Title: PSI 2019/2020 - Quantum Gravity Part 1 - Lecture 5

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Canonical analysis of 3D Palatini action

$$S = \int \underbrace{\tilde{\epsilon}^{ab} e^i_b}_{E^a_i} \partial_0 A_{ai} - e_{0i}\underbrace{\left(\tfrac{1}{2} F^i_{ab} \tilde{\epsilon}^{ab}\right)}_{\mathcal{F}^i} - A^i_0 \underbrace{\left(\partial_a e_{bi} + \epsilon_{ijm} A^j_a e^m_b\right)\tilde{\epsilon}^{ab}}_{\frac{1}{2} T_{abi}\tilde{\epsilon}^{ab}} d^3x\, dt$$

$p \quad q$

$\mathcal{G}_i = \partial_a E^a_i$

Canonical analysis of 3D Palatini action

$$S = \int \left(\underbrace{\tilde{\epsilon}^{ab} e^i_b}_{E^a_i} \partial_0 A_{ai} - \cdots - e_0{}^i \underbrace{\left(\tfrac{1}{2} F^i_{ab} \tilde{\epsilon}^{ab} \right)}_{\Sigma^i} - A^i_0 \underbrace{\left(\partial_a e_{bi} + \epsilon_{ijk} A^j_a e^k_b \right) \tilde{\epsilon}^{ab}}_{\frac{1}{2} T_{abi} \tilde{\epsilon}^{ab}} \right) d^3x\, dt$$

$p \quad q$

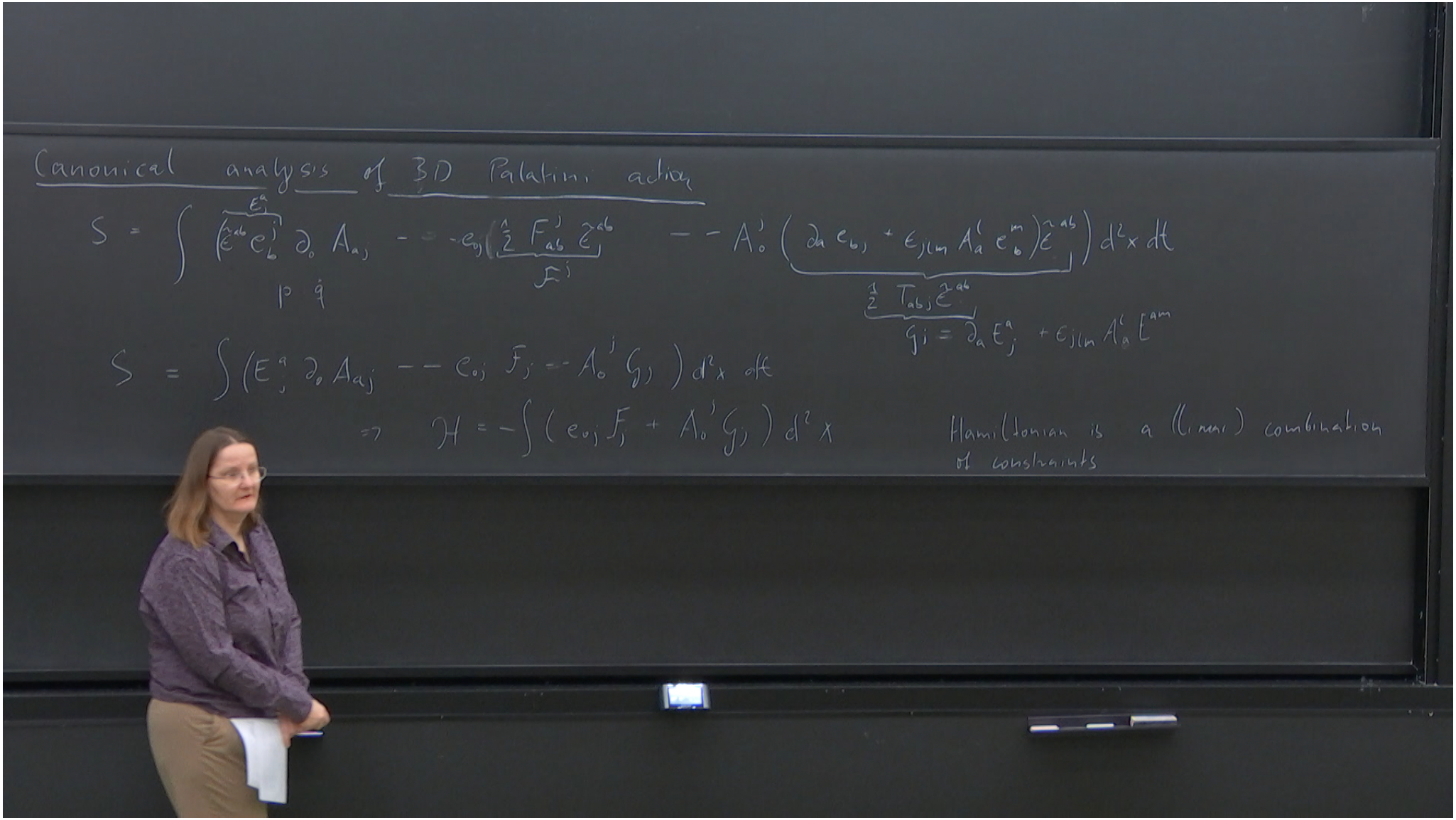
$$G_i = \partial_a E^a_i + \epsilon_{ijk} A^j_a E^{ak}$$

Canonical analysis of 3D Palatini action

$$S = \int \left(\underbrace{\tilde{E}^{ab}_i}_{p} \underbrace{e^i_b}_{q} \partial_0 A_{aj} - - e_{0j} \underbrace{\tfrac{1}{2} F^j_{ab} \tilde{\epsilon}^{ab}_j}_{\mathcal{E}^j} - - A^j_0 \underbrace{\left(\partial_a e_{bj} + \epsilon_{jlm} A^l_a e^m_b \right) \tilde{\epsilon}^{ab}_j}_{\frac{1}{2} T_{abj} \tilde{\epsilon}^{ab}_j} \right) d^2x \, dt$$

$$G_j = \partial_a E^a_j + \epsilon_{jlm} A^l_a E^{am}$$

$$S = \int \left(E^a_j \partial_0 A_{aj} - - e_{0j} \mathcal{E}_j - - A_0^j G_j \right) d^2x \, dt$$


Canonical analysis of 3D Palatini action
$S = \int \left(\underbrace{\tilde{\epsilon}^{ab} e^i_b}_{E^a_i} \partial_0 A_{ai} - e_{0i} \underbrace{\tfrac{1}{2} F^i_{ab} \tilde{\epsilon}^{ab}}_{F^i} - A^i_0 \underbrace{\left(\partial_a e_{bi} + \epsilon_{ilm} A^l_a e^m_b \right) \tilde{\epsilon}^{ab}}_{\tfrac{1}{2} T_{abi} \tilde{\epsilon}^{ab}} \right) d^2x\, dt$
p q
$G_i = \partial_a E^a_i + \epsilon_{ilm} A^l_a E^{am}$
$S = \int \left(E^a_i \partial_0 A_{ai} - e_{0i} F_i - A^i_0 G_i \right) d^2x\, dt$
$\Rightarrow \quad \mathcal{H} = -\int \left(e_{0i} F_i + A^i_0 G_i \right) d^2x$
Hamiltonian is a (linear) combination of constraints

$$S = \int \left(E^a_j \, \partial_0 A_{aj} - e_{0j} F_j - A_0^j G_j \right) d^2x \, dt$$

$$\Rightarrow \quad \mathcal{H} = -\int \left(e_{0j} F_j + A_0^j G_j \right) d^2x$$

Hamiltonian is a ((of constraints => to

Can still consider flow generated by $\mathcal{H}$

$$\frac{d}{ds} f(s) = \{ f, \mathcal{H} \}(s)$$

Flow generated by Gauß constraint:

$$\delta_\Lambda E^a_j(x) = \{ E^a_j(x), \int \Lambda^k \left(\partial_b E^b_k + \epsilon_{klm} A^l_b E^{bm} \right)(y) \, d^2y \}$$

$$= -\epsilon_{jmk} E^{am} \Lambda^k(x)$$

$$\delta_\Lambda A^j_a(x) = \{ A^j_a(x), \int \left(-E^b_k \partial_b \Lambda^k + \epsilon_{klm} \Lambda^k A^l_b E^{bm} \right)(y) \, d^2y \} =$$

$$\left(E^a_j \partial_0 A_{aj} - - e_{0j} F_j - - A_0^j G_j\right) d^3x \text{ ok}$$

$$G_j = \partial_a E^a_j + \epsilon_{jlm} A^l_a E^{am}$$

$$\Rightarrow \quad \mathcal{H} = -\int \left(e_{0j} F_j + A_0^j G_j\right) d^3x$$

Hamiltonian is a (linear) combination of constraints $\Rightarrow$ totally constrained

$$\frac{d}{ds} f(s) = \{f, \mathcal{H}\}(s)$$

Gauss constraint:

$$\delta_\Lambda E^a_j(x) = \left\{ E^a_j(x), \int \Lambda^k \left(\partial_b E^b_k + \epsilon_{klm} A^l_b E^{bm}\right)(y)\, d^3y \right\}$$

$$= -\epsilon_{jmk} E^{am} \Lambda^k(x)$$

$$\delta_\Lambda A^j_a(x) = \left\{ A^j_a(x), \int \left(-E^b_k \partial_b \Lambda^k + \epsilon_{klm} \Lambda^k A^l_b E^{bm}\right)(y)\, d^3y \right\} = -\left(\partial_a \Lambda^j + \epsilon^j{}_{lm} A^l_a \Lambda^m\right)(x)$$

$\Rightarrow$ Gauss constraint generates rotations.

straint. $\quad \delta_N E^a_j$

$$\delta_\Lambda A^i_a(x) = \{ A^i_a(x), \int (-E^b_k \partial_b \Lambda^k + \epsilon_{klm} \Lambda^k A^l_b E^{blm}) / \sqrt{q} \, d^3y \} =$$

$\Rightarrow$ Gauss constraint generates rotations.

flatness constraint:

$$\delta_N E^a_i = \{ E^a_i(x), \int N_k \tilde{\epsilon}^{cb} (\partial_c A^k_b + \frac{1}{2} \epsilon^k{}_{ml} A^m_c A^l_b) d^3y \}$$

$$= \tilde{\epsilon}^{ca} (\partial_c N_i + \epsilon_{ikm} A^m_c N^k)$$

$$\delta_N A^i_a = \{ A^i_a(x), \int F^i(A) \, d^3y \} = 0$$

flatness generates translation symmetry.

by Gauß constraint:

$$\delta_\Lambda E^a_j(x) = \{ E^a_j(x), \int \Lambda^k (\partial_b E^b_k + \epsilon_{klm} A^l_b E^{bm})(y)\, d^2y \}$$

$$= - \epsilon_{jmk} E^{am} \Lambda^k(x)$$

$$\delta_\Lambda A^j_a(x) = \{ A^j_a(x), \int (- E^b_k \partial_b \Lambda^k + \epsilon_{klm} \Lambda^k A^l_b E^{bm})(y)\, d^2y \} = - (\partial_a \Lambda^j + \epsilon^j{}_{lm} A^l_a \Lambda^m)(x)$$

⇒ Gauß constraint generates rotations.

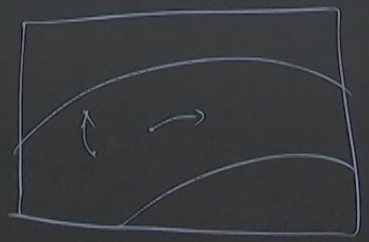
constraint:

$$\delta_N E^a_j = \{ E^a_j(x), \int N_k \tilde{\epsilon}^{cb} (\partial_c A^k_b + \tfrac{1}{2} \epsilon^k{}_{mn} A^m_c A^n_b)\, d^2y \}$$

$$= \tilde{\epsilon}^{ca} (\partial_c N_j + \epsilon_{jmk} A^m_c N^k)$$

$$\delta_N A^j_a = \{ A^j_a(x), \int F^j(A)\, d^2y \} = 0$$

Flatness generates translation symmetry.



Need to check: Time evolution preserves the constraints

$$\{G_i, H\} \approx 0$$

"weakly vanishing" = vanishing on $\mathcal{C}$.

Combinations of constraint

$$\{F^i, H\} \approx 0$$

$$\{G[\Lambda'], G[\Lambda]\} = \{ \int \left(-E^b_k \partial_b (\Lambda')^k + \epsilon_{klm} (\Lambda')^k A^l_b E^{bm} \right)(y) d^2x \,,\, \int \Lambda^l \left(\partial_c E^c_j + \epsilon_{jpq} A^p_c E^{cq} \right)(y) d^2y \}$$

$$= -\int \Big(\partial_b (\Lambda')^b \epsilon_{kml} E^{bm} \Lambda^l - \epsilon_{klm} (\Lambda')^k \left(\partial_b \Lambda^l + \epsilon^{lpq} A^p_b \Lambda^q \right) E^{bm}$$

$$- \epsilon_{klm} (\Lambda')^k A^l_b \epsilon^m{}_{pq} E^{bp} \Lambda^q \Big) d^2x$$

Use: $\epsilon_{klm} \epsilon_{lpq} + \epsilon_{kpl} \epsilon_{lmq} + \epsilon_{pml} \epsilon_{lkq} = 0$

$$\{G[\Lambda'], G[\Lambda]\} = \{ \int (-E^b_{ik}\,\partial_b(\Lambda')^k + \epsilon_{ikm}(\Lambda')^k A^l_b E^{bm})(y)\,d^2x\,,\ \int \Lambda^j(\partial_c E^c_j + \epsilon_{jpq} A^p_c E^{cq})(y)\,d^2y \}$$

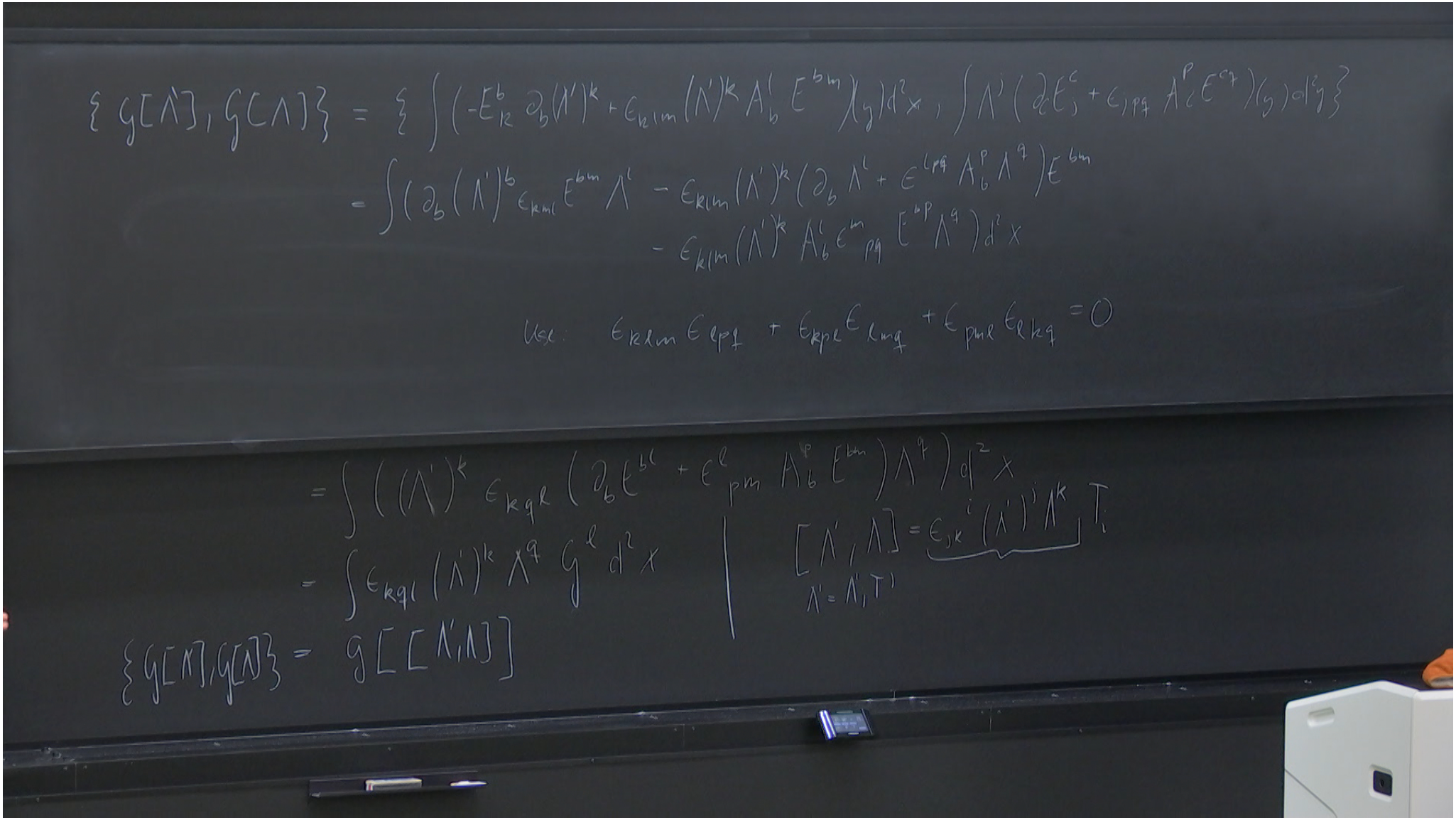
$$= -\int (\partial_b(\Lambda')^b \epsilon_{kml} E^{bm} \Lambda^l - \epsilon_{klm}(\Lambda')^k(\partial_b \Lambda^l + \epsilon^{lpq} A^p_b \Lambda^q) E^{bm}$$
$$- \epsilon_{klm}(\Lambda')^k A^l_b \epsilon^m{}_{pq} E^{bp} \Lambda^q)\,d^2x$$

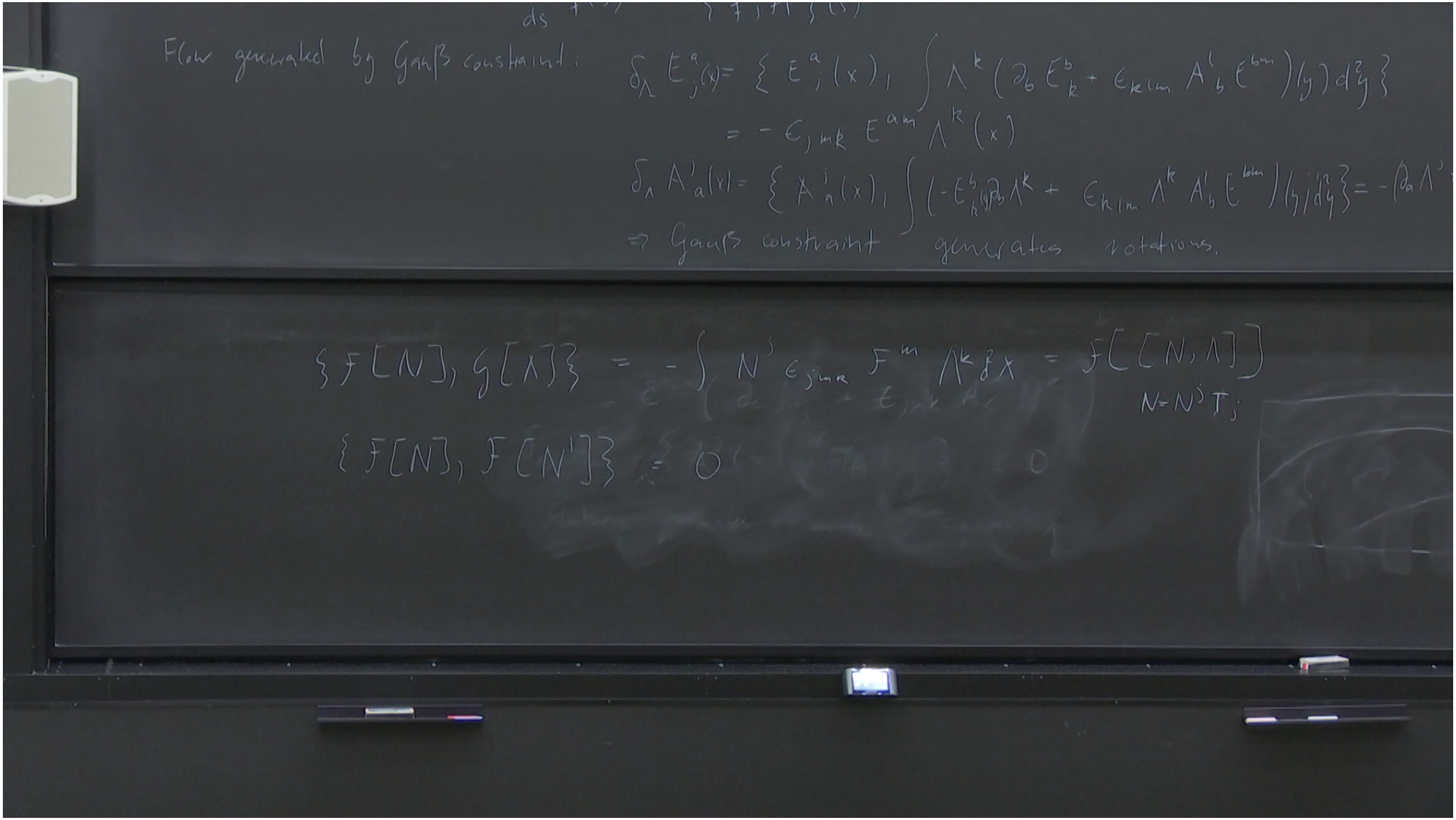
Use: $\epsilon_{klm}\epsilon_{lpq} + \epsilon_{kpl}\epsilon_{lmq} + \epsilon_{pml}\epsilon_{lkq} = 0$

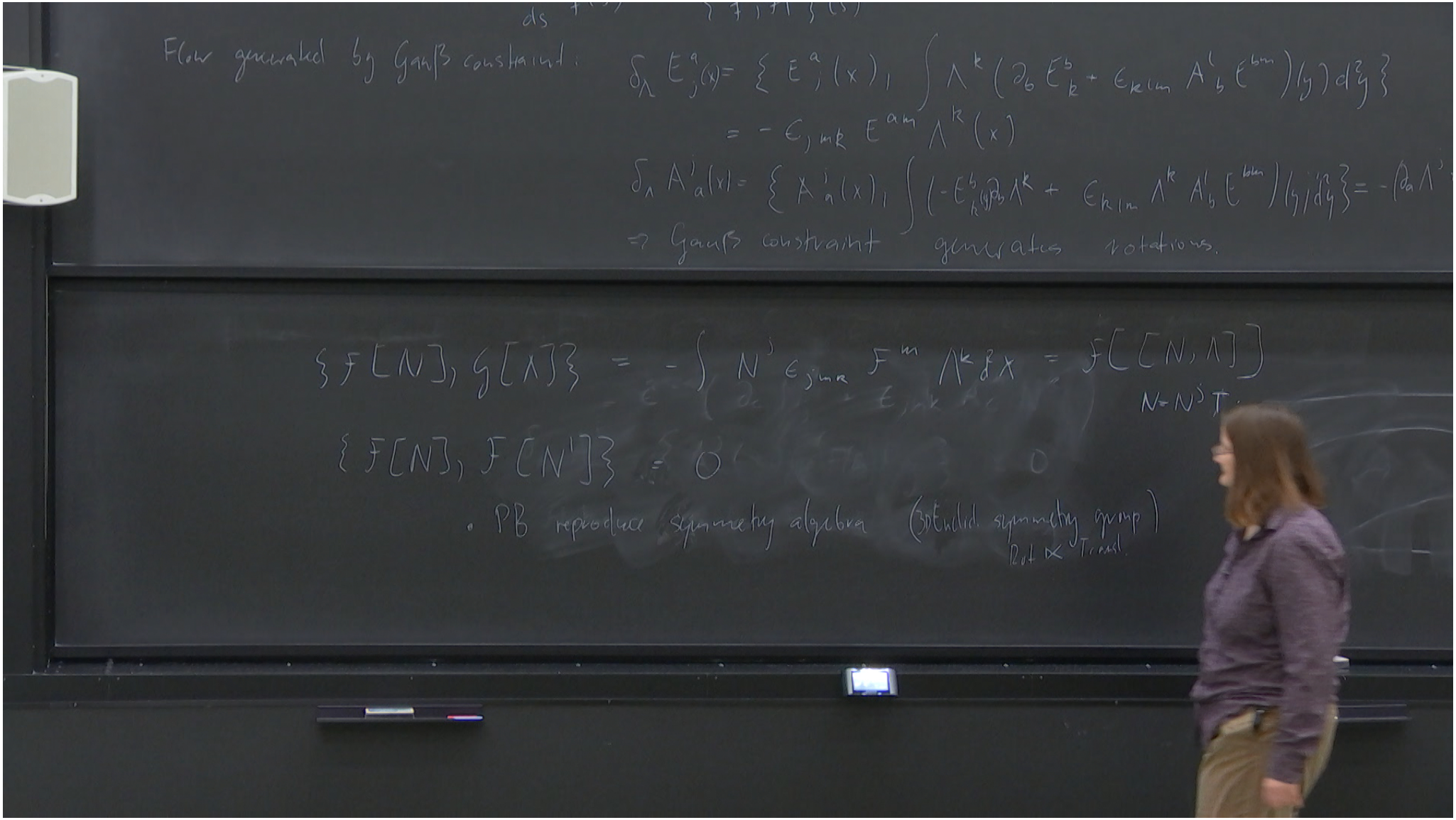
$$= \int ((\Lambda')^k \epsilon_{kql}(\partial_b E^{bl} + \epsilon^l{}_{pm} A^p_b E^{bm})\Lambda^q)\,d^2x$$

$$= \int \epsilon_{kql}(\Lambda')^k \Lambda^q G^l\,d^2x \qquad [\Lambda', \Lambda] = \epsilon_{jk}{}^i(\Lambda')^j \Lambda^k T_i$$
$$\Lambda' = \Lambda, T)$$

$$= G[[\Lambda', \Lambda]]$$


$\{G[\Lambda'], G[\Lambda]\} = \{ \int (-E^b_{ik}\,\partial_b(\Lambda')^k + \epsilon_{klm}(\Lambda')^k A^l_b E^{bm})(y)\,d^3x\,,\ \int \Lambda^j(\partial_c E^c_j + \epsilon_{jpq} A^p_c E^{cq})(y)\,d^3y \}$
$= \int (\partial_b(\Lambda')^b \epsilon_{kml} E^{bm} \Lambda^l - \epsilon_{klm}(\Lambda')^k(\partial_b \Lambda^l + \epsilon^{lpq} A^p_b \Lambda^q) E^{bm}$
$- \epsilon_{klm}(\Lambda')^k A^l_b \epsilon^m{}_{pq} E^{bp} \Lambda^q)\,d^3x$
Use: $\epsilon_{klm}\epsilon_{lpq} + \epsilon_{kpl}\epsilon_{lmq} + \epsilon_{pml}\epsilon_{lkq} = 0$
$= \int ((\Lambda')^k \epsilon_{kql}(\partial_b E^{bl} + \epsilon^l{}_{pm} A^p_b E^{bm})\Lambda^q)\,d^3x$
$= \int \epsilon_{kql}(\Lambda')^k \Lambda^q G^l\,d^3x$
$[\Lambda', \Lambda] = \epsilon_{jk}{}^i (\Lambda')^j \Lambda^k T_i$
$\Lambda' = \Lambda_1 T$
$\{G[\Lambda], G[\Lambda]\} = G[[\Lambda', \Lambda]]$


Flow generated by Gauß constraint:
$\delta_\Lambda E^a_j(x) = \{ E^a_j(x), \int \Lambda^k (\partial_b E^b_k + \epsilon_{klm} A^l_b E^{bm})(y)\, d^3y \}$
$= -\epsilon_{jmk} E^{am} \Lambda^k(x)$
$\delta_\Lambda A^j_a(x) = \{ A^j_a(x), \int (-E^b_k \partial_b \Lambda^k + \epsilon_{klm} \Lambda^k A^l_b E^{bm})(y)\, d^3y \} = -(\partial_a \Lambda^j$
$\Rightarrow$ Gauß constraint generates rotations.
$\{F[N], G[\Lambda]\} = -\int N^j \epsilon_{jmk} F^m \Lambda^k \delta x = F[[N,\Lambda]]$
$N = N^j T_j$
$\{F[N], F[N']\} = 0$


Flow generated by Gauß constraint:
$\delta_\Lambda E^a_j(x) = \{ E^a_j(x), \int \Lambda^k (\partial_b E^b_k + \epsilon_{klm} A^l_b E^{bm})(y)\, d^3y \}$
$= -\epsilon_{jmk} E^{am} \Lambda^k(x)$
$\delta_\Lambda A^j_a(x) = \{ A^j_a(x), \int (-E^b_k \partial_b \Lambda^k + \epsilon_{klm} \Lambda^k A^l_b E^{bm})(y)\, d^3y \} = -\partial_a \Lambda^j$
⇒ Gauß constraint generates rotations.
$\{ F[N], G[\Lambda] \} = -\int N^j \epsilon_{jmk} F^m \Lambda^k d^3x = F[[N, \Lambda]]$
$N = N^j T_j$
$\{ F[N], F[N'] \} = 0$
• PB reproduce symmetry algebra (3d Euclid. symmetry group)
Rot ⋉ Transl.

rated by Gauss constraint:

$$\delta_\Lambda E^a_j(x) = \{ E^a_j(x), \int \Lambda^k (\partial_b E^b_k + \epsilon_{klm} A^l_b E^{bm})(y)\, d^3y \}$$

$$= - \epsilon_{jmk} E^{am} \Lambda^k(x)$$

$$\delta_\Lambda A^j_a(x) = \{ A^j_a(x), \int (- E^b_k \partial_b \Lambda^k + \epsilon_{klm} \Lambda^k A^l_b E^{bm})(y)\, d^3y \} = - (\partial_a \Lambda^j + \epsilon^j{}_{lm} A^l_a \Lambda^m)$$

$\Rightarrow$ Gauss constraint generates rotations.

$$\{ F[N], G[\Lambda] \} = - \int N^j \epsilon_{jmn} E^m \Lambda^k \delta x = F[[N, \Lambda]]$$

$N = N^j T_j$

$$\{ F[N], F[N'] \} = 0$$

- PB reproduce symmetry algebra (3D Euclid. symmetry group) $\mathrm{Rot} \ltimes \mathrm{Transl.}$
- Constraint algebra is "closed" / constraints are first class.

