Title: PSI 2019/2020 - Relativistic Quantum Information Part 1 - Lecture 4

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Collection: PSI 2019/2020 - Relativistic Quantum Information Part 1

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Notions of thermality

+ Gibbs thermality (canonical ensamble): Thermal states

  - Based on 2nd Law

  - $\hat{\rho} = \frac{1}{Z(\beta)} e^{-\beta \hat{H}}, \quad Z(\beta) := \mathrm{tr}\left(e^{-\beta \hat{H}}\right)$

A quantum system in a Gibbs state satisfies **detailed balance**:

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For a Gibbs state that is the Boltzmann factor $e^{-\beta(\Omega_j - \Omega_i)}$

+ For example, for a qubit with $\hat{H} = \Omega\, \hat{\sigma}^{+}\hat{\sigma}^{-}$:

$$\left.\begin{aligned} p^{+} &= \frac{1}{Z}\langle e|e^{-\beta\hat{H}}|e\rangle = \frac{e^{-\beta\hat{H}}}{Z} \\ p^{-} &= \frac{1}{Z}\langle g|e^{-\beta\hat{H}}|g\rangle = \frac{1}{Z} \end{aligned}\right\} \quad \frac{p^{+}}{p^{-}} = e^{-\beta\Omega}$$

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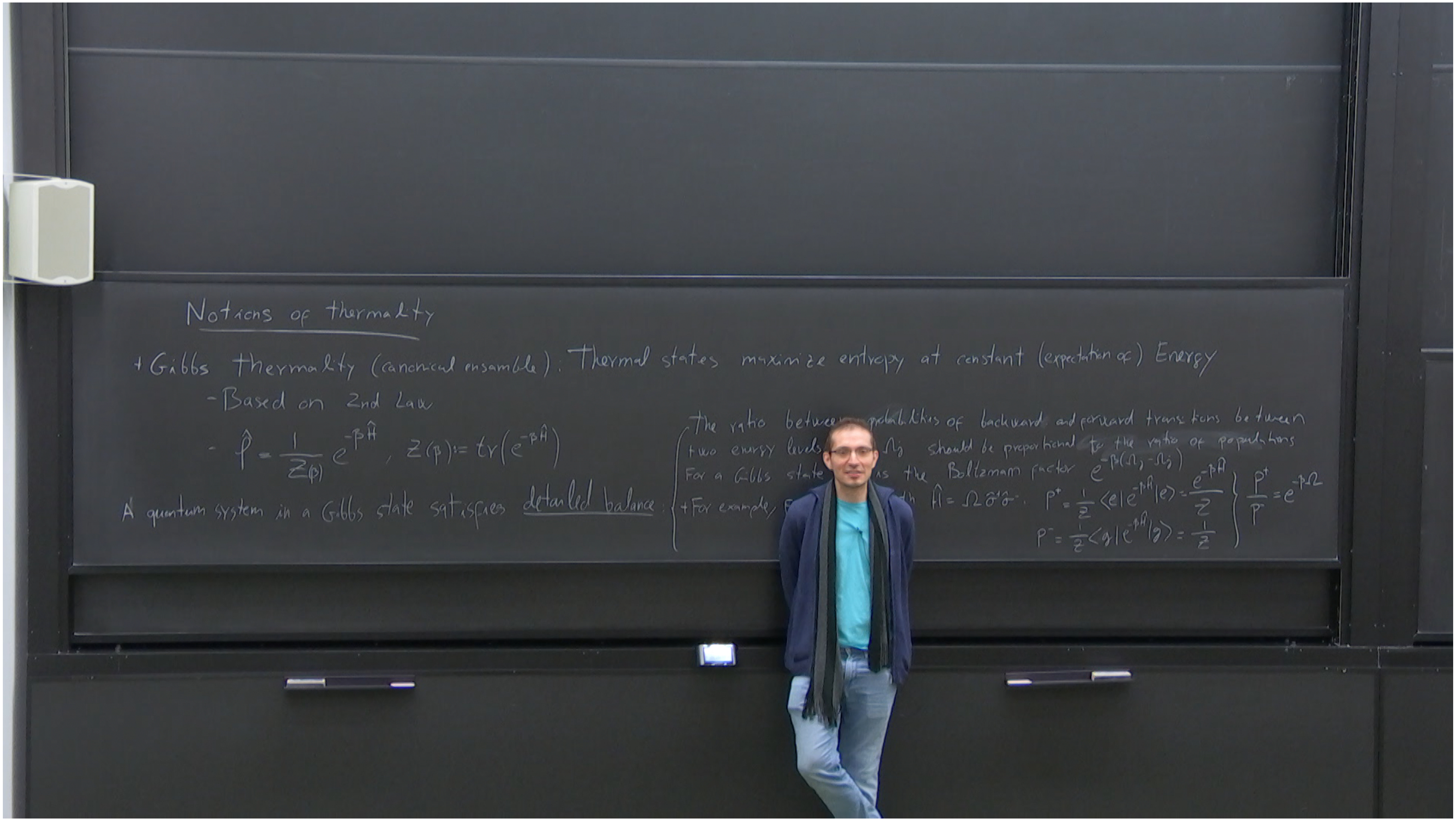

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<u>KMS Thermality</u> (Kubo-Martin-Schwinger)

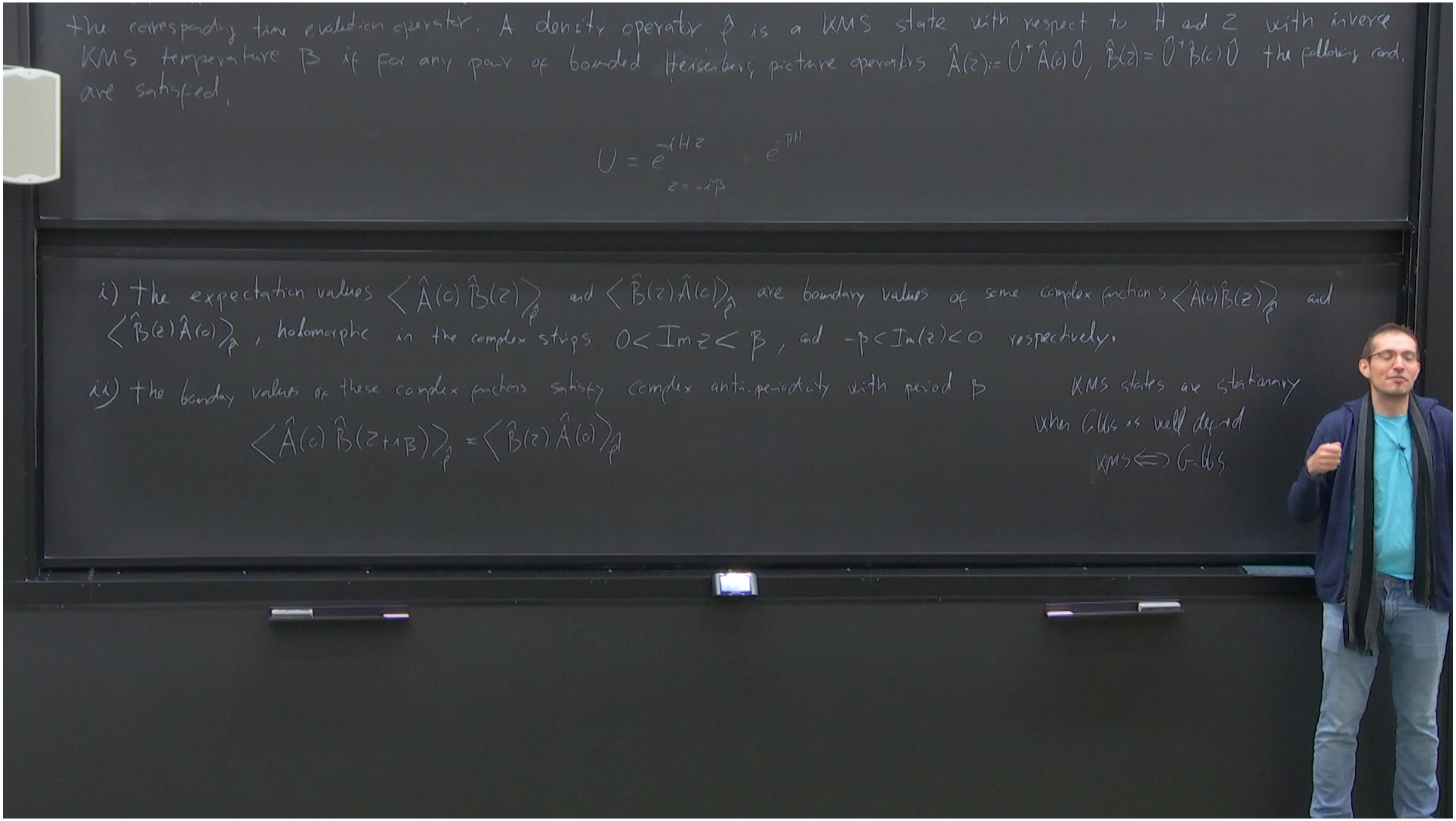
Let $\hat{H}$ be a Hamiltonian that generates time-translations parametrized by a time parameter τ, and let $\hat{U} = e^{-i\hat{H}\tau}$ be the corresponding time evolution operator. A density operator $\hat{\rho}$ is a KMS state with respect to $\hat{H}$ and τ with inverse KMS temperature β if for any pair of bounded Heisenberg picture operators $\hat{A}(\tau) := \hat{U}^{\dagger} \hat{A}(0) \hat{U}$, $\hat{B}(\tau) = \hat{U}^{\dagger} \hat{B}(0) \hat{U}$ the following cond. are satisfied:

Let $\hat{H}$ be a Hamiltonian that generates time-translations parametrized by a time parameter z, and let $\hat{U} = e$ be the corresponding time evolution operator. A density operator $\hat{\rho}$ is a KMS state with respect to $\hat{H}$ and z with inverse KMS temperature β if for any pair of bounded Heisenberg picture operators $\hat{A}(z) := \hat{U}^\dagger \hat{A}(0) \hat{U}$, $\hat{B}(z) = \hat{U}^\dagger \hat{B}(0) \hat{U}$ the following con are satisfied.

i) the expectation values $\langle \hat{A}(0)\hat{B}(z)\rangle_{\hat{\rho}}$ and $\langle \hat{B}(z)\hat{A}(0)\rangle_{\hat{\rho}}$ are boundary values of some complex functions $\langle \hat{A}(0)\hat{B}(z)\rangle_{\hat{\rho}}$ and $\langle \hat{B}(z)\hat{A}(0)\rangle_{\hat{\rho}}$, holomorphic in the complex strips $0 < \mathrm{Im}\, z < \beta$, and $-\beta < \mathrm{Im}(z) < 0$ respectively.

ii) the boundary values of these complex functions satisfy complex anti-periodicity with period β

$$\langle \hat{A}(0)\hat{B}(z+i\beta)\rangle_{\hat{\rho}} = \langle \hat{B}(z)\hat{A}(0)\rangle_{\hat{\rho}}$$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when Gibbs is well defined
KMS $\Longleftrightarrow$ Gibbs

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+ KMS $\Rightarrow$ Stationarity. Stationarity: a state $\hat\rho$ is stationary if $\langle \hat{B}(z) \rangle_{\hat\rho} = \langle \hat{B}(0) \rangle_{\hat\rho}$ for any $\hat{B}$

KMS ii) with $\hat{A} = \mathbb{1}$. For convenience $D := \{ z \in \mathbb{C} : 0 < \operatorname{Im} z < \beta \}$

i) $\Rightarrow$ $f(z) := \langle \hat{B}(z) \rangle_{\hat\rho}$ holomorphic on D and continuous on the boundary ∂D

ii) $f(z + i\beta) = f(z)$

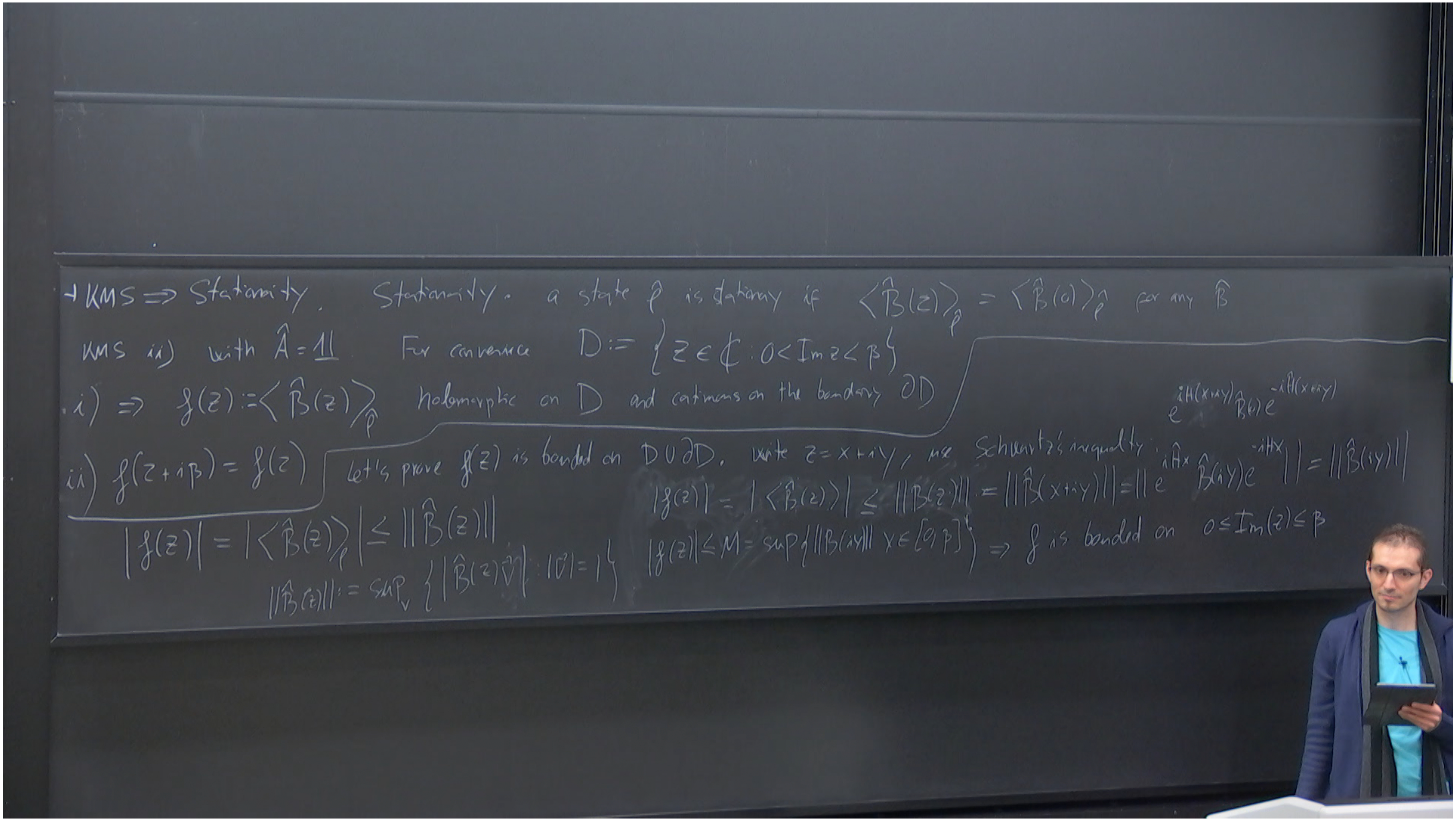
Let's prove $f(z)$ is bounded on $D \cup \partial D$. Write $z = x + iy$, use Schwartz's inequality:

$$|f(z)| = |\langle \hat{B}(z) \rangle| \leq ||\hat{B}(z)|| = ||\hat{B}(x+iy)|| = ||e^{i\hat{H}x} \hat{B}(iy) e^{-i\hat{H}x}|| = ||\hat{B}(iy)||$$

$$|f(z)| = |\langle \hat{B}(z) \rangle_{\hat\rho}| \leq ||\hat{B}(z)||$$

$$|f(z)| \leq M = \sup \{ ||\hat{B}(iy)|| \; y \in [0, \beta] \} \Rightarrow f \text{ is bounded on } 0 \leq \operatorname{Im}(z) \leq \beta$$

$$||\hat{B}(z)|| := \sup_{\vec v} \{ |\hat{B}(z) \vec{v}| : |\vec{v}| = 1 \}$$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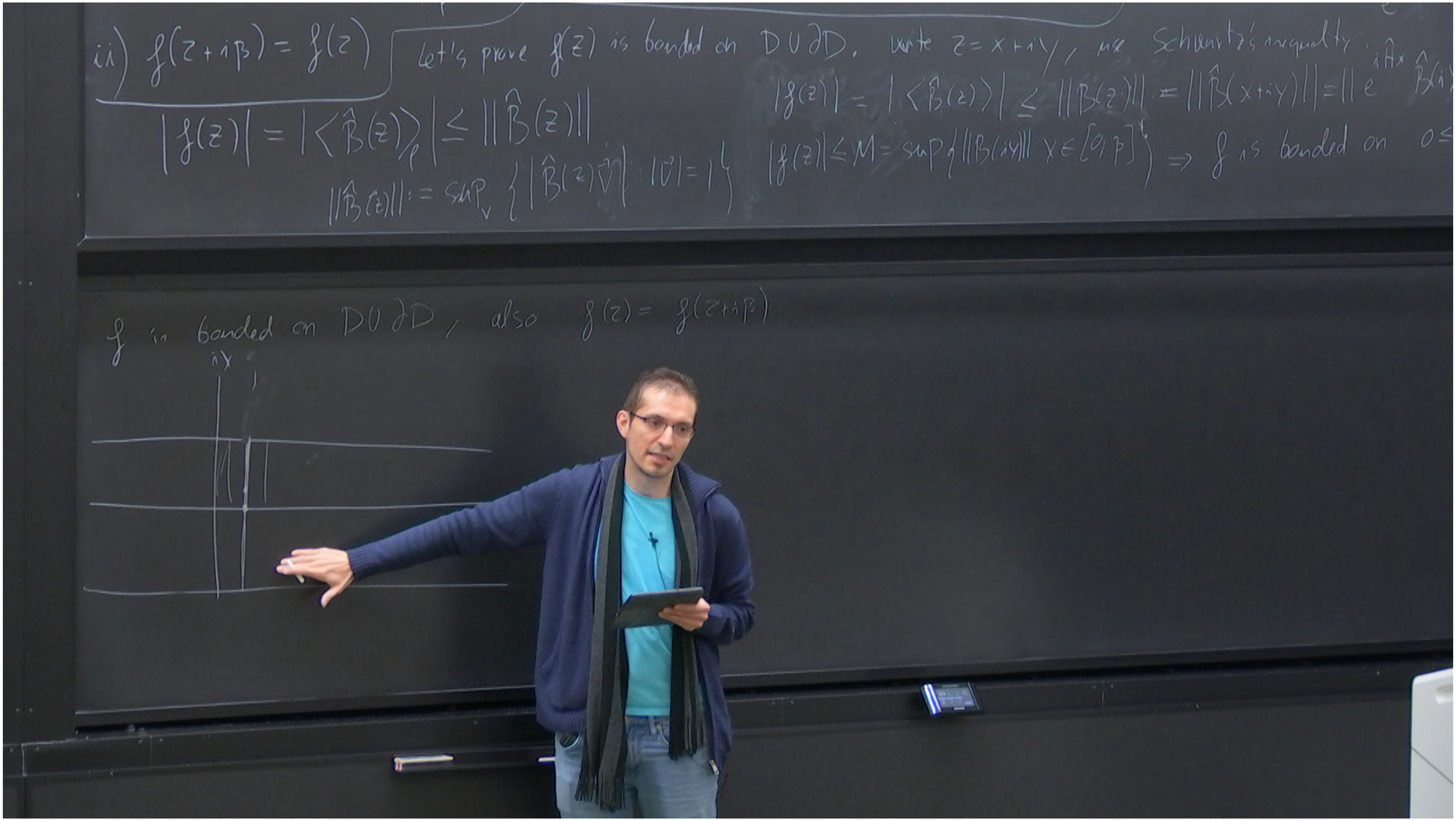
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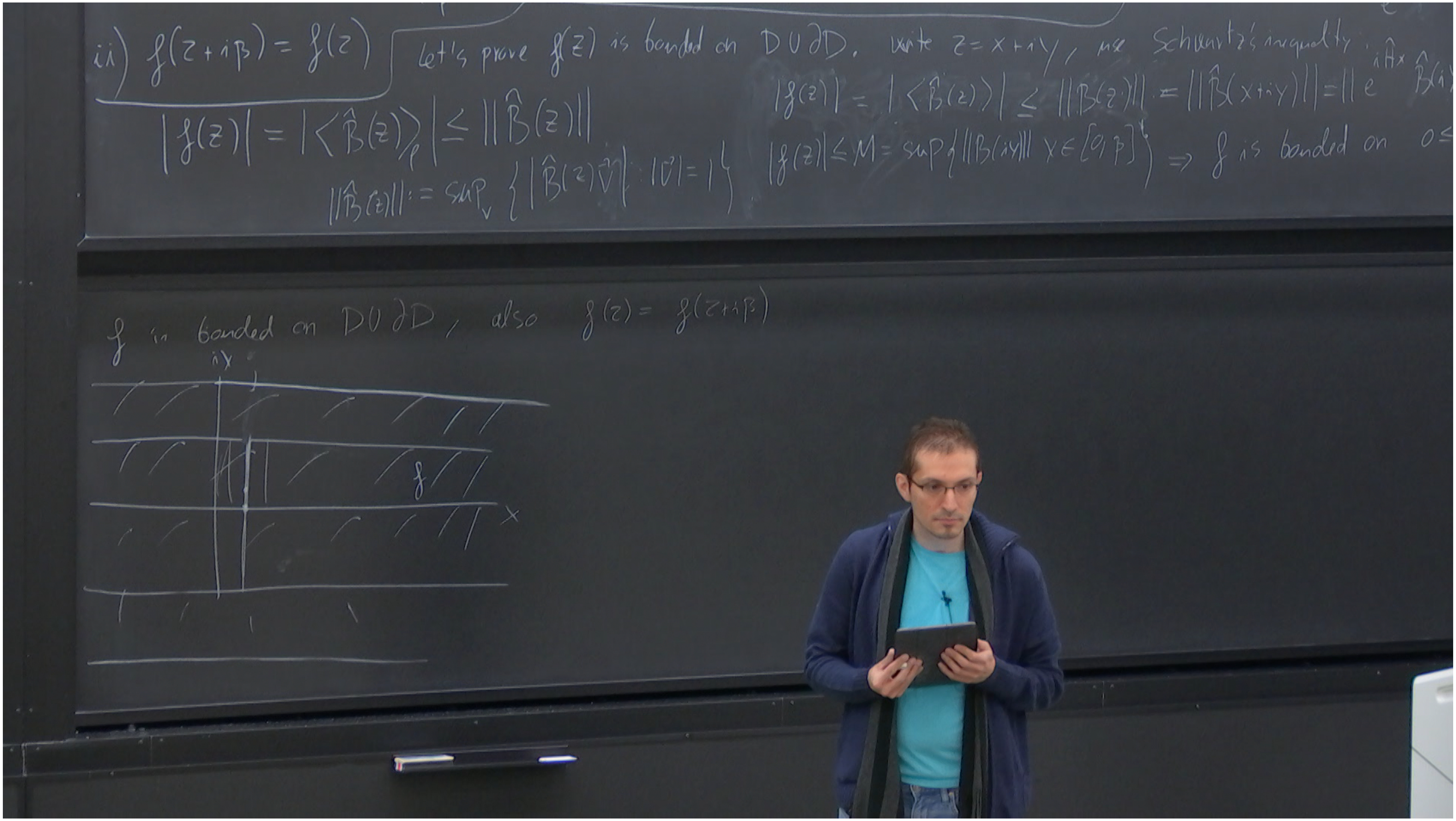
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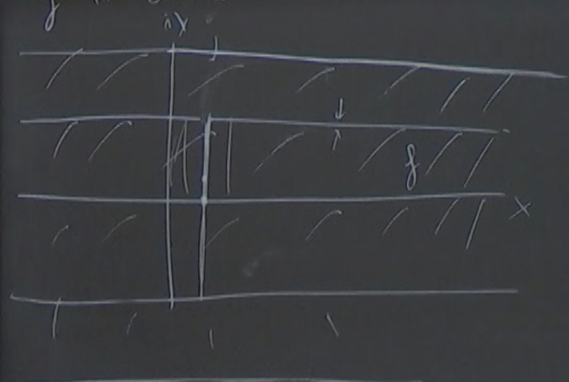

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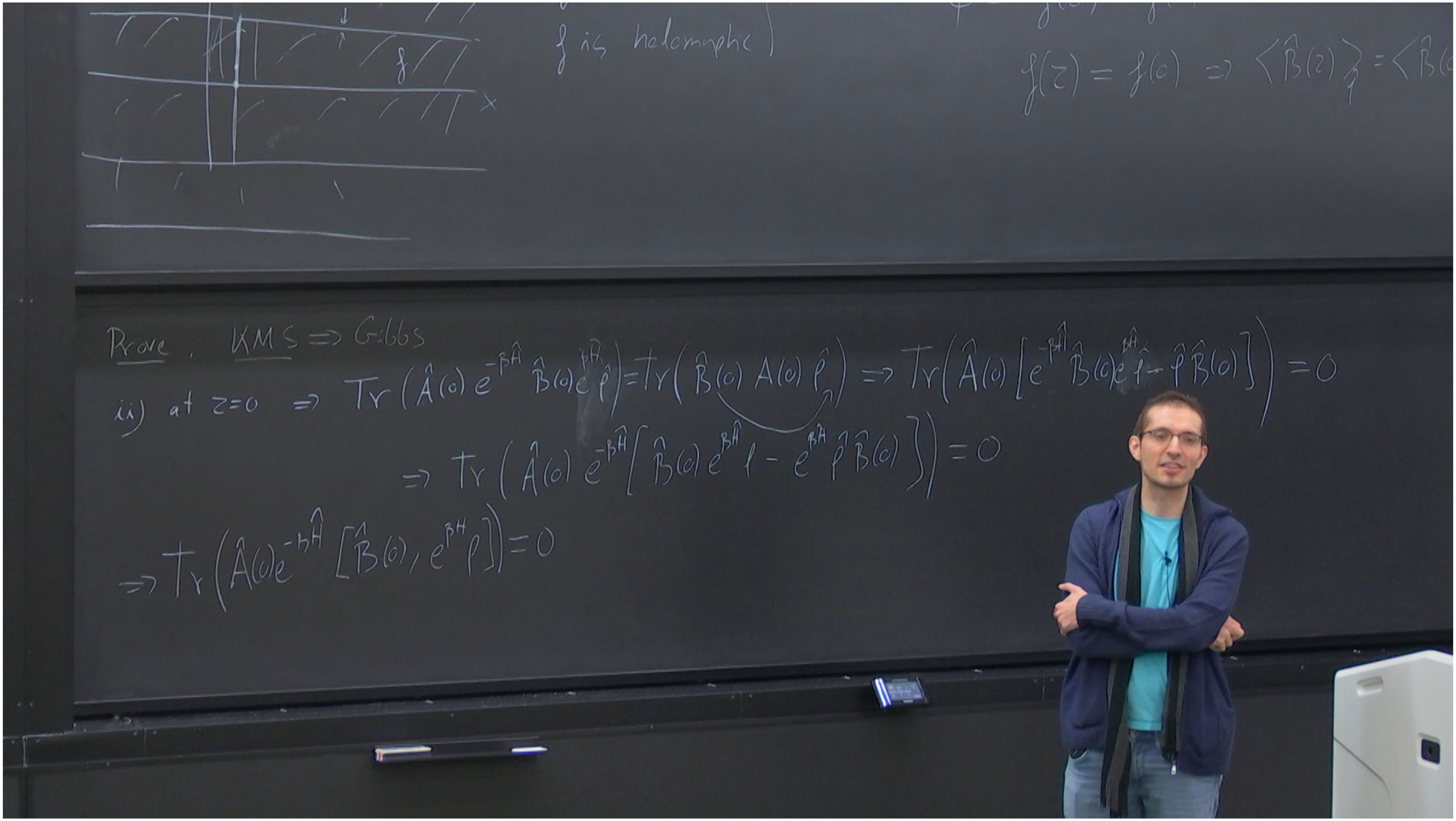
f is holomorphic
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Prove: KMS ⇒ Gibbs
ii) at z=0 $\Rightarrow \mathrm{Tr}(\hat{A}(0) e^{-\beta \hat{H}} \hat{B}(0) e^{\beta \hat{H}} \hat{\rho}) = \mathrm{Tr}(\hat{B}(0) A(0) \hat{\rho}) \Rightarrow \mathrm{Tr}(\hat{A}(0) [e^{-\beta \hat{H}} \hat{B}(0) e^{\beta \hat{H}} \hat{\rho} - \hat{\rho} \hat{B}(0)]) = 0$
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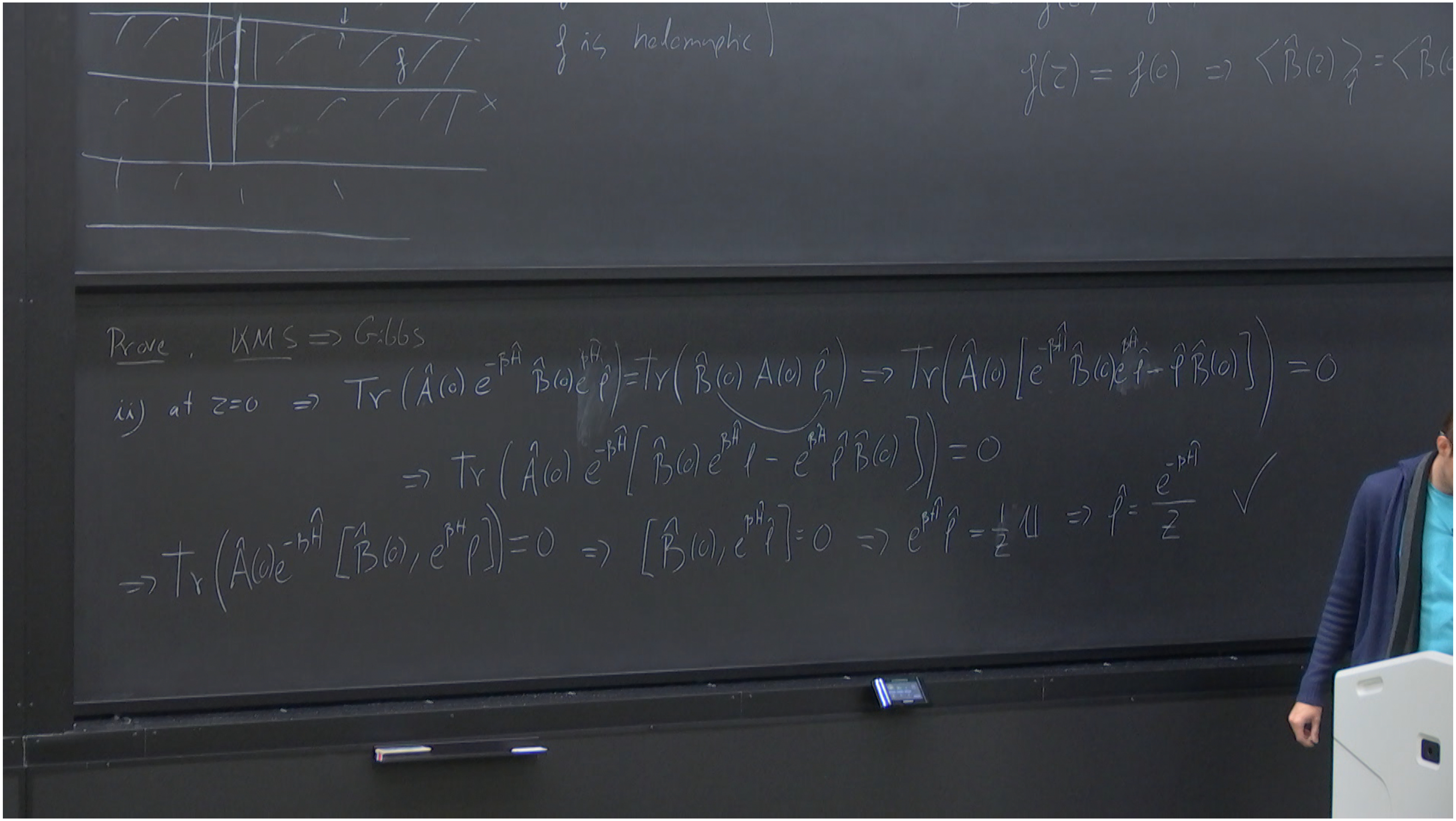
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