Title: Homology of the affine Grassmannian and quantum cohomologies

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Abstract: Let G be a complex reductive group, and X be any smooth projective G-variety. In this talk, we will construct an algebra homomorphism from the G-equivariant homology of the affine Grassmannian Gr_G to the G-equivariant quantum cohomology of X. The construction uses shift operators in quantum cohomologies. We will also discuss the possible extension to the loop rotation equivariant setting and the relation with the Peterson isomorphism when X is the flag variety associated with G. This is based on joint work with Alexander Braverman.

Homology of Gr_G and Quantum Cohomology.

j.w. A. Braverman.

1) Motivation.

$G \curvearrowright X/\mathbb{C}$ smooth proj.

- Construct an action of $H_*^{G(\mathcal{O})}(Gr_G)$ of $QH_G^*(X)$.
- Extension to the general Coulomb branch alg
- Relation with the Peterson Iso when $X = G/B$

n.

· Well known to the experts. (Givental ...)

Claimed by Teleman.

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2) Shift operators in quantum cohomology

$T^*(G/B)$ [BMO]

$\mathrm{Hilb}_n(\mathbb{C}^2)$ [OP]

Nakajima varieties [MO]

toric variety [Iritani]

$X/\mathbb{C}$, smooth proj. $H^*(X) \ni \gamma_1, \gamma_2, \gamma_3$

Gromov-Witten inv.

quantum product: $\star$

$$(\gamma_1 \star \gamma_2, \gamma_3) := (\gamma_1 \cup \gamma_2, \gamma_3) + \sum_{\beta \in H_2(X,\mathbb{Z})_{eff}} q^\beta \langle \gamma_1, \gamma_2, \gamma_3 \rangle^X_{0,3,\beta}$$

$QH^*(X) = (H^*(X)[[q^\beta \mid \beta \in H_2(X,\mathbb{Z})_{eff}, \star) =$ quantum coh. ring of X.

associative comm alg

quantum conn $H^2(X) \times H^*(X)$

$\{\phi_i\} \in H^2(X)$

$q^\beta(\phi_i) = q^{\langle \beta, \phi_i \rangle}$

$$\nabla_i := \frac{\partial}{\partial \phi_i} + \frac{1}{z} \phi_i \star$$

∇ is flat connection.

$$\frac{\partial}{\partial \phi_i}(q^\beta) = \langle \beta, \phi_i \rangle q^\beta$$

$G \circlearrowright X$. $T \subseteq G$ maximal torus.

$\left(\kappa : \mathbb{C}^* \to T \text{ a one-parameter subgroup}\right) \rightsquigarrow \left(\begin{array}{l}\text{shift operator } \mathbb{S}_\kappa \\ \text{acting on the q.c.}\end{array}\right)$.

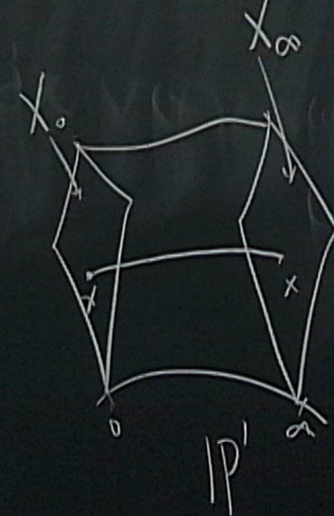
$$\left((\mathbb{C}^2 \setminus \{0\}) \times X\right) \big/ \mathbb{C}^* =: \widetilde{X}_\kappa$$

$$\downarrow$$

$$\mathbb{P}^1$$

$\mathbb{C}^* \circlearrowright \mathbb{C}^2 \setminus \{0\}$ diagonally

$\mathbb{C}^* \circlearrowright X$ via κ



$T \times \mathbb{C}^*_z \circlearrowright \widetilde{X}_\kappa$

$\uparrow$

scales the first coordinate in $\mathbb{C}^2 \setminus \{0\}$

$x \in X^T$,

$\sigma_x := [\mathbb{P}^1 \times \{x\}] \in H_2(\widetilde{X}_\kappa, \mathbb{Z})$

Claim: $\exists$ a component $F_{min} \subseteq X^{K(\mathbb{C}^*)}$, s.t.

$$H_2(\tilde{X}_k, \mathbb{Z})_{eff} = H_2(X, \mathbb{Z})_{eff} + [\sigma_{min}].$$

where $\sigma_{min} = \sigma_x$, for any $x \in F_{min}$

Def: $\mathbb{S}_X : H^*_{T\times\mathbb{C}^*_z}(X) \to QH^*_{T\times\mathbb{C}^*_z}(X)$

$$\gamma_1, \gamma_2 \in H^*_{T\times\mathbb{C}^*_z}$$

$$(\mathbb{S}_X(\gamma_1), \gamma_2) := \sum_{\beta\in \mathrm{Eff}(X)} q^\beta \langle \iota_{0*}\gamma_1, \iota_{\infty *}\gamma_2 \rangle^{\widetilde{X}_k}_{0,2,\beta+\sigma_{min}}$$

$\iota_0 : X$

$X/\mathbb{C}$, smooth proj.

quantum product: $\star$

$(\gamma_1 \star \gamma_2, \gamma_3) := ($

$QH^*(X) = (H^*(X)[[q^\beta]]$

quantum conn $H^2(X)\times H^*($

$\{\phi_i\} \in H^2(X)$

Claim: $\exists$ a component $F_{min} \subseteq X^{k(\mathbb{C}^*)}$, s.t.

$$H_2(\tilde{X}_k, \mathbb{Z})_{eff} = H_2(X, \mathbb{Z})_{eff} \oplus [\sigma_{min}].$$

where $\sigma_{min} = \sigma_x$, for any $x \in F_{min}$

].

<u>Def</u>: $\mathbb{S}_{\hbar}: H^*_{T\times\mathbb{C}^*_{\hbar}}(X) \longrightarrow QH^*_{T\times\mathbb{C}^*_{\hbar}}(X) = QH^*_T(X)[\hbar]$

$$\gamma_1 \mapsto \gamma_2$$

$$(\mathbb{S}_{\hbar}(\gamma_1), \gamma_2) := \sum_{\beta\in \mathrm{Eff}(X)} q^{\beta} \langle \iota_{0*}\gamma_1, \iota_{\infty *}\gamma_2 \rangle^{\widetilde{X}_{\hbar}}_{0,2,\beta+\sigma_{\min}}$$

$$\iota_0 : X_0 \simeq X \hookrightarrow \widetilde{X}_{\hbar}$$

$$\iota_\infty : X_\infty \simeq X \hookrightarrow \widetilde{X}_{\hbar}$$

[z] $\cdot\ S_X = \lim_{z \to 0} \mathbb{S}_X(1) \in QH_T^*(X)$

$\quad \parallel$

Seidel elements.

$\cdot\ \lim_{z \to 0} \mathbb{S}_X$ commutes with quantum multiplications.

$\rightsquigarrow \lim_{z \to 0} \mathbb{S}_X = S_X \star$

$(X) = QH^*_T(X)[z]$

$\widetilde{X}_k$

$, (\alpha_* \gamma_2 \rangle_{0,2,\beta+\sigma_{min}}$

- $S_x = \lim_{z \to 0} \mathbb{S}_x(1) \in QH^*_T(X)$

$\quad \|$

Seidel elements.

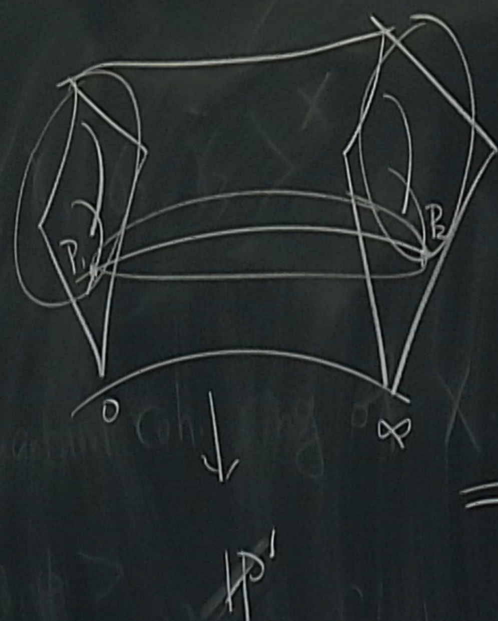
- $\lim_{z \to 0} \mathbb{S}_x$ commutes with quantum multiplications.

$\rightsquigarrow \lim_{z \to 0} \mathbb{S}_x = S_x \star .$

(X)

...antum multiplications.

Localization wrt $T \times \mathbb{C}^*_z$



$= \widetilde{X}_K$

$\Rightarrow S_K = M^{-1} \circ \Delta_K \circ M$

M = fundamental solution of q.d.e

Δ_K = diag. operator.

Properties:

- $[\nabla_\lambda, \mathbb{S}_\kappa] = 0$
- $\mathbb{S}_0 = id$
- $\mathbb{S}_\kappa \circ \mathbb{S}_l = q^{*} \, \mathbb{S}_{\kappa+l}$
- $w \in W$, $\mathbb{S}_{w\kappa} = w \circ \mathbb{S}_\kappa \circ w^{-1}$

$\uparrow$

Weil group action $QH_T^*(X)$

<u>Example</u>: $X = \mathbb{P}^{n-1} \circlearrowleft (\mathbb{C}^*)^n \xleftarrow{e_i} \mathbb{C}^*$

$D_i = \{z_i = 0\} \subseteq \mathbb{P}^{n-1}$

Seidel element $S_i = [D_i]$

$e_1 + e_2 + \cdots + e_n$ acts trivially on X.

$$q \, S_{e_1+e_2+\cdots+e_n} = S_{e_1} \star S_{e_2} \star \cdots \star S_{e_n}$$

$$\| \qquad\qquad\qquad\qquad\qquad$$

$$q = [D_1] \star [D_2] \star \cdots \star [D_n]$$

③ Renormalization.

character lattice.

$\exists$ a map $\mu : X^T \longrightarrow H_2(X,\mathbb{Z}) \otimes X^*(T)$

① Pick an equivariant lift $H^2(X) \rightsquigarrow H^2_T(X)$

$D \rightsquigarrow D^T$

② $p \in X^T$, $\mu(p)(D) := D^T|_p \in X^*(T)$

$p, q \in X^T$, $\mu(p) - \mu(q)$ is well-defined aracter. lattice.

G simply connected, s.s.

① Pick an G-equiv. lift:

$$H^2(X,\mathbb{Z}) \rightsquigarrow H^2_G(X,\mathbb{Z}).$$

Define: $\mathcal{S}'_k = q^{-\langle \mu(x),\, k \rangle} \mathcal{S}_k.$ ($k \in H_2(X,\mathbb{Z})$)

where $x \in F_{\min}(x)$.

Lemma: $\mathcal{S}'_k \circ \mathcal{S}'_l = \mathcal{S}'_{k+l}.$

S.

$\rightsquigarrow H^2_G(X, \mathbb{Z})$.

$H_2(X, \mathbb{Z})$

$\mathcal{D}_K$.

④ The map

$Gr_G = G((t))/G[[t]]$.

Thm (Bezrukavnikov – Finkelberg – Mirkovic).

$$H_*^{G(\mathcal{O})}(Gr_G) \simeq \left(\mathbb{C}[T^* T^\vee]\left[\frac{e^\alpha - 1}{\alpha}\right]\right)^W$$

ψ : $\mathbb{C}$

map.

$G((t))/G[[t]]$.

zrnkarnikov – Finkelberg – Mirkovic).

$(G_G) \simeq \left(\mathbb{C}[T^*T^\vee][\frac{e^{\alpha^\vee}-1}{\alpha}]\right)^W$

$\psi : \mathbb{C}[T^*T^\vee] = \mathbb{C}[T^\vee] \otimes \mathbb{C}[\mathfrak{t}] \to QH_T^*(X)$

$\mathbb{C}[\mathfrak{t}] \xrightarrow{\ \mathrm{id}\ } H_T^*(pt)$

$\mathbb{C}[T^\vee] \ni (\kappa : \mathbb{C}^* \to T) \longmapsto \lim_{z \to 0} S_\kappa'$

Thm ① ψ extends to an alg. homomorphism

$\mathbb{C}[T^*T^\vee][\frac{e^{\alpha^\vee}-1}{\alpha}] \to QH_T^*(X)$

② ψ is Weyl group inv.

(x)

· $(G, N\oplus N^*)$ Coulomb branch $\mathcal{M}(G,N)$

$$\mathbb{C}[\mathcal{M}(G,N)] \longleftrightarrow \mathbb{C}[\mathcal{M}(G,0)] = H_*^{G(\mathcal{O})}(Gr_G) \to QH_G^*(X).$$

$$X\times N \xrightarrow{\pi} X \rightsquigarrow \pi^*: QH^*_{G\times\mathbb{C}^*}(X) \longrightarrow QH^*_{G\times\mathbb{C}^*}(X\times N)_{loc} \text{ alg. homomorphism}$$

· $X = G/B$. Peterson isomorphism (Lam-Shimozono)

$$\begin{tikzcd}
H_*^{T(\mathcal{O})}(Gr_G) \arrow[rr, "\sim", "\text{[BFM]}"'] \arrow[dr, "\wr", "\text{Peterson iso}"'] & \arrow[d, phantom, "\circlearrowleft ?"] & \mathbb{C}[T^*T^\vee][\frac{e^{\alpha^\vee}-1}{\alpha}] \arrow[dl, "\text{limit of shift operators}"] \\
& QH_T^*(G/B)_{loc} &
\end{tikzcd}$$

$G = SL(2,\mathbb{C})$. ✓