Title: Gapped condensation in higher categories

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Series: Mathematical Physics

Date: March 17, 2020 - 12:30 PM

URL: http://pirsa.org/20030111

Abstract: Idempotent (aka Karoubi) completion is used throughout mathematics: for instance, it is a common step when building a Fukaya category. I will explain the n-category generalization of idempotent completion. We call it "condensation completion" because it answers the question of classifying the gapped phases of matter that can be reached from a given one by condensing some of the chemicals in the matter system. From the TFT side, condensation preserves full dualizability. In fact, if one starts with the n-category consisting purely of â„‚ in degree n, its condensation completion is equivalent both to the n-category of n-dualizable â„‚-linear (n-1)-categories and to an n-category of lattice condensed matter systems with commuting projector Hamiltonians. This establishes an equivalence between large families of TFTs and of gapped topological phases. Based on joint work with D. Gaiotto.

Zoom Link:  https://msri.zoom.us/j/226801541. Webinar ID: 226801541

CONDENSATION 1905.09566 jt w/ D. Gaiotto

"idempotent"

Defn: A split surjection (retract) in $\mathcal{C}$ of $X$ onto $Y$ is

$X \twoheadrightarrow Y$

$$\begin{tikzcd} X \arrow[d, bend right, "f"'] \\ Y \arrow[u, bend right, "g"'] \end{tikzcd}$$

s.t. $fg = id_Y$

Then

$$e = g\,f \qquad e^2 = e$$

Basic thm. $(X, e)$ s.t. $e \in End(X)$, $e^2 = e$

determines $(Y, f, g)$ uniquely (if it exists)

up to unique iso

Given some 1-category $\mathcal{C}$. Maybe not all idempotents $e$ extend to split surjections.

Kar($\mathcal{C}$) is built w/

some diagram in $\mathcal{C}$ which I want to act like the splitting of $e$

$$\begin{tikzcd} X \arrow[loop below, "e"] \end{tikzcd}$$

objects = idempotents in $\mathcal{C}$

$$\left( \begin{tikzcd} X' \arrow[loop below, "e'"] \end{tikzcd}, \begin{tikzcd} X \arrow[loop below, "e"] \end{tikzcd} \right)\mathrm{mod} = \left\{ \begin{tikzcd} X' \arrow[loop below, "e'"] & X \arrow[l, "m"'] \arrow[loop below, "e"] \end{tikzcd} \text{ s.t. } \begin{aligned} m &= me \\ &= e'm \end{aligned} \right\}$$

Main facts (Thm of Karoubi?)

(0) $\mathrm{Kar}(\mathcal{C})$ is a category.

(1) $\mathcal{C} \hookrightarrow \mathrm{Kar}(\mathcal{C})$ fully faithful $X \mapsto \left(X \circlearrowleft_{id}\right)$

(2) All idempotents in $\mathrm{Kar}(\mathcal{C})$ split.

(3) $\mathrm{Kar}(\mathcal{C})$ is universal for $(0,1,2)$.

(4) $\mathrm{Kar}(\mathcal{C})$ contains all absolute limits.

limit which can be checked with an equation.

Then $\circlearrowleft$ special Frob alg in $\mathrm{End}(X)$. NOT UNITAL.

$$e = g f \qquad e^2 \underset{\mu}{\overset{\Delta}{\rightleftarrows}} e$$

$\mu$ is assoc.
$\Delta$ is coassoc.
frobenius.

$e^2 = g(fg)f$

$e^3 = g(fg)(fg)f$ $\rightrightarrows e^2 \rightarrow e$

**Basic thm:** $(X, e)$ ← a cond monad

determines $(Y, f, g)$ uniquely (if it exists)

$$Y = \lim\left( \overset{X}{\circlearrowleft_e} \right) = \mathrm{colim}\left( \overset{X}{\circlearrowleft_e} \right)$$

up to unique iso

**Defn:** A condensation monad in $n$-cat is $\overset{X}{\circlearrowleft_e}$ w/ all higher data that would be true if $X \to Y$.

"1-object delooping"

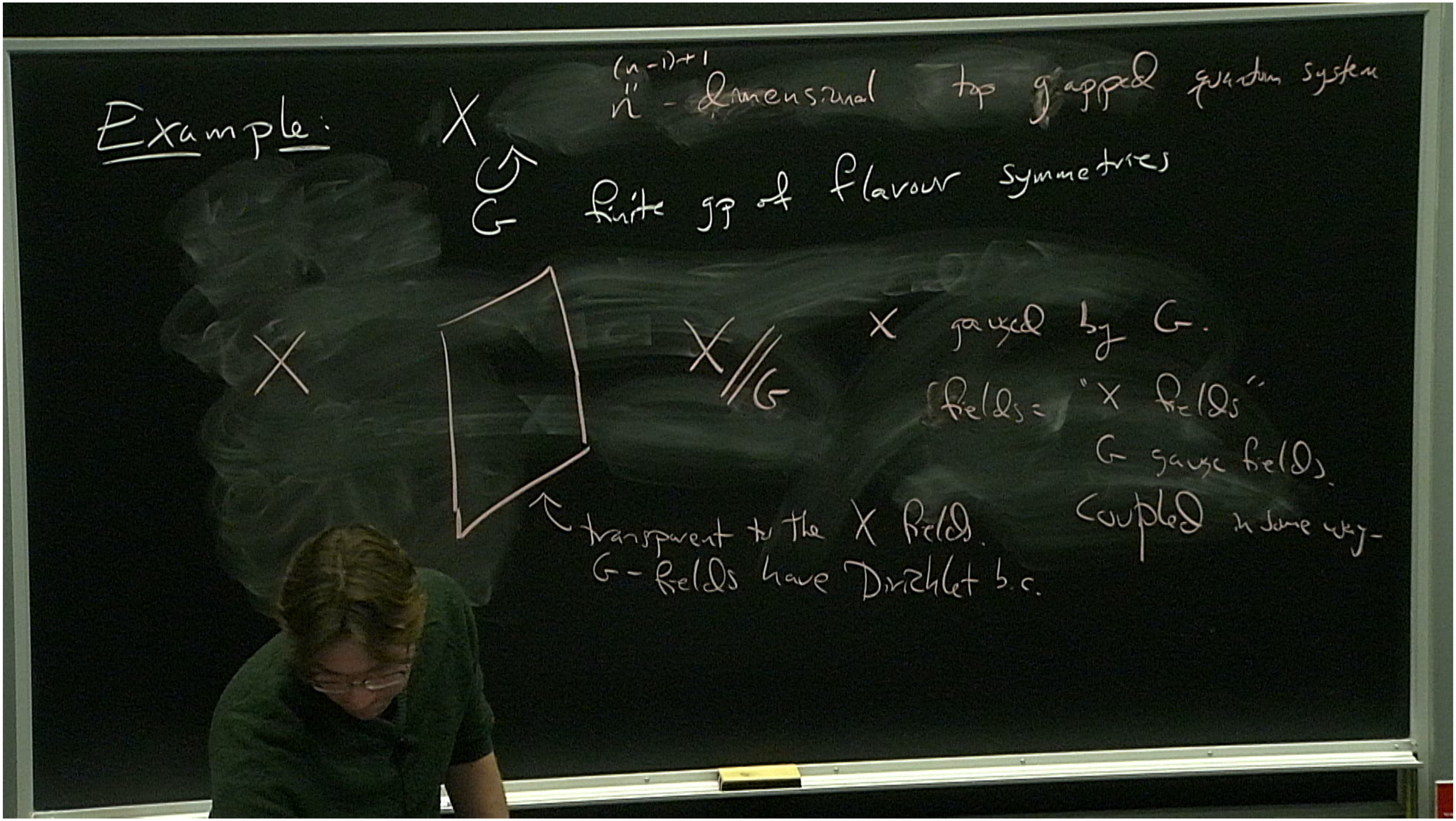
Main example: $R$ is a ring. $\rightsquigarrow$ $\mathcal{B}R$ category w/ obj $= \{\bullet\}$, $\mathrm{End}(\bullet) = R$.

$\rightsquigarrow \mathrm{Mat}(\mathcal{B}R)$ include $\oplus$s

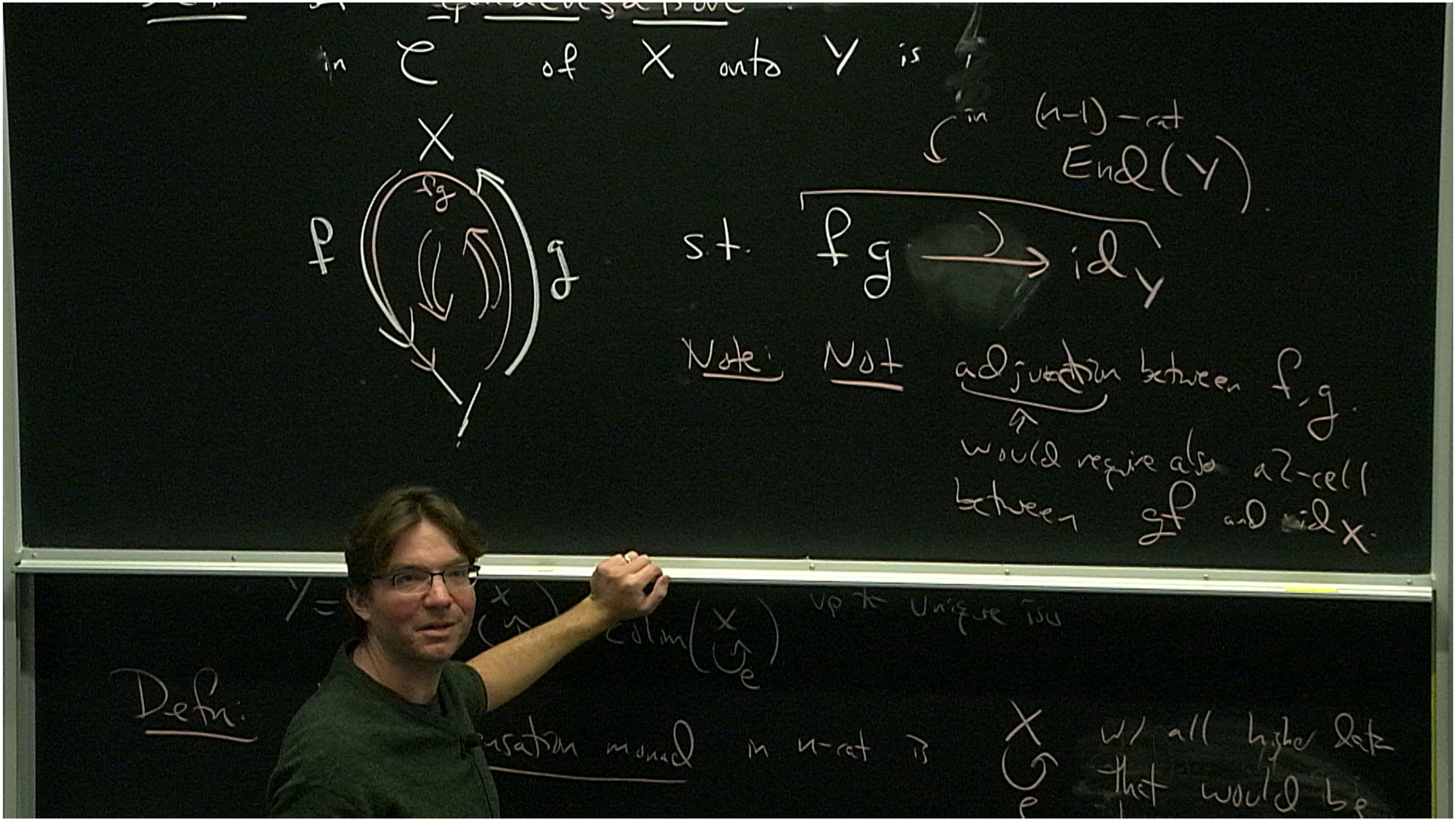
$\rightsquigarrow \mathrm{Kar}(\mathrm{Mat}(\mathcal{B}R)) = \Sigma R.$

$\Sigma R \simeq \{\text{compact projective } R\text{-modules}\} \subseteq \mathrm{Mod}(R).$

And If $R$ is com, then $\Sigma R \simeq \{\text{dualizable } R\text{-modules}\}.$



Example:
X
(n-1)+1
"n"-dimensional top gapped quantum system
G finite gp of flavour symmetries
X
X//G
X gauged by G.
fields = "X fields"
G gauge fields
coupled in some way.
transparent to the X fields.
G-fields have Dirichlet b.c.


in C of X onto Y is
in (n-1)-cat End(Y)
X
fg
f
g
s.t. fg → id_Y
Note: Not adjunction between f, g.
would require also a 2-cell between gf and id_X.
up to unique iso
colim
Def'n:
nsation monad in n-cat is
w/ all higher data that would be

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$\mathcal{C}$ is an n-category ("idempotent")

Defn: A condensation in $\mathcal{C}$ of $X$ onto $Y$ is

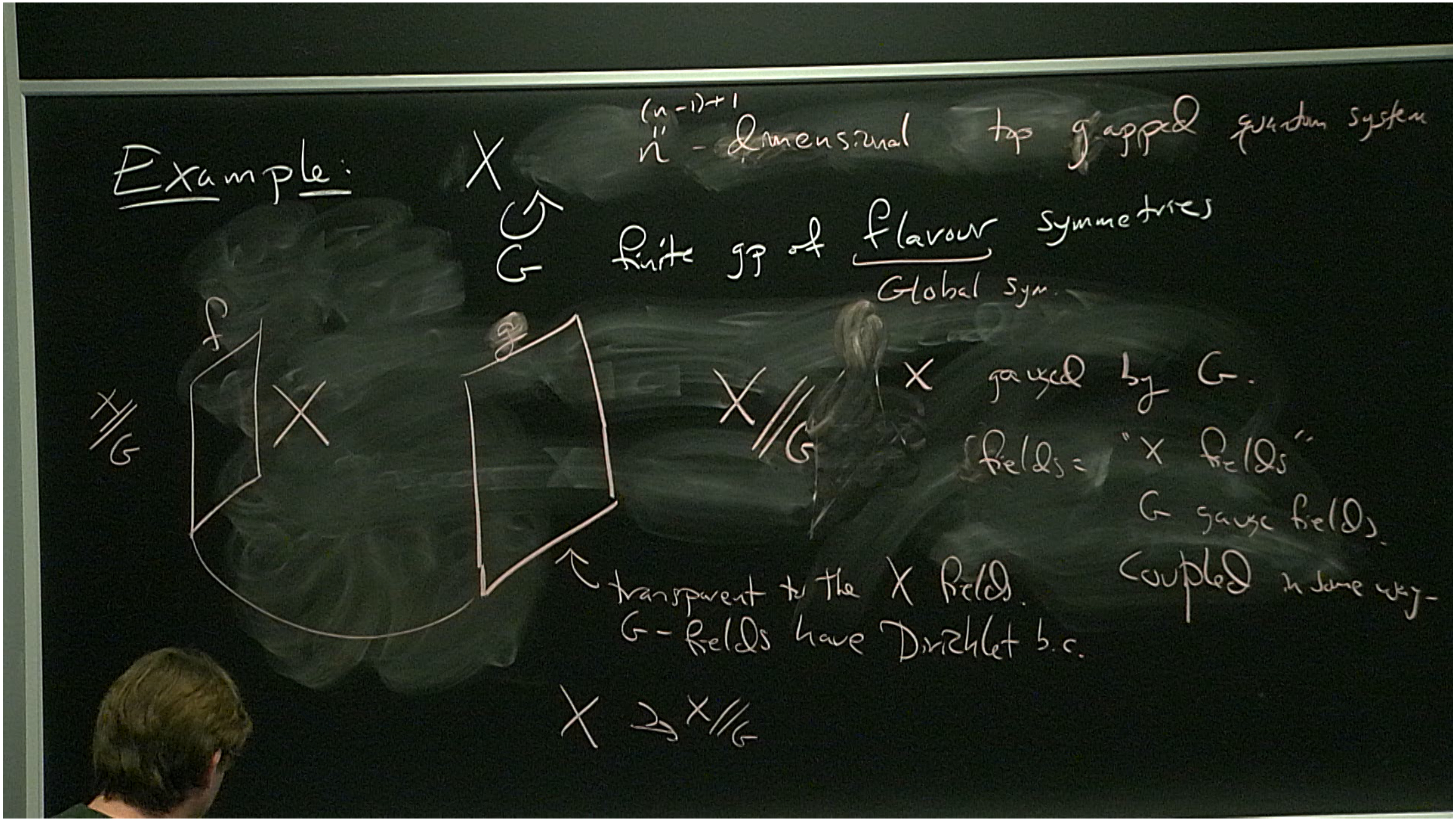
$$\boxed{X \twoheadrightarrow Y}$$

$$\begin{tikzcd} X \arrow[d, bend right, "f"'] \\ Y \arrow[u, bend right, "g"'] \end{tikzcd}$$

s.t. $\overbrace{fg \twoheadrightarrow id_Y}^{\text{in } (n-1)\text{-cat } End(Y)}$.

Note: Not adjunction between $f, g$.

would require also a 2-cell between $gf$ and $id_X$.


Example:
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(n-1)+1
n - dimensional top gapped quantum system
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Global sym.
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X ⇝ X//G

Example: $X$

"$n$" $^{(n-1)+1}$ - dimensional top gapped quantum system

new phase $Y$

$X$

produced by "physical cond".

original phase

gapped top

gapped top.

Example: X "$n$"-dimensional top gapped quantum system

$(n-1)+1$

X

original phase

gapped top

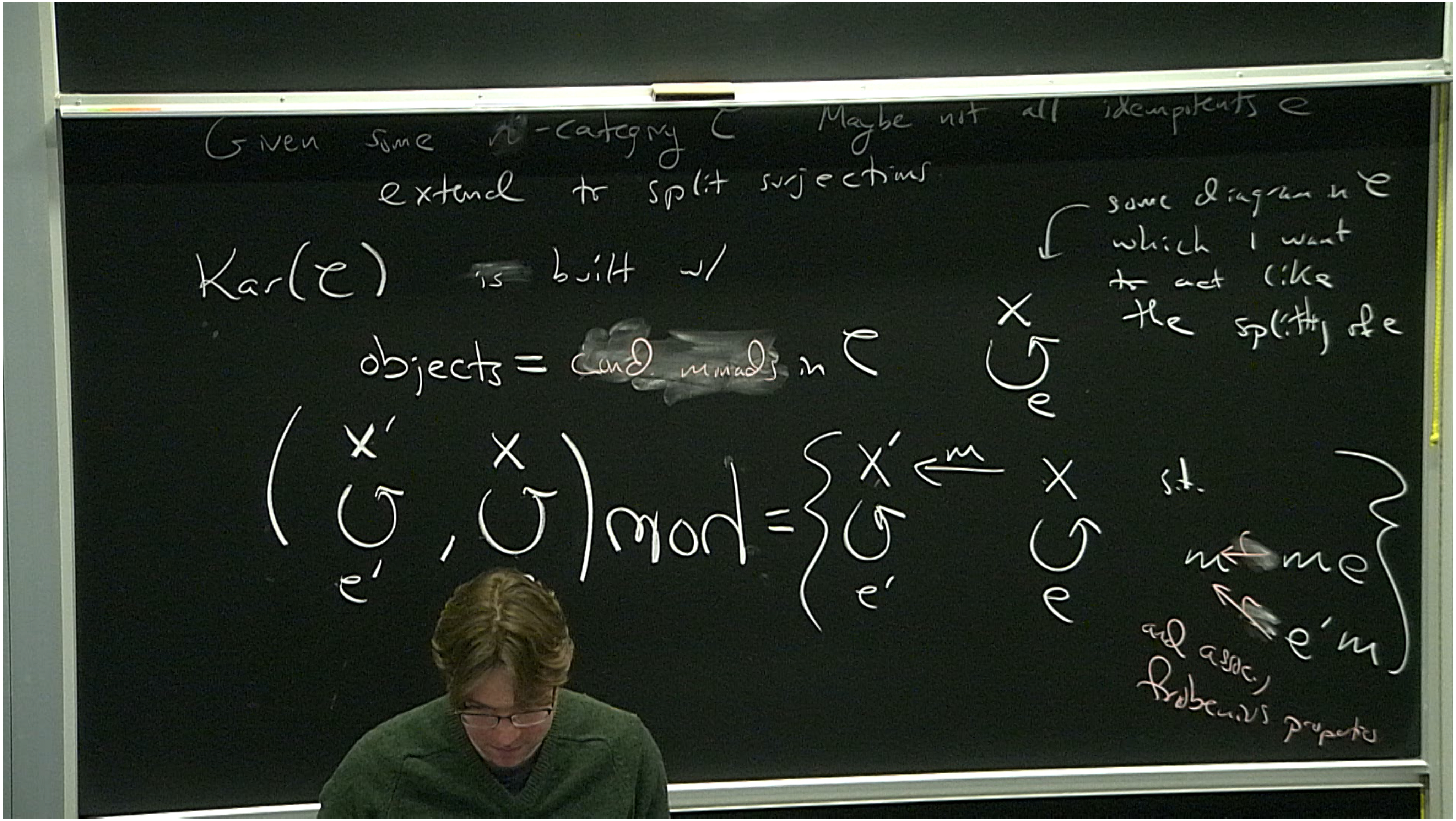
new phase Y

produced by "physical cond".

gapped top.

← might be gapped.

if so, $X \rightsquigarrow Y$.


Given some -category C
Maybe not all idempotents e extend to split surjections
Kar(C) is built w/
objects = cond. monads in C
some diagram in C which I want to act like the splitting of e
X
e
X'
e'
X
e
mod
X' ⇐m X
e'
e
s.t.
me
e'm
and assoc,
Frobenius properties

X

new phase Y

produced by "physical cond".

Y

original phase

gapped top

gapped top.

might be gapped.

if so, $X \rightsquigarrow Y$.

(all hom $(n-1)$-cats cond. complete

Given some $n$-category $\mathcal{C}$. Maybe not all idempotents $e$ extend to split surjections.

some diagram in $\mathcal{C}$

be included in Kar.

Main example:

"1-object delooping"

category w/ obj = $\{\bullet\}$,
$\mathrm{End}(\bullet) = R$.

$R$ is sym $\otimes$ n-cat $\rightsquigarrow BR$

~~is Kar-complete~~

$\rightsquigarrow \mathrm{Kar}\big((\mathrm{Mat})(BR)\big) = \Sigma R$

$(n+1)$-cat.

$$\Sigma R \simeq \left\{ \begin{array}{c} \text{1-dualizable} \\ R \text{ modules} \end{array} \right\} \subseteq \left\{ R\text{-module } n\text{-cats} \right\}$$

And: If $R$ has all duals
then $\Sigma R$ has all duals $= \{\text{f.d. } R\text{-modules}\}$