Title: PSI 2019/2020 - Standard Model and Beyond part 2 - Lecture 5

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Collection: PSI 2019/2020 - Standard Model and Beyond - Part 2

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SM+B #23

Algebras (continued)

$C_\infty(M) - \infty$

$M_n(\mathbb{C}) - 2n^2$

$\{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\} \to \{1, 2, 4, 8\}$

Lie:

$so(n) \to \frac{n(n-1)}{2}$

$su(n) \to n^2 - 1$

$sp(n) \to n(2n+1)$

$g_2 - 14$

$f_4 - 52$

$e_6 - 78$

$e_7 - 133$

$e_8 - 2$

Example 5: Jordan algebras

$$[A_1, A_2] \to \text{Lie}$$

$$\{H_1, H_2\} = H_1 H_2 + H_2 H_1$$

i) $[A, B] = -[B, A]$

ii) $[[A, B], C] \neq [A, [B, C]]$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

$$\{\{H_1, H_2\}, H_3\} \neq \{H_1, \{H_2, H_3\}\}$$

i) $\{X, Y\} = \{Y, X\} = X \circ Y$

ii) $X^2 = \{X, X\}$

$$(X \circ Y) \circ X^2 = X \circ (Y \circ X^2)$$ Jordan id.

Euclidean iii) $X_1^2 + \dots + X_n^2 = 0$

$$X_1 = \dots = X_n = 0$$

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Euclidean iii) $\underline{X_1^2 + \dots + X_n^2 = 0}$

$\underline{X_1 = \dots = X_n = 0}$

Jordan, von Neumann, Wigner:

$\rightarrow$ i) $J\mathrm{spin}(n) = \{(a, \vec{a})\}$

$(a, \vec{a}) + (b, \vec{b}) = (a+b, \vec{a}+\vec{b})$

$(a, \vec{a})(b, \vec{b}) = (ab + \vec{a}\cdot\vec{b}, a\vec{b} + \vec{a}b)$

ii) $\underline{H_n(\mathbb{R})}$, $\underline{\underline{H_n(\mathbb{C})}}$, $\underline{\underline{H_n(\mathbb{H})}}$

iii) $H_3(\mathbb{O})$

$$\begin{pmatrix} \alpha & x_3 & \bar{x}_2 \\ \bar{x}_3 & \beta & x_1 \\ x_2 & \bar{x}_1 & \gamma \end{pmatrix} \leftarrow 3 + 24 = 27$$

$$\bar{x} = a_0 \underset{1}{e_0} - a_1 e_1 - \dots - a_7 e_7$$

$X \circ Y = \frac{1}{2}(XY + YX)$

$X \circ Y = \frac{1}{2}\{X, Y\}$

$\{X, Y\} = XY + YX$

Algebra Automorphisms:

$$\alpha : A \to A \qquad \text{(invertible, linear)}$$

$$\alpha(a_1 a_2) = \alpha(a_1)\,\alpha(a_2)$$

Derivations: $\alpha \approx 1 + D$

$$\Rightarrow \cancel{(a_1 a_2)} + D(a_1 a_2) \approx [a_1 + D(a_1)][a_2 + D(a_2)]$$

$$D : A \to A \qquad \approx \cancel{a_1 a_2} + D(a_1)\,a_2 + a_1\,D(a_2)$$

$$\Rightarrow \boxed{D(a_1 a_2) = D(a_1)\,a_2 + a_1\,D(a_2)} \leftarrow \begin{matrix} D_1 + D_2 \\ [D_1, D_2] \end{matrix}$$

Algebra Automorphisms: $\boxed{\mathrm{Aut}(A)}$

$\alpha : A \to A$ (invertible, linear)

$$\alpha(a_1 a_2) = \alpha(a_1)\,\alpha(a_2)$$

Derivations: $\alpha \approx 1 + D$ $\boxed{\mathrm{Der}(A)}$

$$\cancel{(a_1 a_2)} + D(a_1 a_2) \approx [a_1 + D(a_1)][a_2 + D(a_2)]$$

$D : A \to A$ $$\approx \cancel{a_1 a_2} + D(a_1)\,a_2 + a_1\,D(a_2)$$

$$\Rightarrow \boxed{D(a_1 a_2) = D(a_1)\,a_2 + a_1\,D(a_2)} \leftarrow \begin{matrix} D_1 + D_2 \\ [D_1, D_2] \end{matrix}$$

<u>Exercise:</u> if A is associative or Lie

$a \in A$ $\quad \underline{\underline{D_a = [a, \_\_]}}$ is a derivation

(inner derivations)

$$D_a(x) = \underline{\underline{[a, \underline{\underline{x}}]}} \leftarrow$$

$$D_a([\underline{\underline{x}}, y]) = [D_a(x), y] + [x, D_a(y)] \leftarrow \text{true by Jacobi}$$

$$[a, xy] = [a, x]\, y + x\, [a, y]$$

$$D_{x,y}(\underline{a}) = \underline{\underline{[[x,y],\underline{a}] - 3[x,y,\underline{a}]}}$$

Alternative alg.

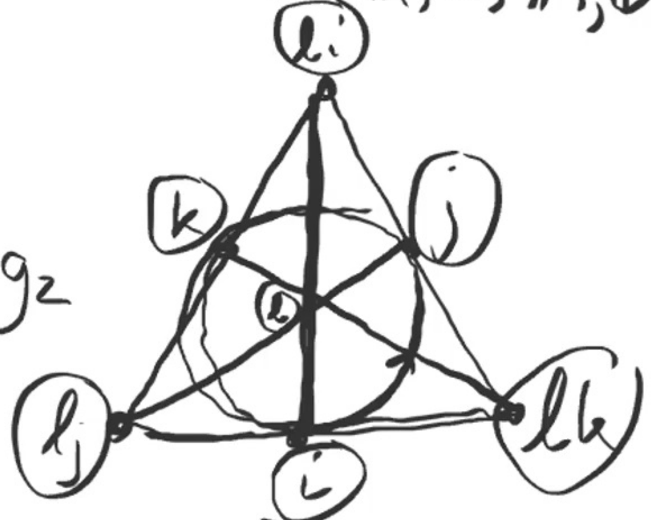
$$[a,b,c] = (ab)c - a(bc)$$

all $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

$$D_{x,y} = -D_{y,x}$$

$$\mathbb{H} : \binom{3}{2} = 3 \longrightarrow so(3) \cong su(2)$$

$$\mathbb{O} : \binom{7}{2} = \frac{7 \cdot 6}{2} = \underline{\underline{21}} - 7 = \underline{\underline{14}} \longrightarrow g_2$$



$$D_{lk,lj} + D_{li,l} + D_{j,k} = 0$$

$$g_2 \underset{G}{=} Der(\mathbb{O})$$

$\mathbb{R} \to 0$

$\mathbb{C} \to 0 \qquad i \to -i \quad \mathbb{Z}_2$

$\mathbb{H} \to so(3)$

$\mathbb{O} \to g_2$

Jordan:

$$SO(n)$$

i) $J\mathrm{spin}(n) \to \mathrm{Spin}(n)$

ii) $H_n(\mathbb{R}) \to SO(n) \leftarrow$

$H_n(\mathbb{C}) \to SU(n) \leftarrow$

$H_n(\mathbb{H}) \to Sp(n) \leftarrow$

iii) $A = H_3(\mathbb{O}) \to \quad \mathrm{Aut}(A) = F_4$

$\underline{\underline{H_n(\mathbb{K})}}$ $\qquad$ $\mathbb{K} = \underline{\underline{\mathbb{R}, \mathbb{C}, \mathbb{H}}}, \underline{\mathbb{O}}$

$\underline{\underline{D_A(X)}} = [A, X]$ $\qquad$ $X \in H_n(\mathbb{K})$

$A \in A_n(\mathbb{K})$

$$[A, \{X, Y\}] = \{[A,X], Y\} + \{X, [A,Y]\}$$

when $\underline{\mathbb{K} = \mathbb{O}}$: if $n=2$

or if $n=3$ and $\mathrm{tr}(A)=0$

Jordan:

$$SO(n)$$

i) $J\mathrm{spin}(n) \to \mathrm{Spin}(n)$

ii) $H_n(\mathbb{R}) \to SO(n) \leftarrow$

$H_n(\mathbb{C}) \to SU(n) \leftarrow$

$H_n(\mathbb{H}) \to SQ(n) \leftarrow$

iii) $A = H_3(\mathbb{O}) \to \begin{cases} \to \mathrm{Aut}(A) = F_4 \\ \to \mathrm{Der}(A) = f_4 \end{cases}$

$$\left[A , \underline{\underline{\begin{pmatrix} \alpha & x_3 & \bar{x}_2 \\ \bar{x}_3 & \beta & x_1 \\ x_2 & \bar{x}_1 & \gamma \end{pmatrix}}} \right]$$

$$\begin{pmatrix} \alpha & x_3 & \bar{x}_2 \\ \bar{x}_3 & \beta & x_1 \\ x_2 & \bar{x}_1 & \gamma \end{pmatrix}$$

$$\begin{pmatrix} \alpha & x_3 & \bar{x}_2 \\ \bar{x}_3 & \beta & x_1 \\ x_2 & \bar{x}_1 & \gamma \end{pmatrix} \qquad \begin{pmatrix} \bullet & \bullet & \bullet \\ \cdot & \bullet & \bullet \\ \cdot & \cdot & \bullet \end{pmatrix} \begin{matrix} 24+14 \\ =38 \end{matrix}$$

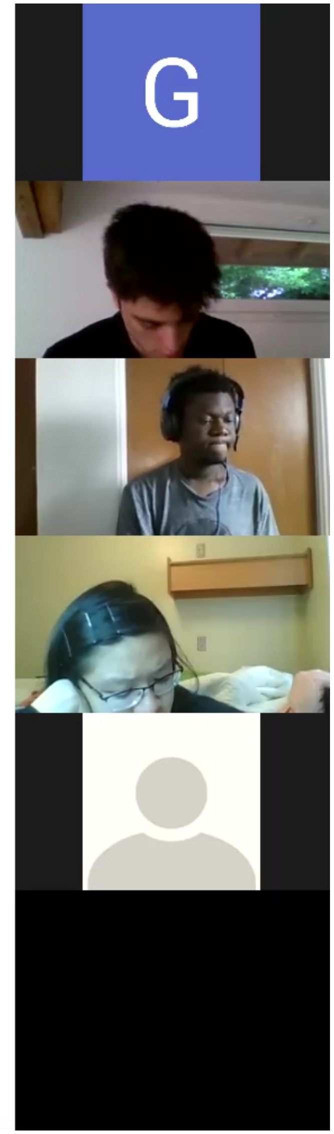
$$\mathrm{Der}[H_3(\mathbb{K})] = \underbrace{A_3'(\mathbb{O})}_{} \oplus \underbrace{\mathrm{Der}(\mathbb{O})}_{}$$

$$\underset{\underset{f_4}{\uparrow}}{52} = \underbrace{38 + 14}_{}$$

Jspin

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}$$

$$\{a\mathbb{1} + a^i\gamma^i, b + b^i\gamma^i\}$$



$$H_2(\mathbb{K}) \qquad H_3(\mathbb{K})$$

$$\parallel \quad \nu = \dim(\mathbb{K})$$

$$J\mathrm{spin}(\nu+1)$$