Title: PSI 2019/2020 - Quantum Gravity Part 1 - Lecture 2

Speakers: Bianca Dittrich

Collection: PSI 2019/2020 - Quantum Gravity Part 1

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Spin connection

$\nabla_\mu \to \partial_\mu, \Gamma^\rho_{\mu\nu}$

$D_\mu \to \partial_\mu, \Gamma, \omega$

Compatibility $D_\mu e^i_\nu = 0$

$$\Rightarrow \omega^i_{\mu k} = e^\nu_k \Gamma^\rho_{\mu\nu} e^i_\rho - e^\nu_k \partial_\mu e^i_\nu$$

$$\Gamma^\rho_{\mu\nu} = e^\rho_i \omega^i_{\mu k} + e^\rho_i \partial_\mu e^i_\nu$$

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What does ω need to satisfy so that Γ is LC:

a) Metricity: $\nabla_\mu g_{\rho\nu} = 0$ b) $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$

a) $\nabla_\mu g^{\alpha\nu} = D_\mu g^{\alpha\nu} = 0 \quad ; \quad D_\mu e^k_\nu = 0$

$0 = D_\mu e^k_\mu e^j_\nu g^{\mu\nu} = D_\mu \delta^{kj}$

def. of D_μ $= \underbrace{\partial_\mu \delta^{kj}}_{0} + \omega_{\mu\ell}{}^{k} \delta^{\ell j} + \omega_{\mu\ell}{}^{j} \delta^{k\ell}$

$\Rightarrow \underbrace{\omega_\mu{}^{kj}}_{\text{antisym. in } (kj)} = \omega_{\mu\ell}{}^{k} \delta^{\ell j} \overset{!}{=} 0$

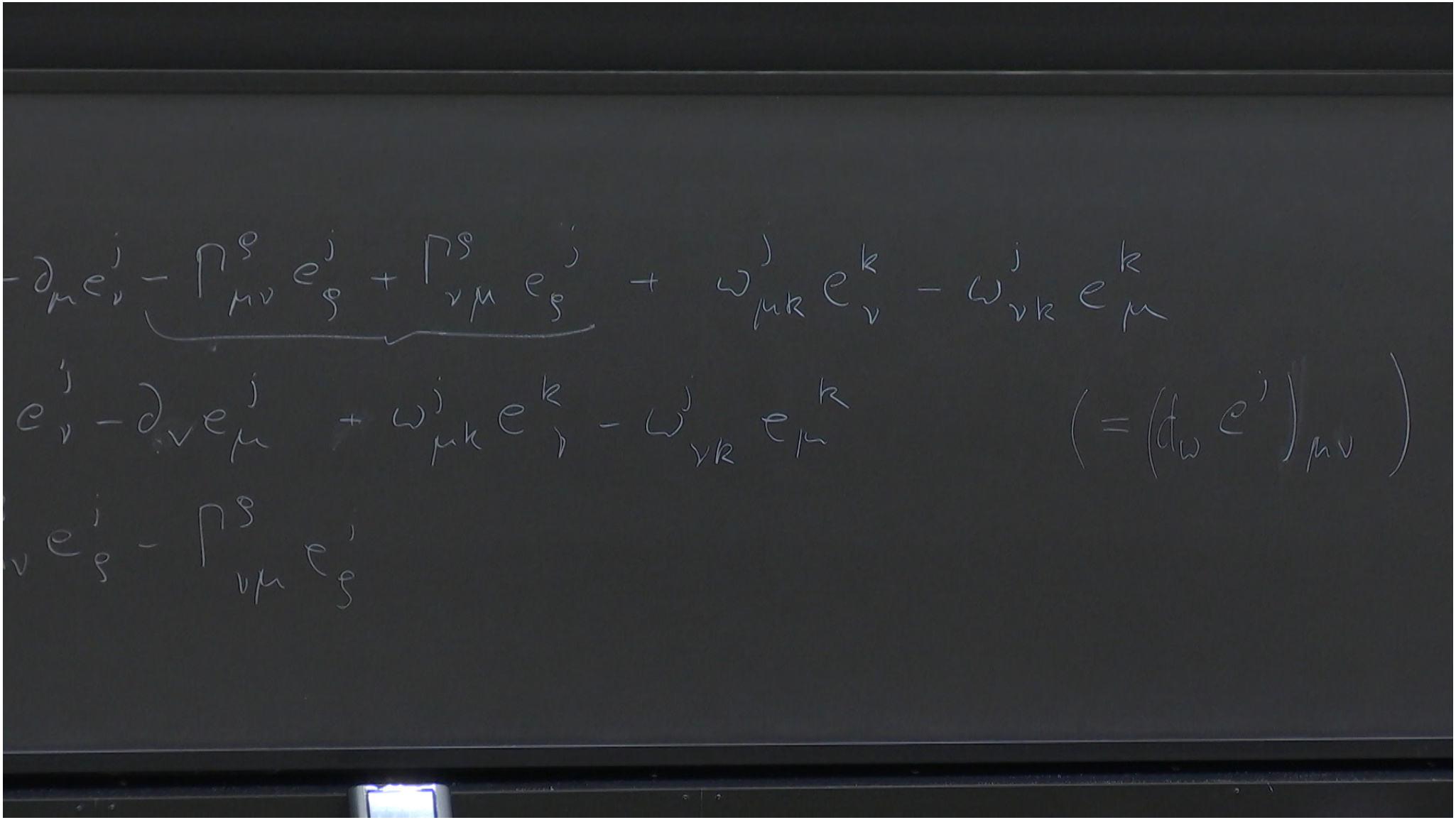
Metricity $\Rightarrow$ Antisymmetry in internal indices of ω

b) Torsion (freeness)

$$0 = D_\mu e^i_\nu - D_\nu e^i_\mu = \partial_\mu e^i_\nu - \partial_\mu e^i_\nu - \Gamma^\rho_{\mu\nu} e^i_\rho + \Gamma^\rho_{\nu\mu}$$

$$\boxed{T^i_{\mu\nu}} := \partial_\mu e^i_\nu - \partial_\nu e^i_\mu + \omega^i_{\mu k}$$

$$= \Gamma^\rho_{\mu\nu} e^i_\rho - \Gamma^\rho_{\nu\mu} e^i_\rho$$


$-\partial_\mu e^j_\nu - \Gamma^\rho_{\mu\nu} e^j_\rho + \Gamma^\rho_{\nu\mu} e^j_\rho + \omega^j_{\mu k} e^k_\nu - \omega^j_{\nu k} e^k_\mu$
$e^j_\nu - \partial_\nu e^j_\mu + \omega^j_{\mu k} e^k_\nu - \omega^j_{\nu k} e^k_\mu \quad \left(= (d_\omega e^j)_{\mu\nu}\right)$
$\nu e^j_\rho - \Gamma^\rho_{\nu\mu} e^j_\rho$

Riemann tensor as fct. of (e, ω)

$$R_{\mu\nu\rho}{}^{\sigma} v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \overbrace{(e^i_\rho v_i)}^{\vec{v}_\rho}$$

$$= \underbrace{(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)(e^i_\rho v_i)}$$

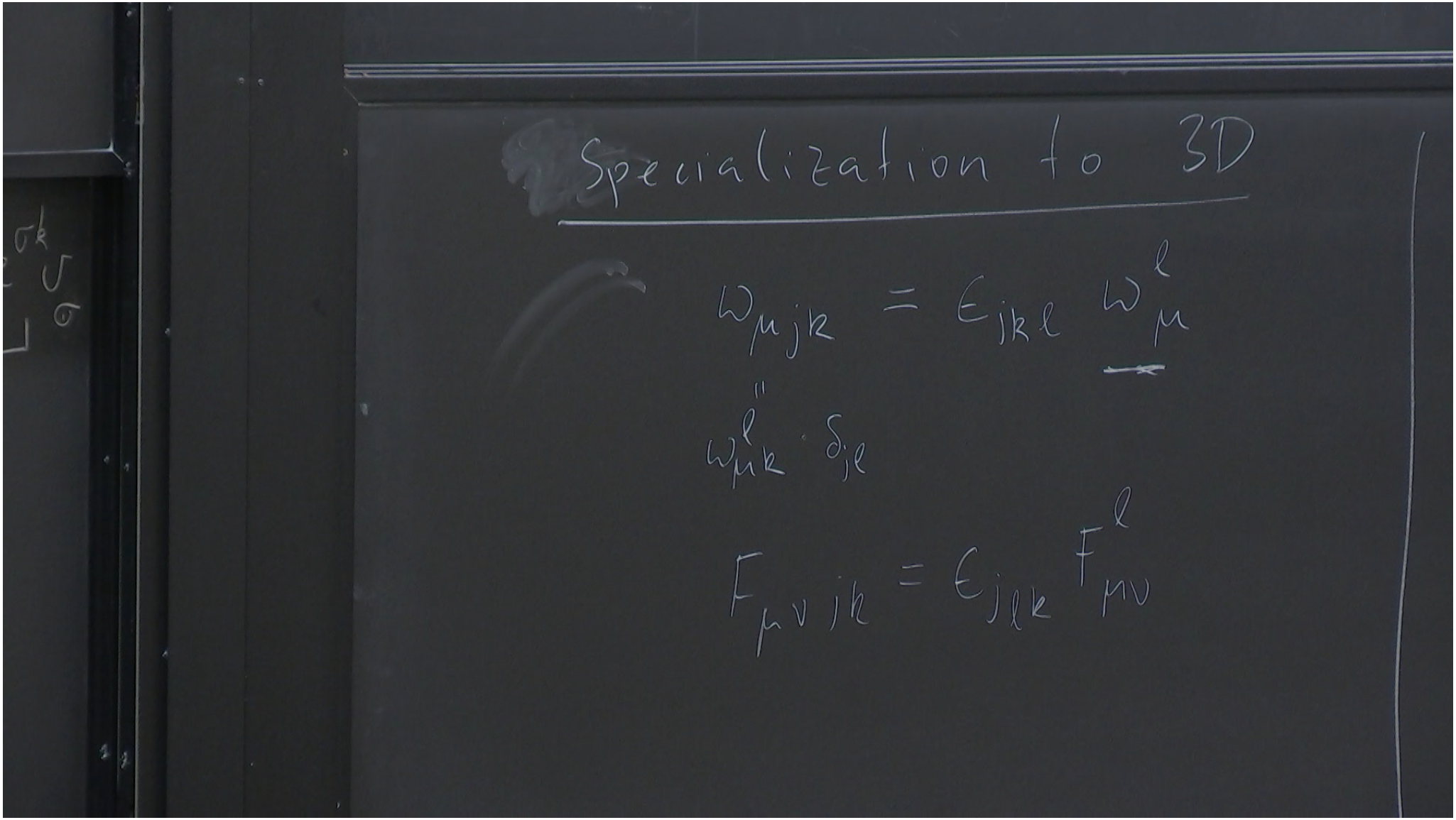
$$- e^i_\rho \nabla_\mu (\partial_\nu v_i - \omega_{\nu i}{}^k v^k) - (\mu \leftrightarrow \nu)$$

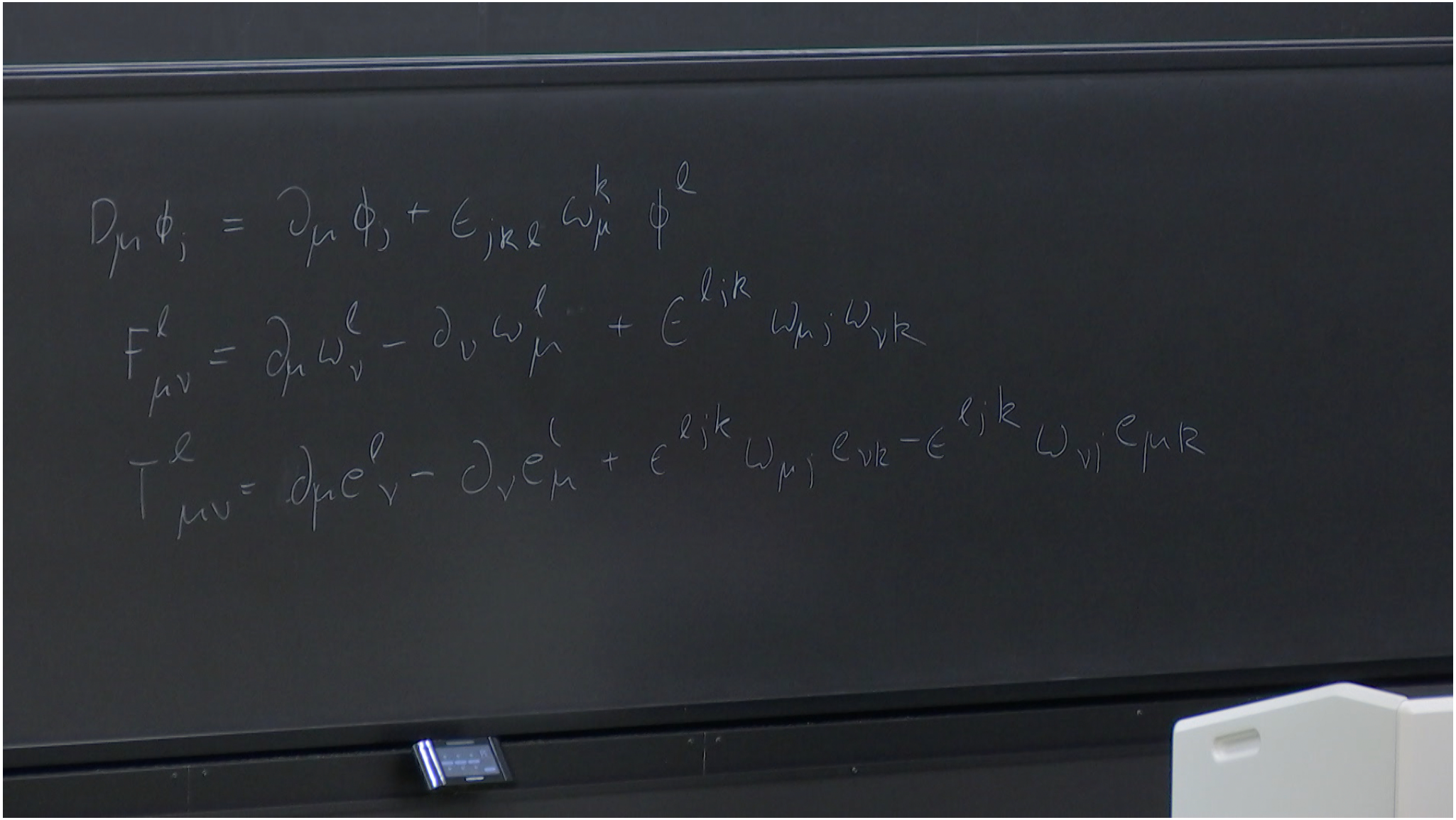
$$= e^{j}_{\beta}\left[\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu j\ell}\,\omega^{\ell}_{\nu k} - \omega_{\nu j\ell}\,\omega^{\ell}_{\mu k}\right] e^{\sigma k}\, v_\sigma$$

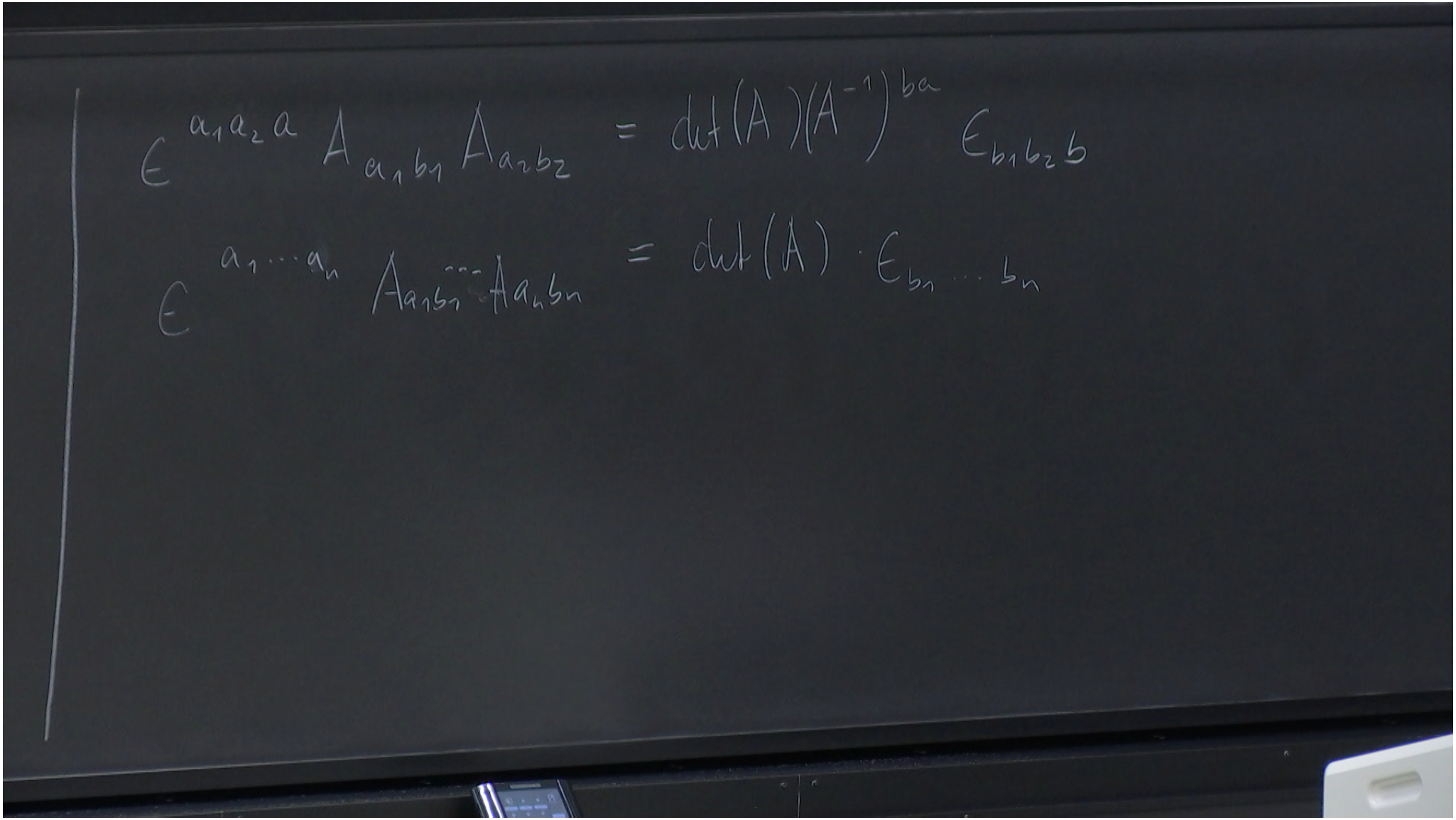
$$R_{\mu\nu\beta}{}^{\sigma} = e^{j}_{\beta}\, e^{\sigma k}\, F_{\mu\nu jk}$$

$$R = R_{\mu\beta}\, g^{\mu\beta} = e^{\mu}_{j}\, e^{\sigma}_{k}\, F^{jk}_{\mu\sigma}$$

$$\boxed{R = e^{\mu}_{j}\, e^{\sigma}_{k}\, F^{jk}_{\mu\sigma}}$$


Specialization to 3D
$\omega_{\mu jk} = \epsilon_{jkl}\,\omega^{l}_{\mu}$
$\omega^{l}_{\mu k}\cdot\delta_{jl}$
$F_{\mu\nu jk} = \epsilon_{jlk}\,F^{l}_{\mu\nu}$


$D_\mu \phi_j = \partial_\mu \phi_j + \epsilon_{jkl} \omega_\mu^k \phi^l$
$F^l_{\mu\nu} = \partial_\mu \omega^l_\nu - \partial_\nu \omega^l_\mu + \epsilon^{ljk} \omega_{\mu j} \omega_{\nu k}$
$T^l_{\mu\nu} = \partial_\mu e^l_\nu - \partial_\nu e^l_\mu + \epsilon^{ljk} \omega_{\mu j} e_{\nu k} - \epsilon^{ljk} \omega_{\nu j} e_{\mu k}$


$\epsilon^{a_1 a_2 a} A_{a_1 b_1} A_{a_2 b_2} = \det(A)(A^{-1})^{ba} \epsilon_{b_1 b_2 b}$
$\epsilon^{a_1 \cdots a_n} A_{a_1 b_1} \cdots A_{a_n b_n} = \det(A) \cdot \epsilon_{b_1 \cdots b_n}$

$$R = \underbrace{e^{\mu}_{j}\, e^{\nu}_{k}\, \epsilon^{jlk}}\, F_{\mu\nu l}$$

$$= -\tilde{\epsilon}^{\mu\nu\sigma}\, e^{l}_{\sigma}\, \underbrace{\det(e^{s}_{m})}_{(\sqrt{g})^{-1}}\, F_{\mu\nu l}$$

$$\Rightarrow \quad \sqrt{g}\, R = -\tilde{\epsilon}^{\sigma\mu\nu}\, e_{\sigma l}\, F^{l}_{\mu\nu}$$

$$\int \sqrt{g}\, R\, d^3x = -\int \underbrace{e_{\sigma l} F^{l}_{\mu\nu}\, \tilde{\epsilon}^{\sigma\mu\nu}}_{\sim \mathrm{tr}(e \wedge F)}\, d^3x$$

$$\epsilon^{a_1 a_2 \ldots}$$

$$\epsilon^{a_1 \ldots}$$

$$S := \frac{1}{2} \int c_{\sigma\ell} F^{\ell}_{\mu\nu}(\omega) \underbrace{\tilde{\epsilon}^{\sigma\mu\nu}}_{\tilde{\epsilon}^{123} = +1} d^3x$$

$$\delta_\epsilon S \Rightarrow F^{\ell}_{\mu\nu} = 0$$

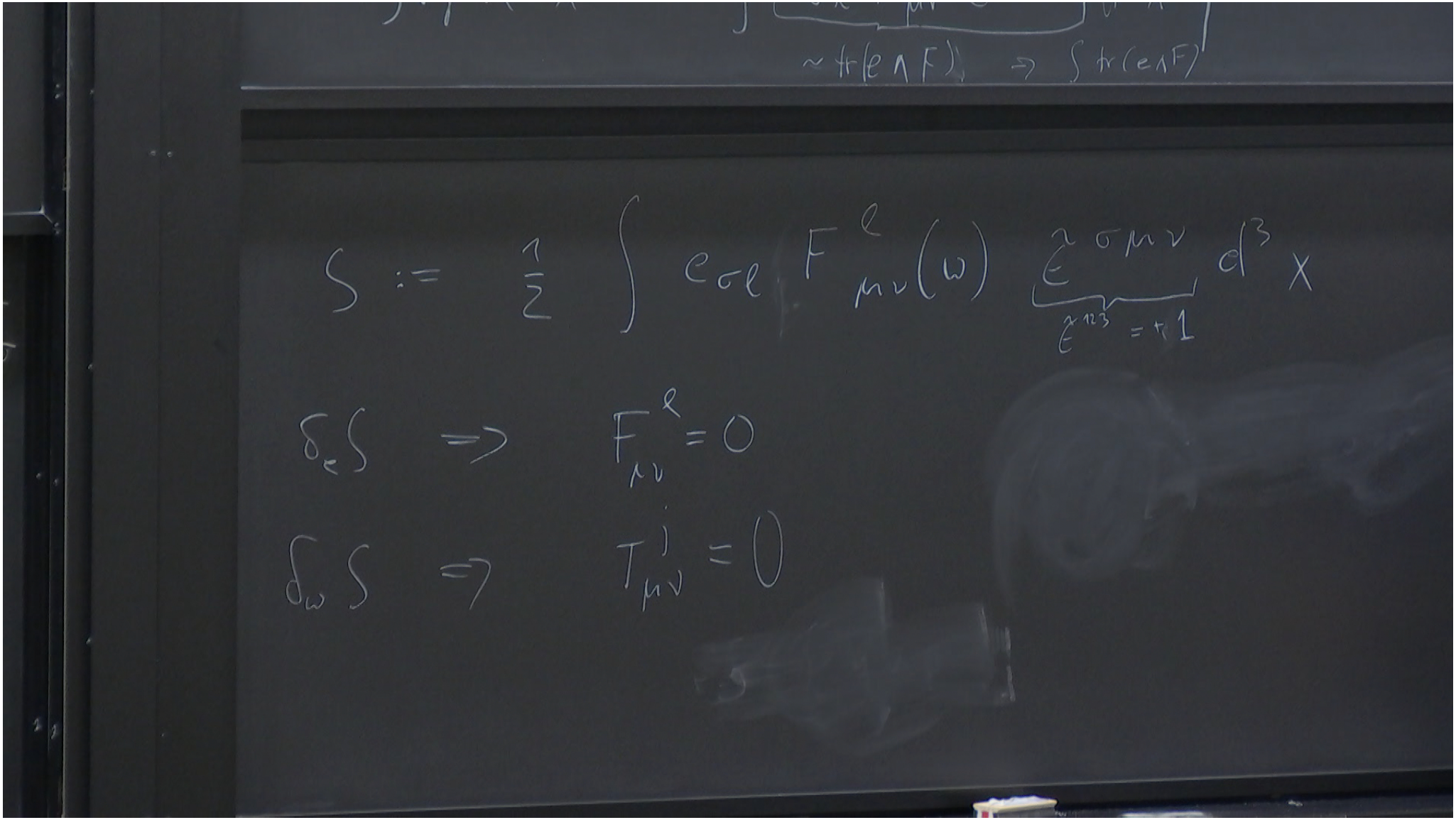
$$\tilde{\epsilon}_{\rho\lambda\tau} = \det(g)^{-1} g_{\rho\sigma} g_{\lambda\mu} g_{\tau\nu} \epsilon^{\sigma\mu\nu}$$

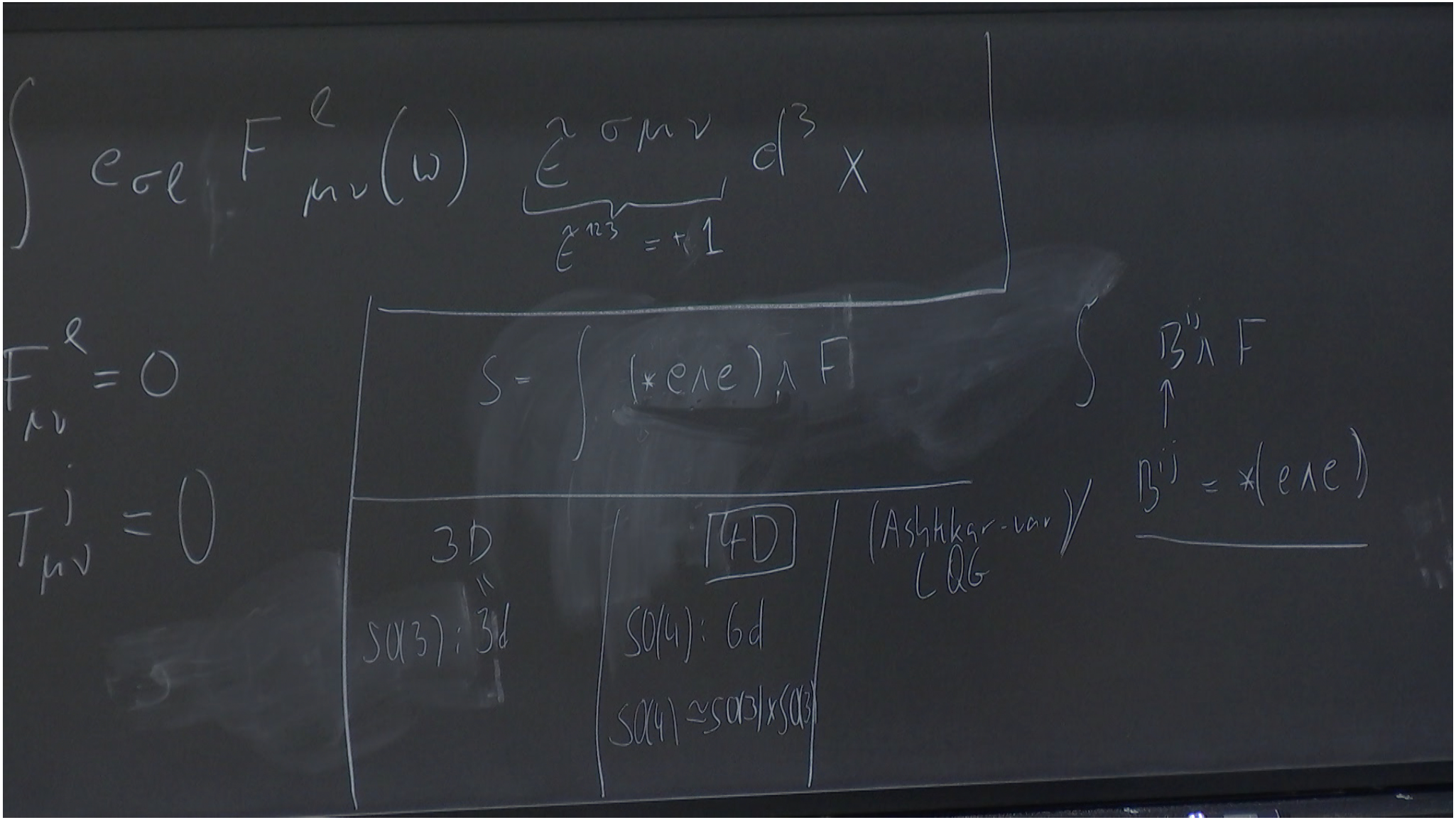
$$S := \frac{1}{2} \int \epsilon_{\sigma\ell} F^{\ell}_{\mu\nu}(\omega)$$

$$\delta_\epsilon S \Rightarrow F^{\ell}_{\mu\nu} = 0$$

$$= -\int \underbrace{c_{\sigma\ell} F^{\ell}_{\mu\nu}}_{\sim \mathrm{tr}(c \wedge F)} \varepsilon^{\sigma\mu\nu} d^3x \quad \Rightarrow \int \mathrm{tr}(c \wedge F)$$

$$\int c_{\sigma\ell} F^{\ell}_{\mu\nu}(\omega) \underbrace{\varepsilon^{\sigma\mu\nu}}_{\in \mathbb{R}^3 \to 1} d^3x$$


$\sim \mathrm{tr}(e \wedge F)$, $\Rightarrow \int \mathrm{tr}(e \wedge F)$
$S := \frac{1}{2} \int e_\sigma e F^e_{\mu\nu}(\omega) \underbrace{\epsilon^{\sigma\mu\nu}}_{\epsilon^{123} = +1} d^3x$
$\delta_e S \Rightarrow F^e_{\mu\nu} = 0$
$\delta_\omega S \Rightarrow T^i_{\mu\nu} = 0$


$\int e_{\sigma\ell} F^{\ell}_{\mu\nu}(\omega) \underbrace{\epsilon^{\sigma\mu\nu}}_{\epsilon^{123}=+1} d^3x$
$F^{\ell}_{\mu\nu} = 0$
$T^{i}_{\mu\nu} = 0$
$S = \int (*e \wedge e) \wedge F$
$\int B^{ij} \wedge F$
$B^{ij} = *(e \wedge e)$
3D
$SO(3) : 3d$
4D
$SO(4) : 6d$
$SO(4) \simeq SO(3) \times SO(3)$
(Ashtekar-Lan
LQG