Title: Affine Beilinson-Bernstein at the critical level for GL_2

Speakers: Sam Raskin

Series: Mathematical Physics

Date: March 05, 2020 - 11:00 AM

URL: http://pirsa.org/20030070

Abstract:

There has long been interest in Beilinson-Bernstein localization for the affine Grassmannian (or affine flag variety). First, Kashiwara-Tanisaki treated the so-called negative level case in the 90's. Some ten years later, Frenkel-Gaitsgory (following work of Beilinson-Drinfeld and Feigin-Frenkel) formulated a conjecture at the critical level and made some progress on it. Their conjecture is more subtle than its negative level counterpart, but also more satisfying. We will review the necessary background from representation theory of Kac-Moody algebras at critical level, formulate the Frenkel-Gaitsgory conjecture, and outline a proof for GL_2. Time permitting, we will discuss how our result provides a test of the Frenkel-Gaitsgory proposal for local geometric Langlands.

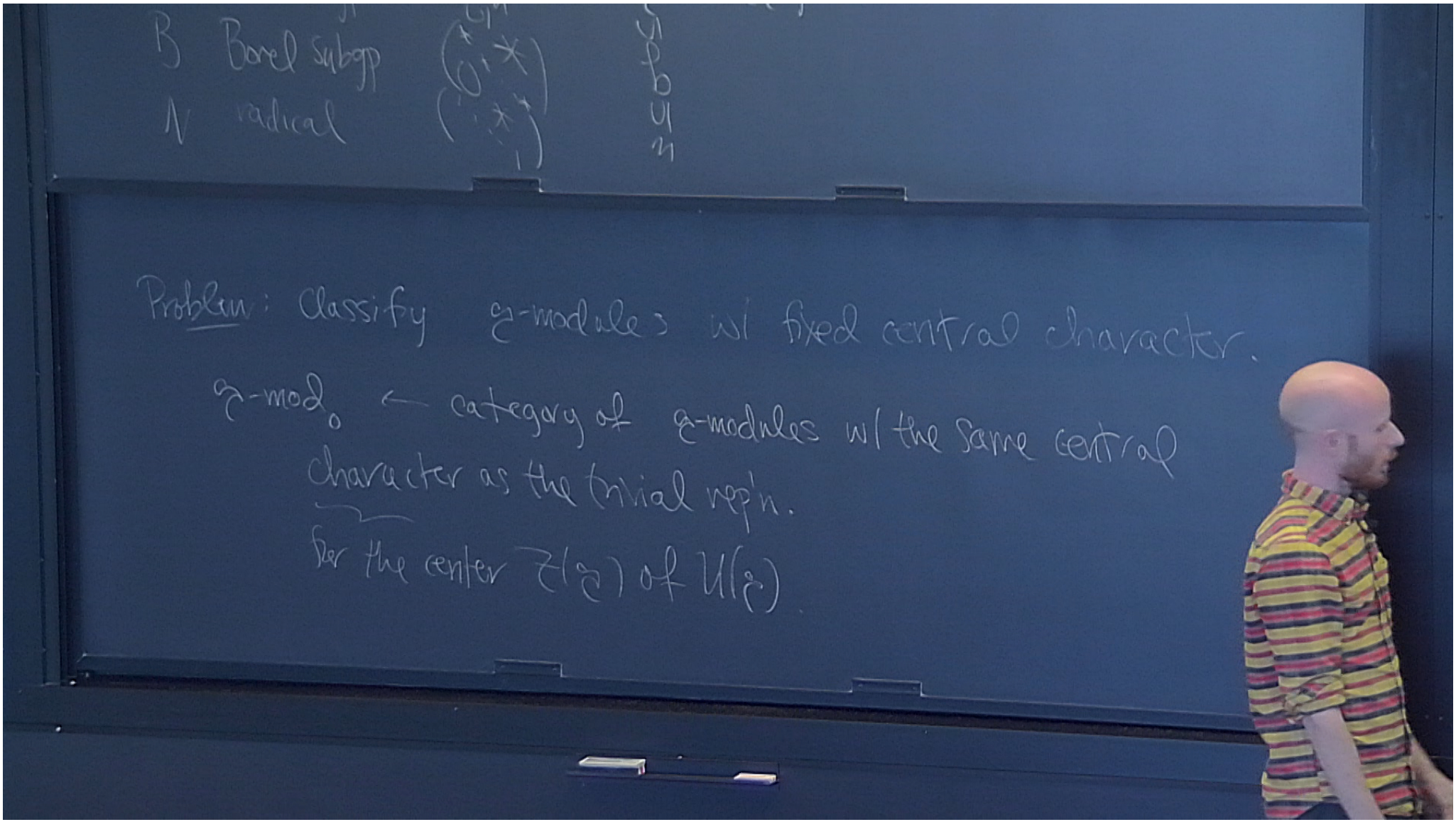
Affine Beilinson-Bernstein

Background on non-affine BB loc.

G reductive gp $\quad GL_n \quad \mathfrak{g} = \mathrm{Lie}(G) \quad M_n$ w/ bracket

B Borel subgp $\quad \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \quad \mathfrak{b}$

N radical $\quad \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \quad \mathfrak{n}$

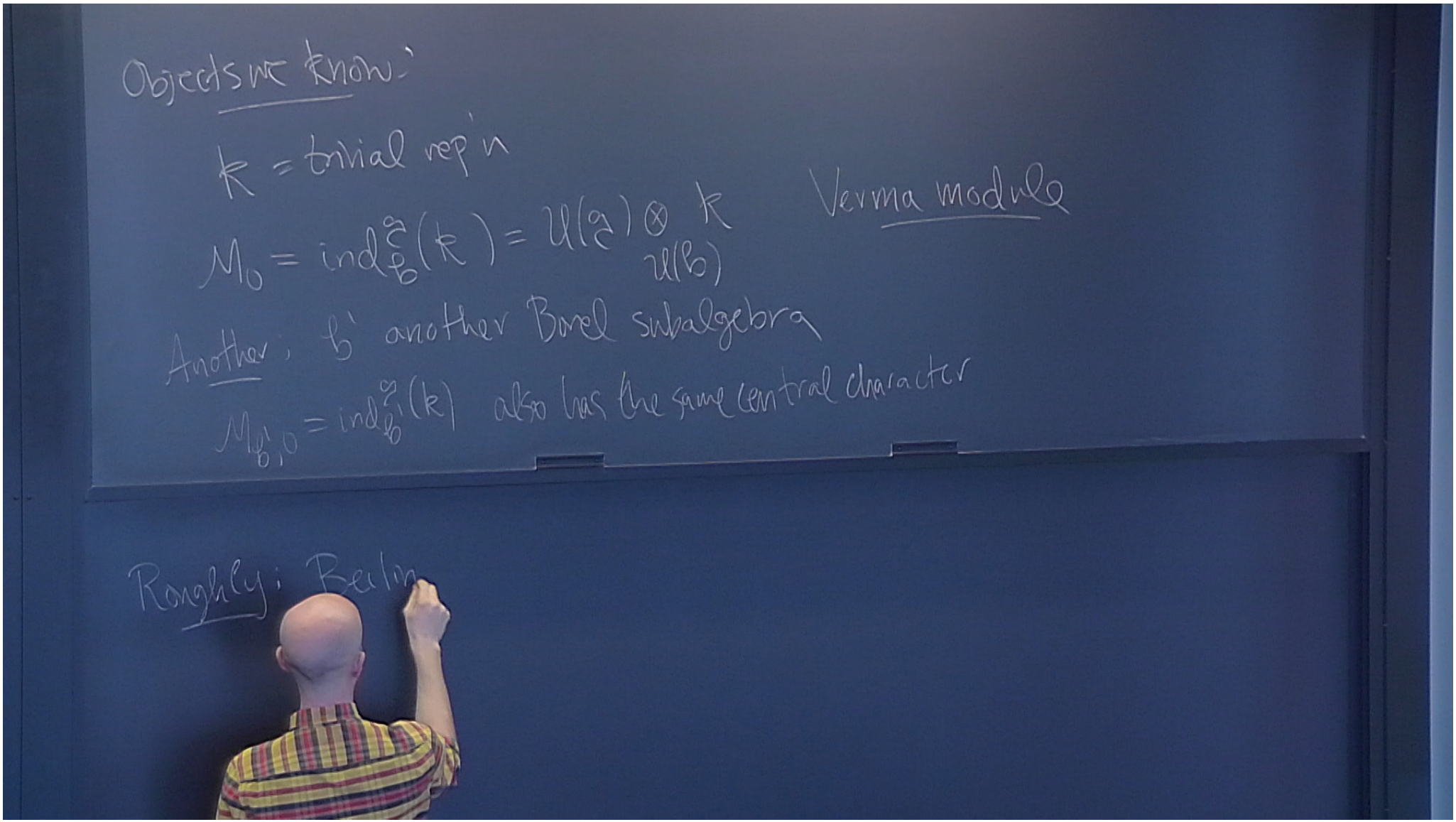

B Borel subgp
N radical
Problem: Classify $\mathfrak{g}$-modules w/ fixed central character.
$\mathfrak{g}$-mod$_0$ ← category of $\mathfrak{g}$-modules w/ the same central character as the trivial rep'n.
for the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$.

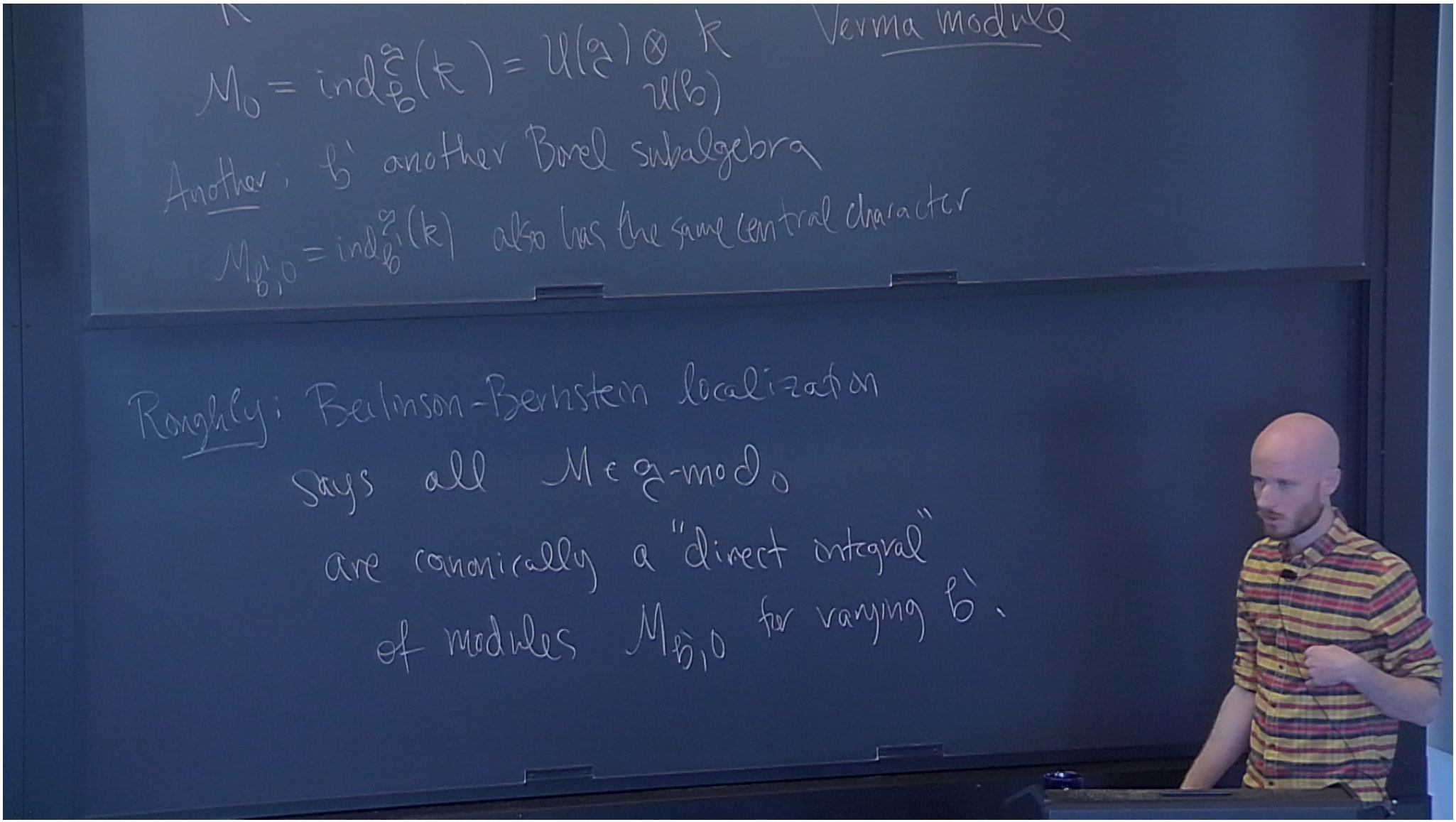
Objects we know:

k = trivial rep'n

$M_0 = \mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(k) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} k$ Verma module

Another: $\mathfrak{b}'$ another Borel subalgebra


Objects we know:
k = trivial rep'n
$M_0 = \mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(k) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} k$
Verma module
Another: $\mathfrak{b}'$ another Borel subalgebra
$M_{\mathfrak{b}',0} = \mathrm{ind}_{\mathfrak{b}'}^{\mathfrak{g}}(k)$ also has the same central character
Roughly: Berlin


Verma module
$M_0 = \mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(k) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} k$
Another: $\mathfrak{b}'$ another Borel subalgebra
$M_{\mathfrak{b}',0} = \mathrm{ind}_{\mathfrak{b}'}^{\mathfrak{g}}(k)$ also has the same central character
Roughly: Beilinson-Bernstein localization
says all $M \in \mathfrak{g}\text{-mod}_0$
are canonically a "direct integral"
of modules $M_{\mathfrak{b}',0}$ for varying $\mathfrak{b}'$.

for the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$.

More precisely:

G/B flag variety

$D(G/B)$ category of D-modules on G/B:

What's a D-module $\mathcal{F}$ on G/B?

$\forall\, \mathcal{U} \subseteq G/B$

I know $\Gamma(\mathcal{U}, \mathcal{F})$ actions of $\mathrm{Fun}(\mathcal{U})$ and $\mathrm{Vect}(\mathcal{U})$ compatible.

$D(G/B)$ category of D-modules on G/B

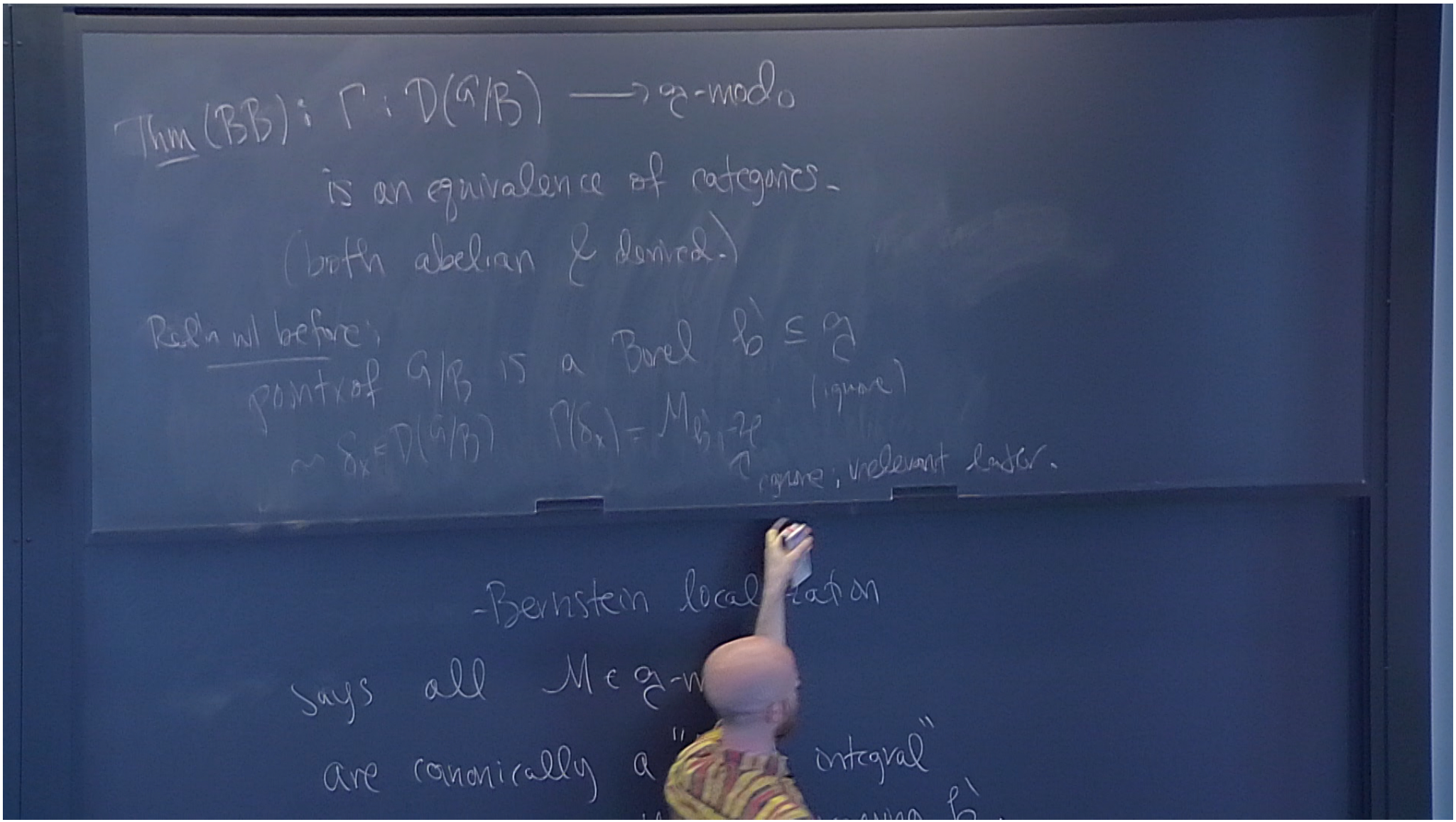
I know $\Gamma(\mathcal{U}, \mathcal{F})$
actions of $\mathrm{Fun}(\mathcal{U})$
and $\mathrm{Vect}(\mathcal{U})$ compatible.

$G \curvearrowright G/B$

$\rightsquigarrow \mathfrak{g} \longrightarrow \mathrm{Vect}(G/B)$

$\mathcal{F} \in D(G/B)$ $\quad \Gamma(G/B, \mathcal{F})$ has an action of $\mathfrak{g}$

<u>Check</u>: Same central char as the trivial rep'n.


Thm (BB): Γ : D(G/B) ⟶ g-mod₀
is an equivalence of categories.
(both abelian & derived.)
Rel'n w/ before:
point x of G/B is a Borel b' ⊆ g
(ignore)
⇝ δx ∈ D(G/B) Γ(δx) = M_{b',-ℋ}
ignore; irrelevant later.
-Bernstein local
says all M ∈ g-m
are canonically a "
integral"

Thm (BBD)

is an equivalence of categories.

(both abelian & derived.)

Rel'n w/ before:

point x of G/B is a Borel $\mathfrak{b}' \subseteq \mathfrak{g}$ (ignore)

$\rightsquigarrow \delta_x \in D(G/B)$ $\Gamma(\delta_x) = M_{\mathfrak{b}', -\rho}$

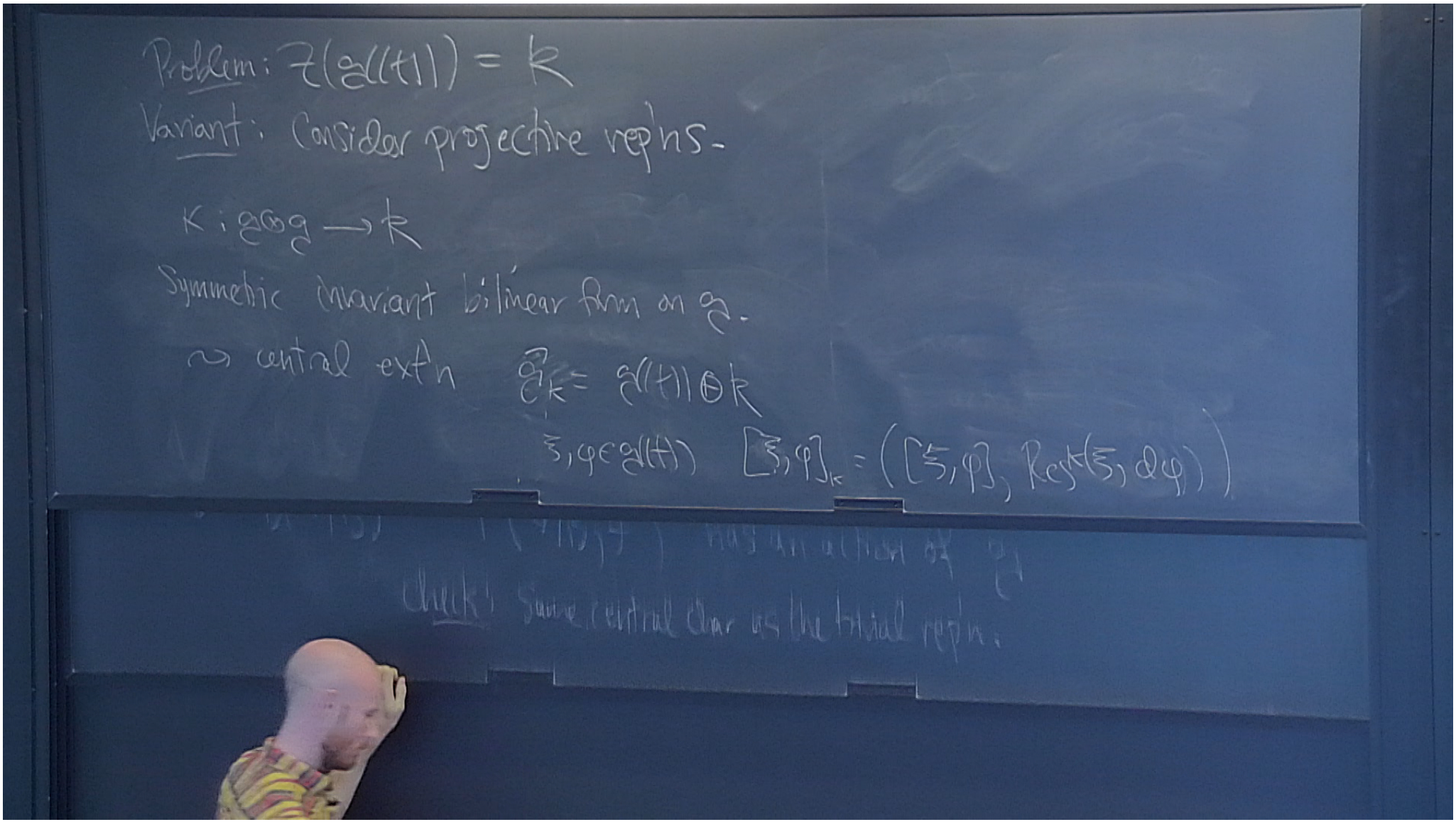
ignore; irrelevant later.

Problem: Find a version for KM algebras.

inf. dim'l Lie algebras.

Notation: $\mathfrak{g}((t)) := \mathfrak{g} \otimes k((t))$

$\mathfrak{g}[[t]] := \mathfrak{g} \otimes k[[t]]$.


Problem: $Z(\mathfrak{gl}((t))) = k$
Variant: Consider projective rep'ns.
$K : \mathfrak{g} \otimes \mathfrak{g} \to k$
Symmetric invariant bilinear form on $\mathfrak{g}$.
$\rightsquigarrow$ central ext'n $\hat{\mathfrak{g}}_K = \mathfrak{gl}((t)) \oplus k$
$\xi, \varphi \in \mathfrak{gl}((t))$ $[\xi, \varphi]_K = ([\xi, \varphi], \mathrm{Res}\, K(\xi, d\varphi))$
check: Same central char as the trivial rep'n.

Symmetric invariant bilinear form on $\mathfrak{g}$.

$\rightsquigarrow$ central ext'n $\hat{\mathfrak{g}}_k = \mathfrak{g}(t) \oplus k$

$\xi, \varphi \in \mathfrak{g}(t)$ $\quad [\xi, \varphi]_k = ([\xi, \varphi], \mathrm{Res}_{t=0}(\xi, d\varphi))$

Feigin-Frenkel theory of the center:

- $Z(U(\hat{\mathfrak{g}}_k)) = k$ if $\mathfrak{g}$ is simple and $k \neq \mathrm{crit} := -\frac{1}{2}$ Killing form.

Symmetric invariant bilinear form on $\mathfrak{g}$.

$\rightsquigarrow$ central ext'n $\hat{\mathfrak{g}}_k = \mathfrak{g}((t)) \oplus k$

$\xi, \varphi \in \mathfrak{g}((t))$ $\quad [\xi, \varphi]_k = \left([\xi, \varphi], \mathrm{Res}_{t=0}(\xi, d\varphi) \right)$

Feigin-Frenkel theory of the center:

- $Z(U(\hat{\mathfrak{g}}_k)) = k$ if $\mathfrak{g}$ is simple and $k \neq \mathrm{crit} := -\frac{1}{2}$ Killing form.
- Center is large for $k = \mathrm{crit}$.

Precisely: $\check{G}$ Langlands dual to G

Drinfeld-Sokolov introduced a space

$\mathrm{Op}_{\check{G}}$ "$\check{G}$-opers on $\mathring{D} = \mathrm{Spec}\, k((t))$".

1) These are $\check{G}$-bundles on $\mathring{D}$ w/ ∇ and extra structure.

e.g.: $\check{G} = GL_n$: data of a cyclic vector for my ∇.

Problem: Find a version for KM algebras.

[illegible] Lie algebras.

Drinfeld-Sokolov introduced a space

$Op_{\check{G}}$ "$\check{G}$-opers on $\mathring{D} = \operatorname{Spec} k((t))$".

1) These are $\check{G}$-bundles on $\mathring{D}$ w/ ∇ and extra structure.

E.g.: $\check{G} = GL_n$: data of a cyclic vector for my ∇.

2) $\check{G} = SL_2$

$$Op_{\check{G}} = \left\{ \nabla = d + \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix} dt \;\middle|\; f \in k((t)) \right\}.$$

3) Form an ∞-dim'l affine space.

Thm (FF):

$$\mathrm{Fun}(\mathrm{Op}_{\check{\mathfrak{g}}}) \simeq Z(\mathcal{U}(\hat{\mathfrak{g}}_{crit}))$$

canonically.

$$([\xi, \varphi], \mathrm{Res}^k(\xi, d\varphi))$$

n-Frenkel theory of the center:

$$Z(\mathcal{U}(\hat{\mathfrak{g}}_k)) = k \text{ if } \mathfrak{g} \text{ is simple}$$

$\mathrm{Fun}(\mathrm{Op}_{\check{G}}) \cong Z(U(\hat{\mathfrak{g}}_{crit}))$

canonically.

$([\xi, \varphi], \mathrm{Res}^*(\xi, d\varphi))$

Construction: $\mathbb{V}_{crit} = \mathrm{ind}_{\mathfrak{g}[[t]]}^{\hat{\mathfrak{g}}_{crit}}(k)$.

Facts. $Z(\hat{\mathfrak{g}}_{crit}) \xrightarrow{\text{surjective}} \mathrm{End}_{\hat{\mathfrak{g}}_{crit}\text{-mod}^{\heartsuit}}(\mathbb{V}_{crit})$

FF $\wr$ $\qquad\qquad\qquad\qquad \wr$

$\mathrm{Fun}(\mathrm{Op}_{\check{G}}) \twoheadrightarrow \mathrm{Fun}(\mathrm{Op}_{\check{G}}^{reg})$

4) $\mathrm{Op}^{reg}_{\check{G}} := \check{G}$-opers on $D = \mathrm{Spec}\, k[[t]]$

$\check{G} = SL_2$ $\left\{ \nabla = d + \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix} dt \;\middle|\; f \in k[[t]] \right\}$

$\mathrm{Op}^{reg}_{\check{G}} \subseteq \mathrm{Op}_{\check{G}}$.

$\check{G} = SL_2 \quad \left\{ \nabla = d + \begin{pmatrix} & \\ 1 & 0 \end{pmatrix} dt \;\middle|\; f \in K[[t]] \right\}$

$Op^{reg}_{\check{G}} \subseteq Op_{\check{G}}$.

$\hat{\mathfrak{g}}_{crit}$-$mod_{reg}$ is the category of $\hat{\mathfrak{g}}_{crit}$-modules

w/ $Z(\hat{\mathfrak{g}}_{crit})$ acts via the quotient

$Fun(Op^{reg}_{\check{G}})$

<u>Problem</u>: Classify modules w/ regular central character.

(new)

Conj (Frenkel-Gaitsgory): Every such module is a direct integral of conjugated vacuum rep'ns.

Thm (R.): True for $G = GL_2$.

Thm (R.): True for $G = GL_2$.

In more detail:

$G((t)) \leftarrow$ loop gp of G

$\cup\!|$

$G[[t]] \leftarrow$ arc subgp

think: invertible matrices w/ Laurent entries

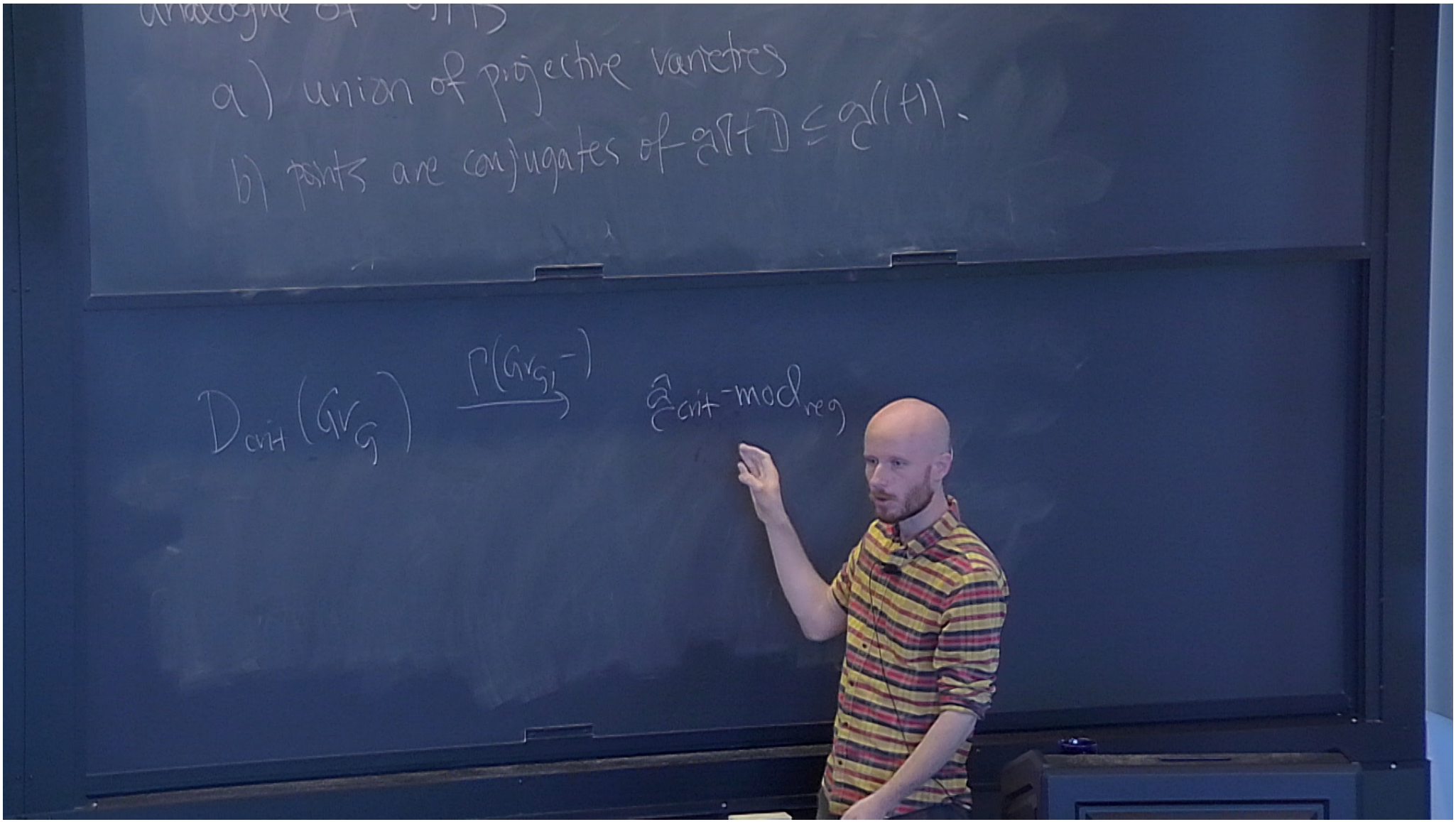
Laurent coeffs $\rightsquigarrow$ ∞-dim'l "manifold"

$Gr_G = G((t))/G[[t]]$

analogue of G/B:

a) union of projective varieties

b) points are conjugates of $\mathfrak{g}[[t]] \leq \mathfrak{g}((t))$.


a) union of projective varieties
b) points are conjugates of $\mathfrak{g}[[t]] \subseteq \mathfrak{g}((t))$.
$D_{crit}(Gr_G) \xrightarrow{\Gamma(Gr_G, -)} \hat{\mathfrak{g}}_{crit}\text{-mod}_{reg}$

analogue of ...

a) union of projective varieties

b) points are conjugates of $\mathfrak{g}[[t]] \subseteq \mathfrak{g}((t))$.

Conj: (FG)

$$D_{crit}(Gr_G) \otimes_{\mathrm{Rep}\,\check{G}} \mathrm{QCoh}(\mathrm{Op}_{\check{G}}^{reg}) \xrightarrow{\Gamma(Gr_G, -)} \hat{\mathfrak{g}}_{crit}\text{-mod}_{reg}$$

is an equivalence (t-exact).

Rem: 1) $\mathrm{Rep}\,\check{G} \xrightarrow{\text{geom Satake}} D_{crit}(Gr_G)^{G(O)} \curvearrowright D_{crit}(Gr_G) \cap G(K)$

$G[[t]] \leftarrow$ arc subgp

w/ Laurent entries

Laurent coeffs $\rightsquigarrow$ ∞-dim'l "manifold"

b) $\mathrm{Op}^{reg}_{\check{G}} \xrightarrow{\text{forgetting structure}} \mathrm{LocSys}_{\check{G}}(D) = B\check{G}$

$\mathrm{Rep}\,\check{G} \longrightarrow \mathrm{QCoh}(\mathrm{Op}^{reg}_{\check{G}})$

sym. mon. functor.

$G[[t]] \leftarrow$ loop gp of G

$\cup\!\!\mid$

$G[[t]] \leftarrow$ arc subgp

w/ Laurent entries

Laurent coeffs $\rightsquigarrow$ ∞-dim'l "manifold"

b) $\mathrm{Op}^{reg}_{\check{G}} \xrightarrow{\text{forgetting structure}} \mathrm{LocSys}_{\check{G}}(D) = B\check{G}$

$\mathrm{Rep}\,\check{G} \longrightarrow \mathrm{QCoh}(\mathrm{Op}^{reg}_{\check{G}})$

sym. mon. functor. (using universal $\check{G}$-bundle on $\mathrm{Op}^{reg}_{\check{G}}$)

sym. mon. functor. (using universal $\check{G}$-bundle on $Op_{\check{G}}^{reg}$)

c) Deep thm of Beilinson-Drinfeld

that the functor I wrote makes sense.

analogue of ...

a) union of projective varieties

b) points are conjugates of $\mathfrak{g}[[t]] \subseteq \mathfrak{g}((t))$.

Conj (FG)

$$D_{crit}(Gr_G) \otimes_{\mathrm{Rep}\,\check{G}} \mathrm{QCoh}(\mathrm{Op}^{reg}_{\check{G}}) \xrightarrow{\Gamma(Gr_G, -)} \hat{\mathfrak{g}}_{crit}\text{-mod}_{reg}$$

is an equivalence (t-exact).

Rem: 1) $\mathrm{Rep}\,\check{G} \xrightarrow{\text{geom Satake}} D_{crit}(Gr_G)^{G(O)} \curvearrowright D_{crit}(Gr_G) \cap G(K)$

sym. mon. functor. (using universal $\check{G}$-bundle on $Op^{reg}_{\check{G}}$)

c) Deep thm of Beilinson-Drinfeld

that the functor I wrote makes sense.

<u>Thm</u> (R.): Conjecture is true for GL_2 (or SL_2 or ...)

Includes t-exactness.

Conj: (FG)

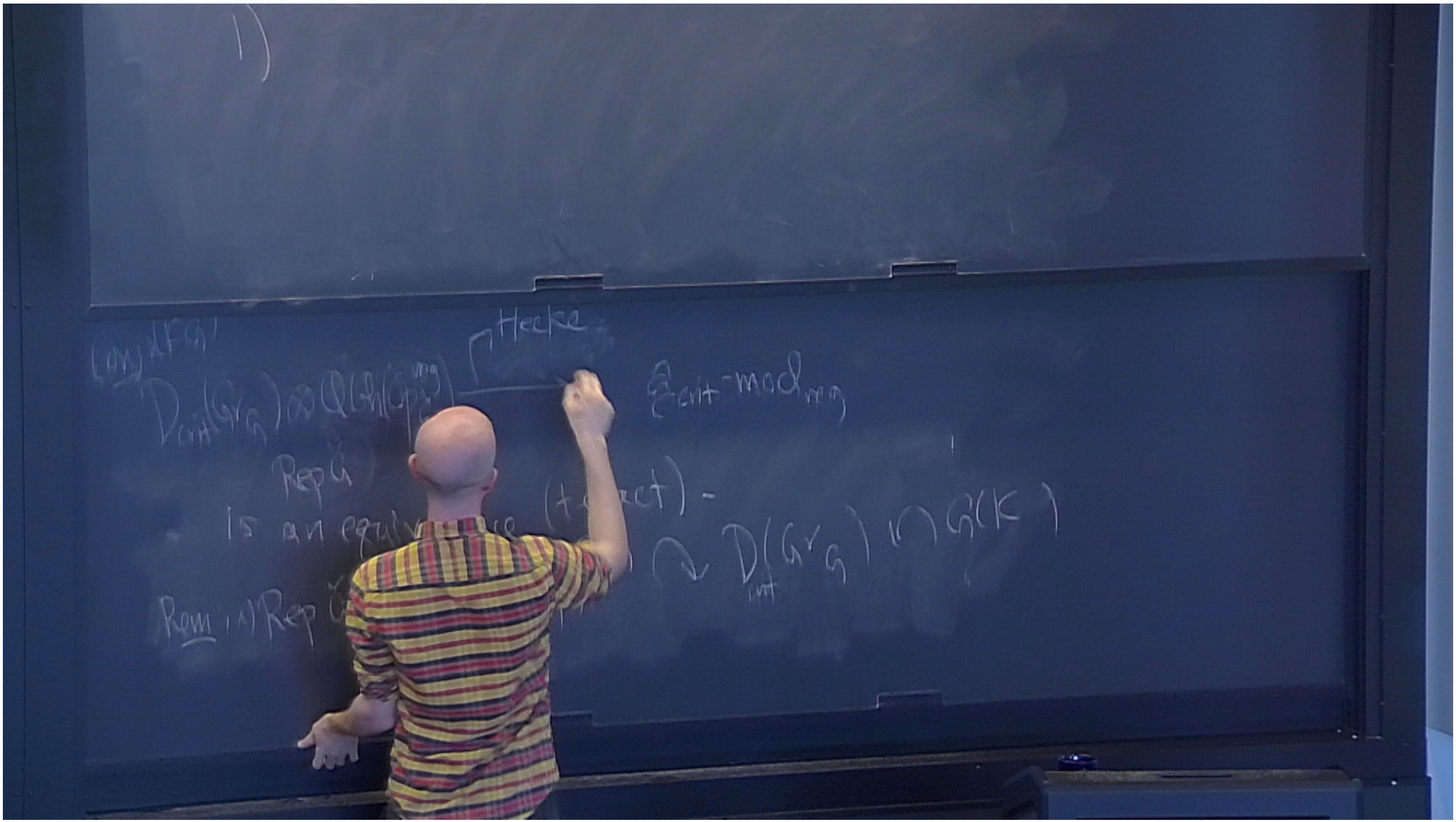
$$D_{crit}(Gr_G) \otimes_{\mathrm{Rep}\,\check{G}} \mathrm{QCoh}(\mathrm{Op}^{reg}_{\check{G}}) \xrightarrow{\Gamma(Gr_G, -)} \hat{\mathfrak{g}}_{crit}\text{-mod}_{reg}$$

is an equivalence (t-exact).

Rem: 1) $\mathrm{Rep}\,\check{G} \xrightarrow{\text{geom Satake}} D_{crit}(Gr_G)^{G(O)} \curvearrowright D_{crit}(Gr_G) \cap G(K)$

$Gr_G = G((t))/G[[t]]$

... of G/B


Hecke
Rep G
is an equivalence
Rem

$D_{crit}(Gr_G) \otimes \mathrm{Coh}(Op_{\check{G}}) \longrightarrow \mathcal{C}_{crit}\ \mathrm{mod}_{reg}$

$\mathrm{Rep}\,\check{G}$

is an equivalence (t-exact).

Rem: 1) $\mathrm{Rep}\,\check{G} \xrightarrow{\text{geom Satake}} D_{crit}(Gr_G)^{G(\mathcal{O})} \curvearrowright D_{crit}(Gr_G) \cap G(K)$

Builds on previous work of Frenkel-Gaitsgory:

1) Γ^{Hecke} is fully-faithful (derivedly)

2) Γ^{Hecke} is an equivalence when restricted to $\mathring{I}$-eq objects $\mathring{I} = G[[t]] \times_G N$

Folk thm (R.):

$$N((t)) \xrightarrow{\Psi} \mathbb{G}_a$$

E.g.: $G = GL_2$ $\quad N = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} = \mathbb{G}_a$

$$f \in \mathbb{G}_a((t)) \longmapsto \mathrm{Res}(f\,dt)$$

$$\mathrm{Op}^{\mathrm{reg}}_{\check{G}} \xrightarrow{\text{forgetting structure}} \mathrm{LocSys}_{\check{G}}(D) = B\check{G}$$

$$\mathrm{Rep}\,\check{G} \longrightarrow \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}$$

$N((t)) \longrightarrow \mathbb{G}_a$

E.g.: $G = GL_2$ $\quad N = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} = \mathbb{G}_a$

$f \in \mathbb{G}_a((t)) \longmapsto \mathrm{Res}(f\,dt)$

In a suitable sense:

Γ^{Hecke} is an equivalence on $\underbrace{(N((t)), \psi)}_{\text{Whittaker.}}$-equivariant objs

Pf: $\mathrm{Whit}(D(Gr_G)) = D(Gr_G)^{N((t)),\psi} \underset{\text{Frenkel Gaitsgory Vilonen}}{\simeq} \mathrm{Rep}(\check{G})$

$\mathrm{Whit}\left(D(Gr_G) \otimes_{\mathrm{Rep}\,\check{G}} QCoh(Op^{reg}_{\check{G}})\right) = QCoh(Op^{reg}_{\check{G}})$

1) Γ^{Hake} is fully-faithful (deriving)

2) Γ^{Hake} is an equivalence when restricted to $\mathring{I}$-eq objects $\mathring{I} = G[[t]] \rtimes_G N$

Thm (R.)

$$\mathrm{Whit}(\hat{\mathfrak{g}}_\kappa\text{-mod}) \simeq \mathcal{W}_\kappa\text{-mod}$$

$\kappa = \mathrm{crit}$: $\mathrm{RHS} \overset{FF}{=} \mathrm{QCoh}(\mathrm{Op}_{\check{G}})$.

$$\mathrm{Whit}(\hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}) = \mathrm{QCoh}(\mathrm{Op}_{\check{G}}) \otimes_{\mathrm{QCoh}(\mathrm{Op}_{\check{Z}})} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}) = \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}).$$

Pf: $\mathrm{Whit}(D(Gr_G)) = D(Gr_G)^{N((t)),\psi}$ Frenkel Gaitsgory Vilonen $\simeq \mathrm{Rep}(\check{G})$ Whittaker.

$$\mathrm{Whit}\left(D(Gr_G) \otimes_{\mathrm{Rep}\check{G}} QCoh(Op^{reg}_{\check{G}})\right) = QCoh(Op^{reg}_{\check{G}})$$

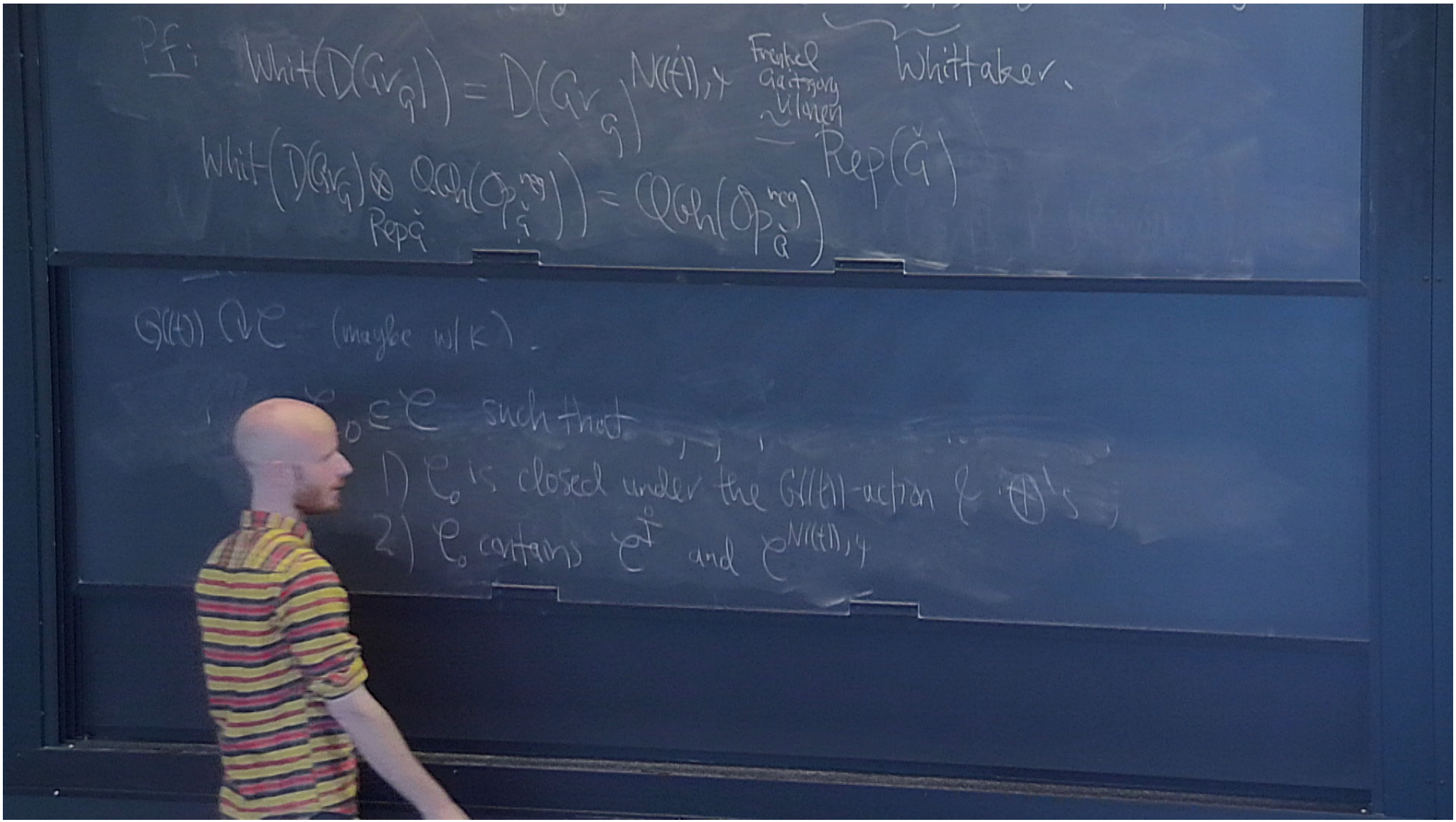
Thm (R.) $G = PGL_2$

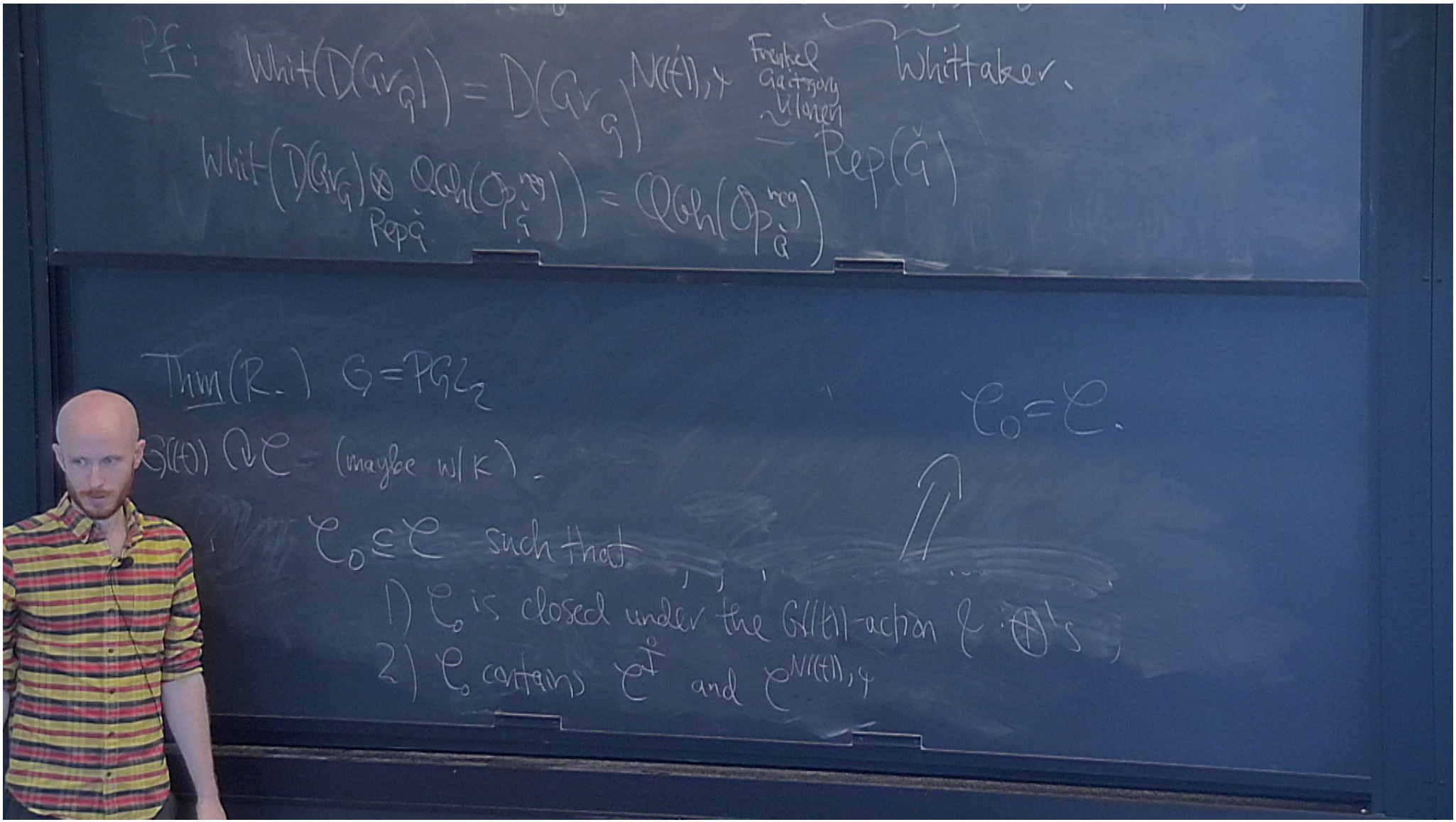
$G((t)) \curvearrowright \mathcal{C}$ (maybe w/ κ).

Then $\mathcal{C}$ is the min'l subcategory $\mathcal{C}_0$ of itself such that:

1) $\mathcal{C}_0$ is closed under the $G((t))$-action & $\oplus$'s;

2) $\mathcal{C}_0$ contains $\mathcal{C}^{\mathring{I}}$ and $\mathcal{C}^{N((t)),\psi}$


Pf: Whit(D(Gr_G)) = D(Gr_G)^{N((t)),ψ}
Frenkel Gaitsgory Ulanen
Whittaker.
≃ Rep(Ǧ)
Whit(D(Gr_G) ⊗_{Rep Ǧ} QCoh(Op^{reg}_Ǧ)) = QCoh(Op^{reg}_Ǧ)
G((t)) ↷ C (maybe w/ κ).
C_0 ⊂ C such that
1) C_0 is closed under the G((t))-action & ⊕'s,
2) C_0 contains C^{I} and C^{N((t)),ψ}


Pf: Whit(D(Gr_G)) = D(Gr_G)^{N((t)),ψ}
Frenkel Gaitsgory Vilonen
Whittaker.
Rep(Ǧ)
Whit(D(Gr_G) ⊗_{Rep Ǧ} QCoh(Op^{reg}_{Ǧ})) = QCoh(Op^{reg}_{Ǧ})
Thm (R.) G = PGL_2
G((t)) ↷ C (maybe w/ K).
C_0 = C.
C_0 ⊆ C such that
1) C_0 is closed under the G((t))-action & ⊕'s
2) C_0 contains C^{Ǧ} and C^{N((t)),ψ}