Title: Categorified sheaf theory and the spectral Langlands TQFT

Speakers: German Stefanich

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Abstract: It is expected that the Betti version of the geometric Langlands program should ultimately be about the equivalence of two 4-dimensional topological field theories. In this talk I will give an overview of ongoing work in categorified sheaf theory and explain how one can use it to describe the categories of boundary conditions arising on the spectral side.

Categorified Sheaf Theory and the Spectral Langlands TQFT

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March 12, 2020
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Geometric Langlands via TQFT

Setup:

- $k = \overline{k}$, $\mathrm{char}(k) = 0$.
- C smooth proper curve
- G reductive group, $G^{\vee}$ Langlands dual group

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Geometric Langlands fits into an equivalence of 4d TQFT

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Question: What are these TQFT?

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Question: What are these TQFT? What are their 3-categories of boundary conditions?

TQFT via Linearization

Example

Let be a X set.

$$\mathcal{O}(X) = k^X \text{ vector space} \rightsquigarrow \text{1d TQFT } \chi$$

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Let X be a scheme.

$$\text{QCoh}(X) \text{ (dg) 1-category of quasicoherent sheaves}$$

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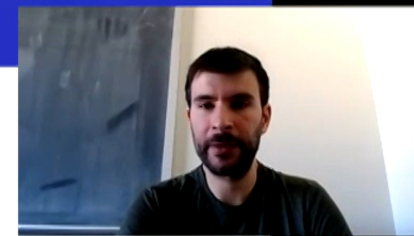
$$\text{Map}(S^1, X) = X \times_{X \times X} X = \text{Spec}(\mathcal{O}(X) \otimes_{\mathcal{O}(X) \otimes \mathcal{O}(X)} \mathcal{O}(X)) \text{ (for affine X)}$$

Sheaves of Categories



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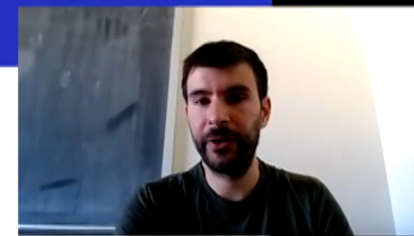
- Let $i : \{x\} \to X$ be a point. Then $\mathrm{QCoh}(X) \curvearrowright \mathrm{Vect}$ via $\mathcal{F} \cdot V = i^*\mathcal{F} \otimes V$ (skyscrapper sheaf).
- $\mathrm{QCoh}(X) \curvearrowright \mathrm{QCoh}(Y)$ for $Y \to X$ scheme over X.

The 3d B-Model

Definition

Let $2\,\mathrm{QCoh}(X) = \mathrm{QCoh}(X)\text{-mod}$ be the 2-category of sheaves of categories on X.

The 3d B-Model



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Let $2\,\mathrm{QCoh}(X) = \mathrm{QCoh}(X)\text{-mod}$ be the 2-category of sheaves of categories on X.

This serves as the boundary conditions of a 3d TQFT χ.

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This serves as the boundary conditions of a 3d TQFT χ.

$$\chi(S^1) = \mathrm{QCoh}(\mathrm{Map}(S^1, X))$$
$$\chi(\Sigma) = \mathcal{O}(\mathrm{Map}(\Sigma, X))$$

Going higher

Definition

Let X be a scheme and $n \geq 1$. Define $n\,\mathrm{QCoh}(X)$ to be the n-category of $(n-1)$-categories $\mathcal{C}$ acted on by $(n-1)\,\mathrm{QCoh}(X)$.

This serves as the boundary conditions for an $(n+1)$-dimensional TQFT χ.

$$\chi(M^j) = (n-j)\,\mathrm{QCoh}(\mathrm{Map}(M^j, X))$$

Back to geometric Langlands

Betti Geometric Langlands: $(k = \mathbb{C})$

Global$(\Sigma = C)$	$\mathrm{Sh}_{\mathcal{N}}(\mathrm{Bun}_G(C))$	$\approx$	$\mathrm{QCoh}(\mathrm{LocSys}^{b}_{G^\vee}(C))$
Local(S^1)	Automorphic 2-category	$\approx$	$2\,\mathrm{QCoh}(G^\vee/G^\vee)$
Fully local (pt)	Automorphic 3-category	$\approx$	$3\,\mathrm{QCoh}(BG^\vee)$

Theorem

Let $G = T$ be a torus. Then $3\,\mathrm{Loc}(BT) = 3\,\mathrm{QCoh}(BT^\vee)$.

Proof.

Use categorified Fourier theory:

$$(\mathrm{Loc}(\Lambda_T), *) = (\mathrm{QCoh}(BT^\vee), \otimes)$$
$$(2\,\mathrm{Loc}(T), *) = (2\,\mathrm{QCoh}(BT^\vee), \otimes)$$
$$(3\,\mathrm{Loc}(BT), *) = (3\,\mathrm{QCoh}(BT^\vee), \otimes)$$

□

What about other groups?

Problem:

$$\mathrm{Sh}_{\mathcal{N}}(\mathrm{Bun}_G(C)) \neq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{betti}}_{G^\vee}(C))$$

Arinkin-Gaitsgory: modify spectral side.
Better conjecture:

$$\mathrm{Sh}_{\mathcal{N}}(\mathrm{Bun}_G(C)) = \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}^{\mathrm{betti}}_{G^\vee}(C))$$

Want: $3\,\mathrm{IndCoh}_{\mathcal{N}}(BG^\vee)$ and $2\,\mathrm{IndCoh}_{\mathcal{N}}(G^\vee/G^\vee)$

What is IndCoh?

Definition

Let X be a finite type scheme.

$$\mathrm{IndCoh}(X) = \mathrm{Ind}\,(\,\mathrm{Coh}(X))$$

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Definition

Let X be a finite type scheme.

$$\mathrm{IndCoh}(X) = \underbrace{\mathrm{Ind}}_{\text{add colimits}} \big(\underbrace{\mathrm{Coh}(X)}_{D^b\,\mathrm{Coh}(X)} \big)$$

c.f:

$$\mathrm{QCoh}(X) = \mathrm{Ind}\big(\underbrace{\mathrm{Perf}(X)}_{\text{perfect complexes}} \big)$$

When X is singular have $\mathrm{QCoh}(X) \subsetneq \mathrm{IndCoh}(X)$.

What do we expect from nIndCoh?

Let $f : X \to Y$ be a map, and $\pi : X \to \mathrm{pt}$ the projection to the point.

$\mathrm{QCoh}(X)$	$\mathrm{IndCoh}(X)$
f^*, f_*	$f^!, f_*$
$\pi_*\pi^*(k) = \mathcal{O}(X)$	$\pi_*\pi^!(k) = \omega(X)$

One level up...

$2\,\mathrm{QCoh}(X)$	$2\,\mathrm{IndCoh}(X)$
f^*, f_*	$f^!, f_*$
$\pi_*\pi^*(\mathrm{Vect}) = \mathrm{QCoh}(X)$	$\pi_*\pi^!(\mathrm{Vect}) = \mathrm{IndCoh}(X)$

2IndCoh

Definition

Let X be a finite type scheme. Then $2\,\mathrm{IndCoh}(X)$ is the 2-category generated by

- Objects "$\mathrm{IndCoh}(Y)$" for Y scheme of finite type over X.
- Morphisms $\mathrm{Hom}(\text{"}\mathrm{IndCoh}(Y)\text{"}, \text{"}\mathrm{IndCoh}(Z)\text{"}) = \mathrm{IndCoh}(Y \times_X Z)$.

Suffices to consider Y proper over X and smooth over k. Therefore:

$$2\,\mathrm{IndCoh}(X) = \underbrace{\bigcup_{\substack{Y \text{ smooth over } k \\ \text{proper over } X}} \mathrm{IndCoh}(Y \times_X Y)\text{-mod}}_{\text{Arinkin-Gaitsgory}}$$

nIndCoh

Definition

Let X be a finite type scheme. Then $n\,\mathrm{IndCoh}(X)$ is the n-category generated by

- Objects "$(n-1)\,\mathrm{IndCoh}(Y)$" for Y scheme of finite type over X.
- Morphisms $\mathrm{Hom}($"$(n-1)\,\mathrm{IndCoh}(Y)$", "$(n-1)\,\mathrm{IndCoh}(Z)$"$) = (n-1)\,\mathrm{IndCoh}(Y \times_X Z)$.

"$n\,\mathrm{IndCoh}(X)$" $\in (n+1)\,\mathrm{IndCoh}(\mathrm{pt})$ serves as the boundary conditions for an $(n+1)$-dimensional TQFT χ.

$$\chi(M^j) = \text{“}(n-j)\,\mathrm{IndCoh}(\mathrm{Map}(M^j, X))\text{”}$$

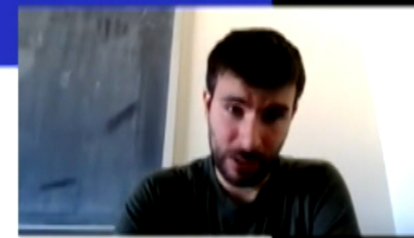
2IndCoh(X) vs Rozansky-Witten theory of T^*X

Let X be a smooth scheme. When $n = 2$ this yields a 3d TQFT χ such that

$$\chi(\mathrm{pt}) = \text{“}2\,\mathrm{IndCoh}(X)\text{”}$$
$$\chi(S^1) = \mathrm{IndCoh}(\mathrm{Map}(S^1, X)) = \mathrm{QCoh}(T^*X[2])$$
$$\chi(S^2) = \omega(\mathrm{Map}(S^2, X)) = \mathcal{O}(T^*X[2])$$

This is consistent with the Rozansky-Witten theory of T^*X.

nIndCoh as a categorified D-module



Want to define

$$3\,\mathrm{IndCoh}_{\mathcal{N}}(BG^\vee) \subset 3\,\mathrm{IndCoh}(BG^\vee)$$
$$2\,\mathrm{IndCoh}_{\mathcal{N}}(G^\vee/G^\vee) \subset 2\,\mathrm{IndCoh}(G^\vee/G^\vee)$$

Key: $n\,\mathrm{IndCoh}(X)$ is a categorified D-module on X. For nice enough X, this gives a sheafification of $n\,\mathrm{IndCoh}(X)$ over $T^*X[n-2]$.

2IndCoh as a categorified D-module

Let X be a smooth scheme. We have an action

$$2\,\mathrm{IndCoh}(X \times X) \curvearrowright 2\,\mathrm{IndCoh}(X)$$

Taking endomorphisms of the unit:

$$\mathrm{End}(\text{“}\mathrm{IndCoh}(X)\text{”}) = \mathrm{IndCoh}(X \times_{X \times X} X) = \mathrm{IndCoh}(\mathrm{Map}(S^1, X))$$

is an E_2-category acting on $2\,\mathrm{IndCoh}(X)$. Take endomorphisms of its units again:

$$\mathrm{End}(\omega_X) = \omega(X \times_{\mathrm{Map}(S^1,X)} X) = \omega(\mathrm{Map}(S^2, X))$$

is an E_3-algebra acting on $2\,\mathrm{IndCoh}(X)$.
Koszul duality:

$$\omega(\mathrm{Map}(S^2, X)) = \mathcal{O}(T^*X[2])^q$$

2IndCoh as a categorified D-module

Consequences:

- $\mathcal{O}(T^*X[2])^q$ is the algebra of local observables of "2 IndCoh(X)".
- 2 IndCoh(X) sheafifies over T^*X.

Example

Let Y be proper over X and smooth over k. Then "IndCoh(Y)" is supported on the image of the zero section of Y across the Lagrangian correspondence

$$T^*Y \leftarrow Y \times_X T^*X \rightarrow T^*X$$

- supp "IndCoh(X)" $= 0$
- supp "IndCoh($\{x\}$)" $= T^*_x X$ for $x \in X$

The local spectral 2-category

Let $X = G^\vee / G^\vee = \mathrm{Map}(S^1, G^\vee)$. For $\sigma \in X$, have

$$H^0(T^*_\sigma X) = H^0(S^1, \mathfrak{g}_\sigma) \supset \mathcal{N}_\sigma$$

Definition

$2\,\mathrm{IndCoh}_{\mathcal{N}}(G^\vee/G^\vee)$ is the subcategory of $2\,\mathrm{IndCoh}(G^\vee/G^\vee)$ of objects supported inside $\mathcal{N} = \bigcup_\sigma \mathcal{N}_\sigma$.

3IndCoh as a categorified D-module

Let $X = BG^\vee$. We have an action

$$\omega(\mathrm{Map}(S^3, X)) \curvearrowright 3\,\mathrm{IndCoh}(BG^\vee)$$

Koszul duality:

$$\begin{aligned}\omega(\mathrm{Map}(S^3, X)) &= \mathcal{O}(T^*BG^\vee[3])^q \\ &= \mathcal{O}(\mathfrak{g}[2]/G^\vee) \\ &= \mathcal{O}(\mathfrak{h}[2]//W)\end{aligned}$$

Consequences:

- $\mathcal{O}(\mathfrak{h}[2]/W)$ is the algebra of local observables of the theory with boundary conditions "$3\,\mathrm{IndCoh}(BG^\vee)$" (c.f. Elliott-Yoo)
- $3\,\mathrm{IndCoh}(BG^\vee)$ sheafifies over $\mathfrak{h}//W$.

Definition

$3\,\mathrm{IndCoh}_{\mathcal{N}}(BG^\vee)$ is the subcategory of $3\,\mathrm{IndCoh}(BG^\vee)$ consisting of objects supported at $0 \in \mathfrak{h}//W$.