull-back of $\omega^{2,0}$ has pole of order ≥ 2, or p_i is the intersection of two divisors where the pull-back of $\omega^{2,0}$ has pole of order ≥ 1.
- Keep only those divisors for which the pull-back of $\omega^{2,0}$ has poles of order one: the union of T^*X and these divisors is our M_{log}. Details in arXiv:1303.3253. Since $W_i - W_{i-1} \simeq \mathbb{C}$ we speak about rational case.

Figure: blow-ups

Thursday, October 29, 2015
9:02 AM



Data

- A finite set $S = \{x_i \in X, 1 \leq i \leq n\}$.
- A collections of singular terms $c_\alpha, \alpha \in J$ at these points, each term with multiplicity m_α.

This data give rise to $\mathbb{C}$-linear categories $Hol((c_\alpha)_{\alpha \in J})$ of holonomic D_X-modules with given singular terms $(c_\alpha)_{\alpha \in J}$ at S and $Conn((c_\alpha)_{\alpha \in J}) \subset Hol((c_\alpha)_{\alpha \in J})$-the subcategory of bundles with flat connections.

This is the de Rham side of the RH-correspondence. The RH-correspondence in this context was formulated and proven by Deligne and Malgrange in 80's. We are going to revisit their result and connect them to Fukaya categories.

We will discuss the Betti counterpart $\mathcal{F}(M)$ of the category $Conn((c_\alpha)_{\alpha \in J})$ which is a bit easier.

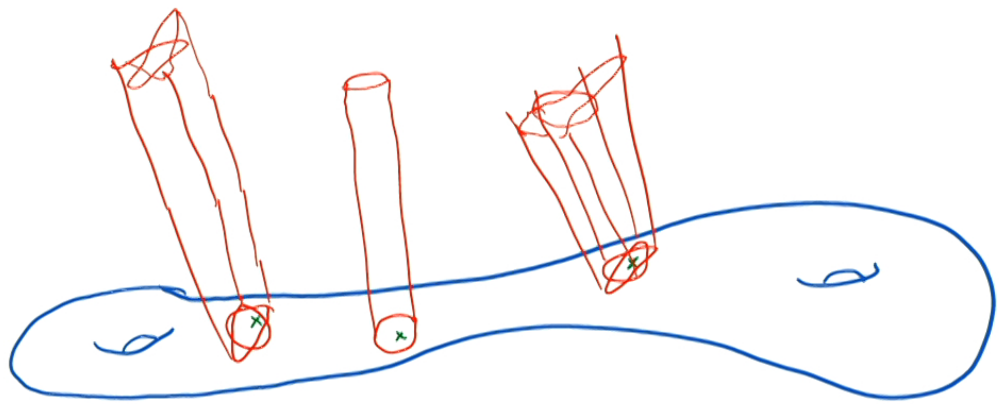
- $\mathcal{F}(M)$ is the category of finite-dimensional modules over certain partially wrapped Fukaya categories.
- The objects of $\mathcal{F}(M)$ are supported on singular Lagrangian submanifolds $L \subset M$ such that the closure $\overline{L} \subset M_{log}$ is compact, moreover $\overline{L} \cap (M_{log} - M)$ consists of finitely many points and L looks as cylinder $\mathbb{C}^* \simeq S^1 \times \mathbb{R}$ near the log-boundary divisors.
- The symplectic form is $\omega = Re(\omega^{2,0})$ and the B-field $B = Im(\omega^{2,0})$.
- The A_∞-structure comes from the formalism of 5 equations.

Note: The category $\mathcal{F}(M)$ admits a description in terms of constructible sheaves on X.

Betti side of the RH-correspondence

Conormal bundles

Thursday, October 29, 2015
5:24 PM



Comments on the RH-correspondence

The RH-correspondence of Deligne-Malgrange can be reformulated as an equivalence $Conn((c_\alpha)_{\alpha\in J}) \cong \mathcal{DM}((c_\alpha)_{\alpha\in J})$ where the Betti category is

$$\mathcal{DM}((c_\alpha)_{\alpha\in J}) := \left(\begin{array}{l} \text{Constructible sheaves } F \text{ on} \\ X - \cup_i(\text{small disc centered at } x_i), \text{ such that} \\ \qquad SS(F) \subset X \cup \cup_{i,\alpha} T^{*,+}_{S^1_\alpha}, \\ \text{where} \\ \qquad S^1_\alpha \ \text{ is a closed cooriented curve } \theta \mapsto e^{Re(c_\alpha/\hbar)} \cdot e^{i\theta} \text{ understood as the image of a very small circle around } x_i. \\ \qquad T^{*,+}_{S^1_\alpha} \subset M \ \text{ is the positive conormal bundle to } S^1_\alpha. \end{array} \right)$$

$\mathcal{DM}((c_\alpha)_{\alpha\in J}) \cong \mathcal{F}(M)$ via e.g. the work of Nadler, Shende, Treumann, Zaslow.

Holonomic modules over quantum tori

Comments on the trigonometric case **b)**:

- The quantum torus, which is the algebra A_q, $q \in \mathbb{C}^*$ with invertible generators x, y and relations $xy = qyx$. It is useful to think that $q = e^{\hbar}$.
- A_q is a deformation quantization of the standard affine symplectic torus $(\mathbb{C}^*)^2$.
- The analog of the cotangent bundle for A_q-modules is $(\mathbb{C}^*)^2$.
- Using the Bernstein-type filtration by the total degree of $x^i y^j$ and a log-extension via a choice of toric surface, we can define good filtrations of A_q-modules and the notion of a holonomic A_q-module.
- The theory of holonomic A_q-modules contains the theory of q-difference equations on $\mathbb{C}^*$.
- There is a local classification (HLT-type theorem) for holonomic A_q-modules due to Ramis, Saloy and Zhang.

Data

- A finite set $S = \{x_i \in X, 1 \leq i \leq n\}$.
- A collections of singular terms $c_\alpha, \alpha \in J$ at these points, each term with multiplicity m_α.

This data give rise to $\mathbb{C}$-linear categories $Hol((c_\alpha)_{\alpha \in J})$ of holonomic D_X-modules with given singular terms $(c_\alpha)_{\alpha \in J}$ at S and $Conn((c_\alpha)_{\alpha \in J}) \subset Hol((c_\alpha)_{\alpha \in J})$-the subcategory of bundles with flat connections.

This is the de Rham side of the RH-correspondence. The RH-correspondence in this context was formulated and proven by Deligne and Malgrange in 80's. We are going to revisit their result and connect them to Fukaya categories.

Let $\mathcal{M}$ be a holonomic A_q-module on $\mathbb{C}^*$ (physicists call such objects quantum spectral curves). The Betti counterpart of $\mathcal{M}$ is its sheaf of solutions $V = \mathrm{sol}(\mathcal{M})$. which can be defined similarly to the case of D-modules. By construction we have

- V is a coherent sheaf on the elliptic curve $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$.
- V comes equipped with two finite filtrations by coherent subsheaves labeled by numbers in $\mathbb{R} \cup \{\infty\}$, and with successive quotients which are semistable and with strictly increasing slopes.

Note: We call such filtrations anti-HN-filtrations since the slopes satisfy the opposite to the Harder-Narasimhan inequalities.

Theorem

The abelian category of holonomic A_q-modules on $\mathbb{C}^*$ is equivalent to the abelian category of coherent sheaves on the elliptic curve E_q, which are endowed with two anti-HN filtrations labeled by $\mathbb{Q} \cup \{\infty\}$.

These anti-HN filtrations are analogs of the Stokes data at $x = 0$ and $x = \infty$ in $\mathbb{C}^*$. The role of the singular terms c_α is now played by rational numbers which should be thought of as slopes of rays defining the fan for toric compactification.

Comments on the Theorem

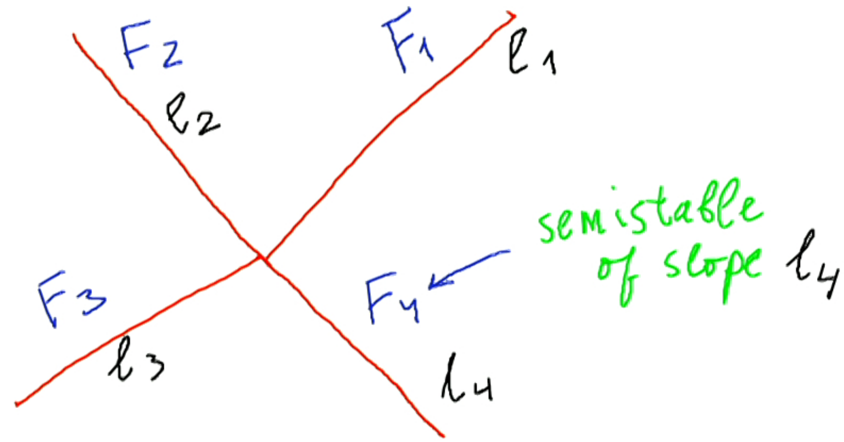
Suppose we are given a coherent sheaf F on $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$ endowed with two filtrations $(F^0_{\leq\lambda})$ and $(F^\infty_{\leq\lambda})$, where $\lambda \in \mathbb{Q} \cup \{\infty\}$ such that

$$0 \subsetneqq F^0_{\leq\lambda^0_{k_m}} \subsetneqq F^0_{\leq\lambda^0_{k_{m-1}}} \subsetneqq \ldots \subsetneqq F^0_{\leq\lambda^0_1} \subseteq F^0_{\leq\infty} = F,$$
$$0 \subsetneqq F^\infty_{\leq\lambda^\infty_{k_m}} \subsetneqq F^\infty_{\leq\lambda^\infty_{k_{m-1}}} \subsetneqq \ldots \subsetneqq F^\infty_{\leq\lambda^\infty_1} \subseteq F^\infty_{\leq\infty} = F,$$

such that

- the labels for the steps of the filtrations are rational numbers equal to the slopes of the successive quotients and are ordered as: $\lambda^0_1 > \ldots > \lambda^0_{k_m}$ (and similarly for the filtration at ∞);
- the quotient $F/F^0_{\leq\lambda^0_1}$ is a torsion sheaf on E_q (and similarly for the filtration at infinity).

Versions of the Betti side

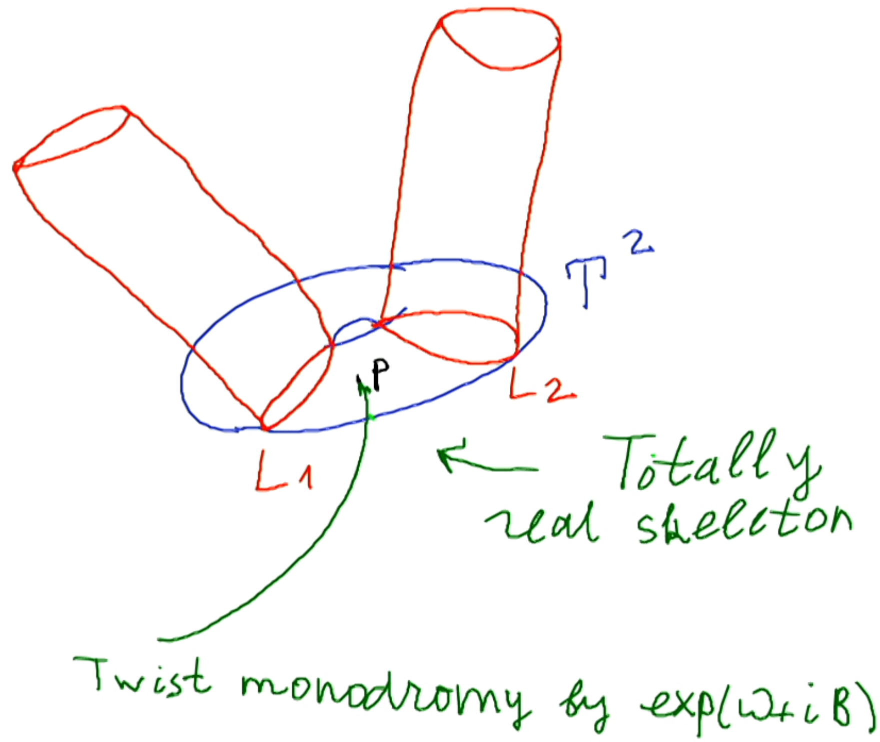


Objects as constructible sheaves over singular $L \subset \mathbb{R}^2$

Remarks

1) There is a generalization for $(\mathbb{C}^*)^{2n}, n > 1$. Then instead of rational rays we have rational Lagrangian fan of cones in $\mathbb{R}^{2n}$, with faces decorated by certain complexes of coherent sheaves on E_q^n.
2) A collection of rational rays l_i gives rise to a collection of Lagrangian submanifolds L_i (closed geodesics) on the standard symplectic torus T^2. The torus T^2 itself is a totally real submanifold in $((\mathbb{C}^*)^2, \omega = Re(\omega^{2,0})$. Then we get a totally real skeleton $L := T^2 \cup (\cup_i T^{*,+}_{L_i}) \subset (\mathbb{C}^*)^2$ which can be depicted similarly to the story with irregular connections on curves. We can consider a (twisted by $\omega + iB$, where $B = Im(\omega^{2,0})$) the derived category of constructible complexes F on T^2 with $SS(F) \subset L$ (twisting: $Mon_p = e^{2\pi i \int_{T^2}(\omega + iB)}$). Conjecturally it is equivalent to $D^b(Hol(A_q - mod))$.

Versions of the Betti side



Elliptic case

In the elliptic case on the de Rham side we have an elliptic curve E_τ endowed with two collections of points: $(p_1, ..., p_k)$ (poles) and $(q_1, ..., q_m)$ (zeros) as well as the shift automorphism $\sigma := \sigma_h : E_\tau \to E_\tau$. Holonomic elliptic module is given by a holomorphic vector bundle V together with a holomorphic isomorphism $A : \sigma^*(V)(\sum_i p_i) \simeq V(\sum_j q_j)$. Elliptic difference equations and some versions of the RH-correspondence were studied by E. Rains, The noncommutative geometry of elliptic difference equations arXiv:1607.08876.

I am going to explain the conjectural RH-correspondence which depends on a choice of *parabolic structure* on de Rham and Betti sides. I did not discuss parabolic versions of the RH-correspondence in rational and trigonometric cases, but they do exist. Also, using the notion of parabolic structure one can develop the non-abelian Hodge theory which generalizes the one of Simpson to our set-up.

Data:

- A smooth 3-dimensional manifold Y with an affine structure, which is isomorphic (as manifold with affine structure) to the standard 3-dimensional torus $(S^1)^3 \simeq \mathbb{R}^3/\mathbb{Z}^3$. We do not assume any orientation on Y.
- A finite subset of points $S = \{p_1, ..., p_N\} \subset Y$. We assume that a choice of orientation on Y gives rise to a splitting $S = S_+ \cup S_-$ (the subsets $S_\pm$ can intersect), and the change of the orientation to the opposite one leads to splitting where S_+ and S_- are interchanged.
A 1-dimensional foliation $\mathcal{F}$ of Y, which carries a transversal complex structure. We assume that $\mathcal{F}$ is locally constant in the affine coordinates on Y.

The data c) can be described by a complex rank 2 subbundle $\mathcal{F}_{\mathbb{C}} \subset T_Y \otimes \mathbb{C}$ or equivalently by a closed $\mathbb{C}$-valued locally constant 1-form κ on Y defined up to a factor from $\mathbb{C}^*$. Then $\mathcal{F} := \mathcal{F}_{\mathbb{R}} = \mathcal{F}_{\mathbb{C}} \cap T_Y$.

Let $y \in Y$ be an arbitrary point, $V := T_y Y \simeq \mathbb{R}^3$, $L := T_y \mathcal{F} \simeq \mathbb{R}$ denote the corresponding tangent spaces. Using the connection on T_Y which defines the affine structure, we can identify tangent spaces for different points y.
Clearly $V/L \simeq \mathbb{C}$. Thus to have the data c) is the same as to have a linear surjective map $V \to \mathbb{C}$ modulo the natural action of the group $\mathbb{C}^*$. It follows that the moduli space of the data c) is naturally isomorphic to $\mathbb{C}\mathbf{P}^2 - \mathbb{R}\mathbf{P}^2$.

We are going to associate with the above data an abelian $\mathbb{C}$-linear category $\mathcal{A} := \mathcal{A}(Y, \mathcal{F}, S_{or})$, where S_{or} means a splitting of S into the union S_+ and S_- for some choice of orientation of Y.

Objects of $\mathcal{A}$ are pairs $(\mathcal{E}, \nabla)$ where $\mathcal{E}$ is a C^∞ complex vector bundle on $Y - S$ and ∇ is a connection along $\mathcal{F}$ which is transversally holomorphic (i.e. the $(0,2)$-part of the curvature ∇^2 vanishes on the quotient $TY/T\mathcal{F}$). Equivalently, such a connection can be described as a $\mathbb{C}$-linear homomorphism at the level of sections of sheaves $\nabla : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E} \otimes_{\mathbb{C}^\infty} \mathcal{F}_\mathbb{C})$ such that $\nabla^2 = 0$ satisfying certain finiteness assumptions, endowed with decorations at the points p_i and satisfying certain semistability conditions.

In the end of the day objects of the category $\mathcal{A}$ can be identified with the 3-periodic monopoles on $\mathbb{R}^3$ with prescribed set of singularities depending on the points $(p_1, ..., p_N)$ (using e.g. B. Charbonneau, J. Hurtubise, Singular Hermitian-Einstein monopoles on the product of a circle and a Riemann surface, arXiv:0812.0221). Choice of a transversal elliptic curve converts objects into holonomic elliptic modules. The moduli stack of categories $\mathcal{A}$ is expected to be isomorphic to $(\mathbb{C}\mathbf{P}^2 - \mathbb{R}\mathbf{P}^2)/SL(3, \mathbb{Z})$.

One defines Fukaya category $\mathcal{F} := \mathcal{F}(M, \omega + iB)$, where
$M = E_\tau \times \mathbb{C}^*, \omega = Re(\omega^{2,0}), B = Im(\omega^{2,0})$.
In terms of wrapping Hamiltonians, the corresponding Hamiltonian vector fields extend to each of the log-divisors $D_\pm \simeq E_\tau$. In order to define the parabolic structure we need to fix two sets of marked points $p_{ij}^{\pm} \in D_\pm$. Then we require that the extended vector fields infinitesimally move each of the marked points $p_{ij}^{\pm}$. Then with two finite sets $S_\pm = \{(p_{ij}^{\pm}, exp(i\theta_{ij}^{\pm}))\} \subset E_\tau \times U(1) := Y \simeq T^3$ we associate the Fukaya category with parabolic structure denoted by $\mathcal{F}_{S_{or}}(M, \omega + iB)$, where $S = S_- \cup S_+$ (objects: Lagrangians which hit $D_\pm$ at $p_{ij}^{\pm}$ and monodromies of local systems are $exp(i\theta_{ij}^{\pm})$).

Conjecture

In the derived sense the category $\mathcal{F}_{S_{or}}(M, \omega + iB)$ is equivalent to the category $\mathcal{A}(Y, \mathcal{F}, S_{or})$ for a certain explicit choice of the 1-dimensional foliation on T^3.

Further problems

- Higher-dimensional case. There are conjectures (e.g. the one for q-difference equations) which in the end of the day should give alternative descriptions of the corresponding Fukaya categories on the Betti side. In categorical terms one speaks about "n-dimensional Calabi-Yau categories with corners".
- Non-abelian Hodge theory in dimension one. Role of harmonic objects is played by dimensional reductions of ASD equations in $4d$, in particular periodic monopoles (Charbonneau, Cherkis, Hurtubise, Mochizuki and many others). In order to compare with de Rham and Betti sides we have to introduce parabolic structures and stability.