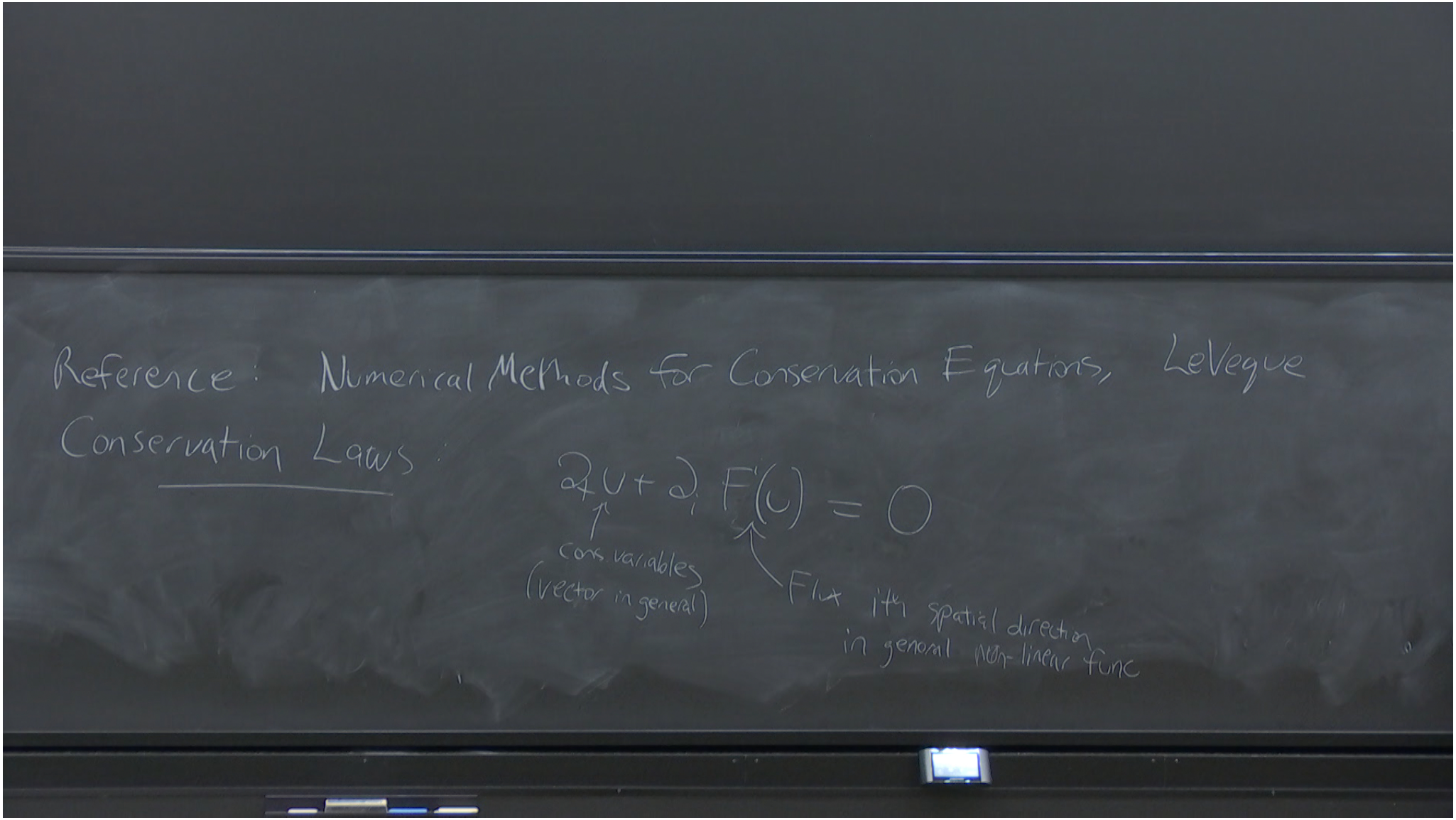
Title: PSI 2019/2020 - Computational Physics - Lecture 12

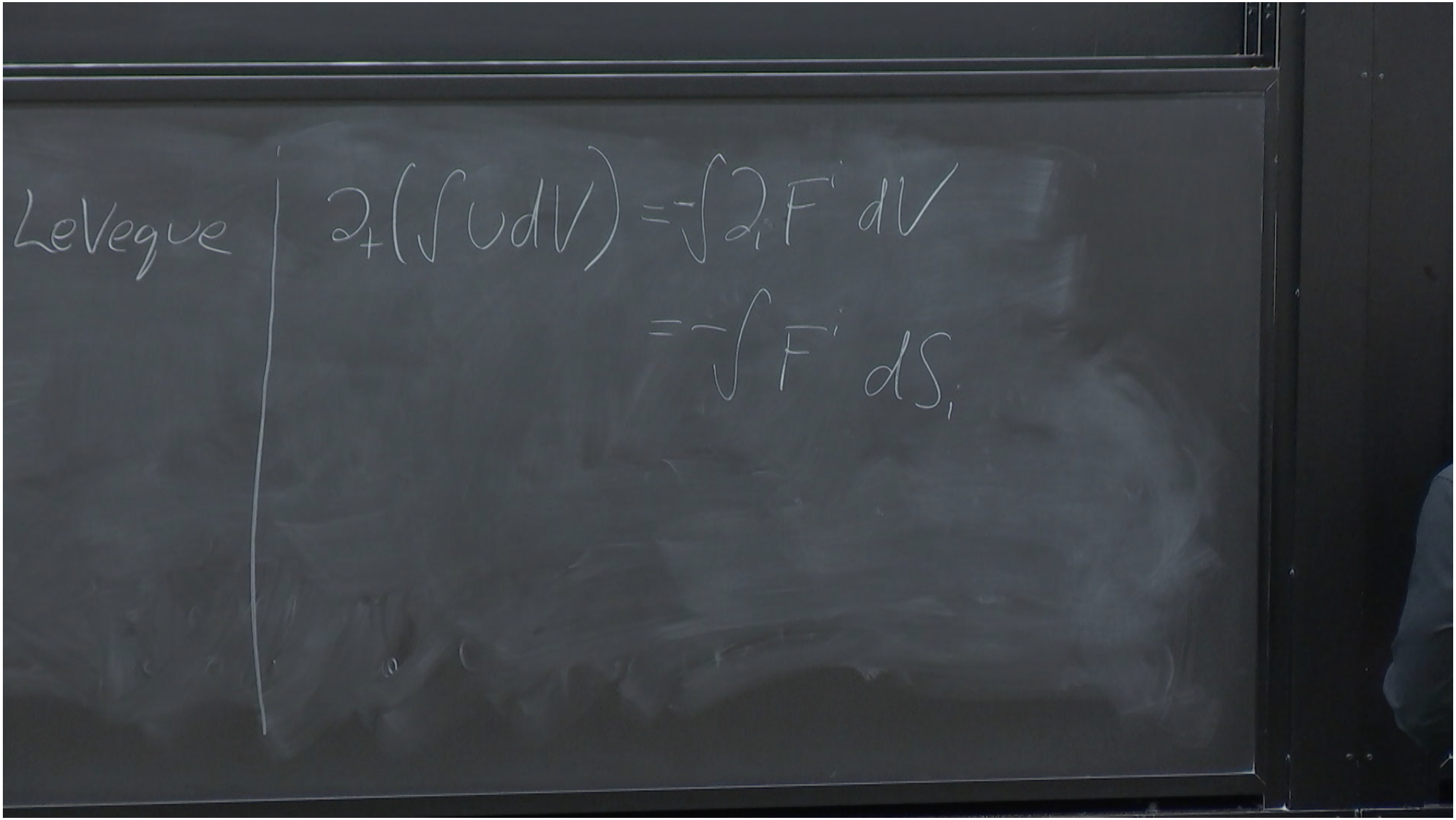
Speakers: Erik Schnetter

Collection: PSI 2019/2020 - Computational Physics

Date: March 02, 2020 - 1:30 PM

URL: http://pirsa.org/20030028


Reference: Numerical Methods for Conservation Equations, LeVeque
Conservation Laws:
$\partial_t U + \partial_i F(U) = 0$
cons. variables (vector in general)
Flux in i^th spatial direction
in general non-linear func


LeVeque
$\partial_t \int_V u\,dV = -\int_S F \cdot dV$
$= -\int_S F \cdot dS$

Euler Eqns:

$$\partial_t \rho + \partial_i (\rho V^i) = 0 \qquad \text{Cons. of mass}$$

$$\partial_t E + \partial_i (V^i (E + P)) = 0 \qquad \text{Cons. of energy}$$

$$\partial_t (\rho v_j) + \partial_i (\rho v^i v_j + \delta^i_j P) = 0 \qquad \text{Cons. of mom.}$$

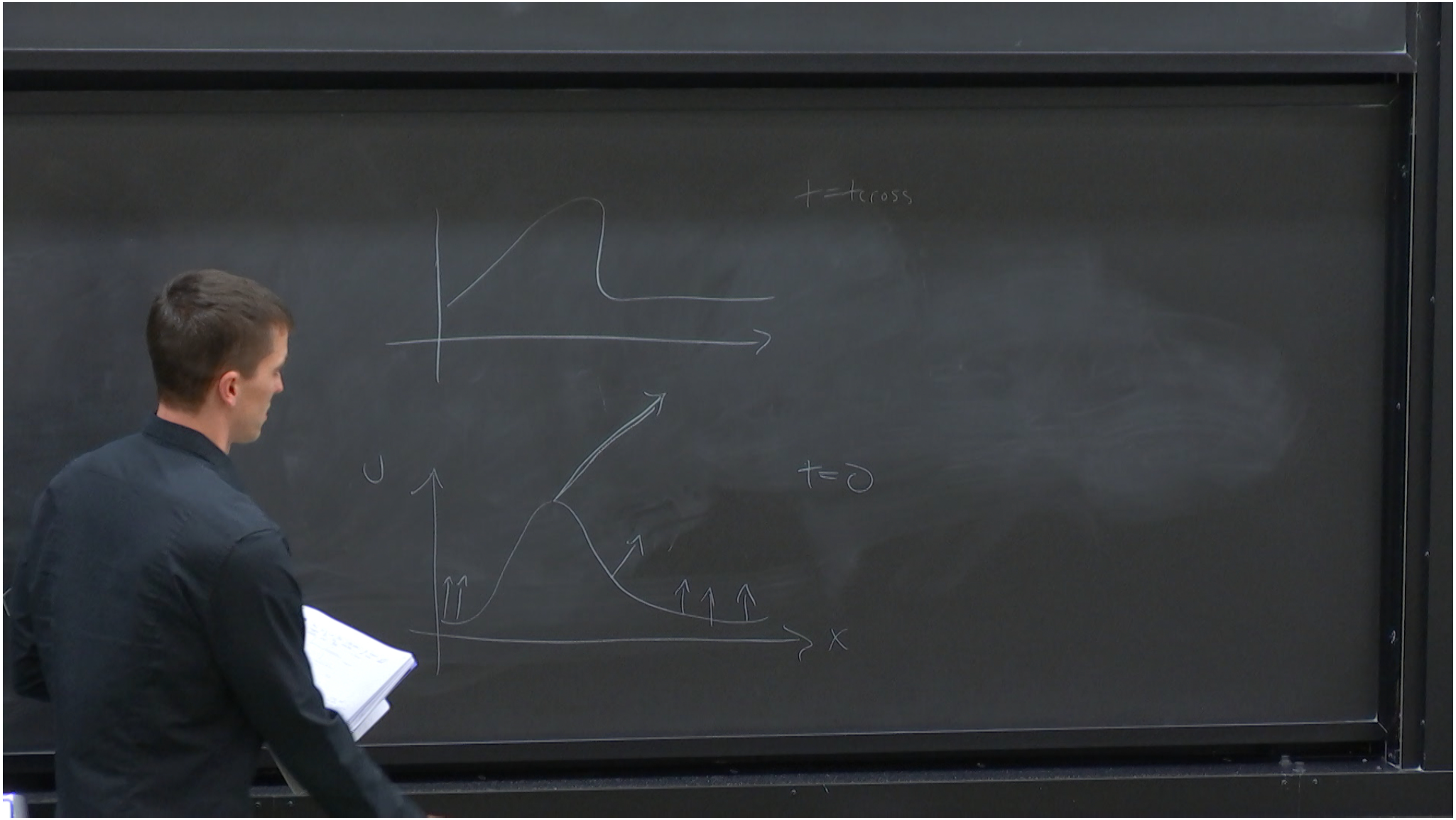
Main difference from other hyperbolic PDEs: Shocks

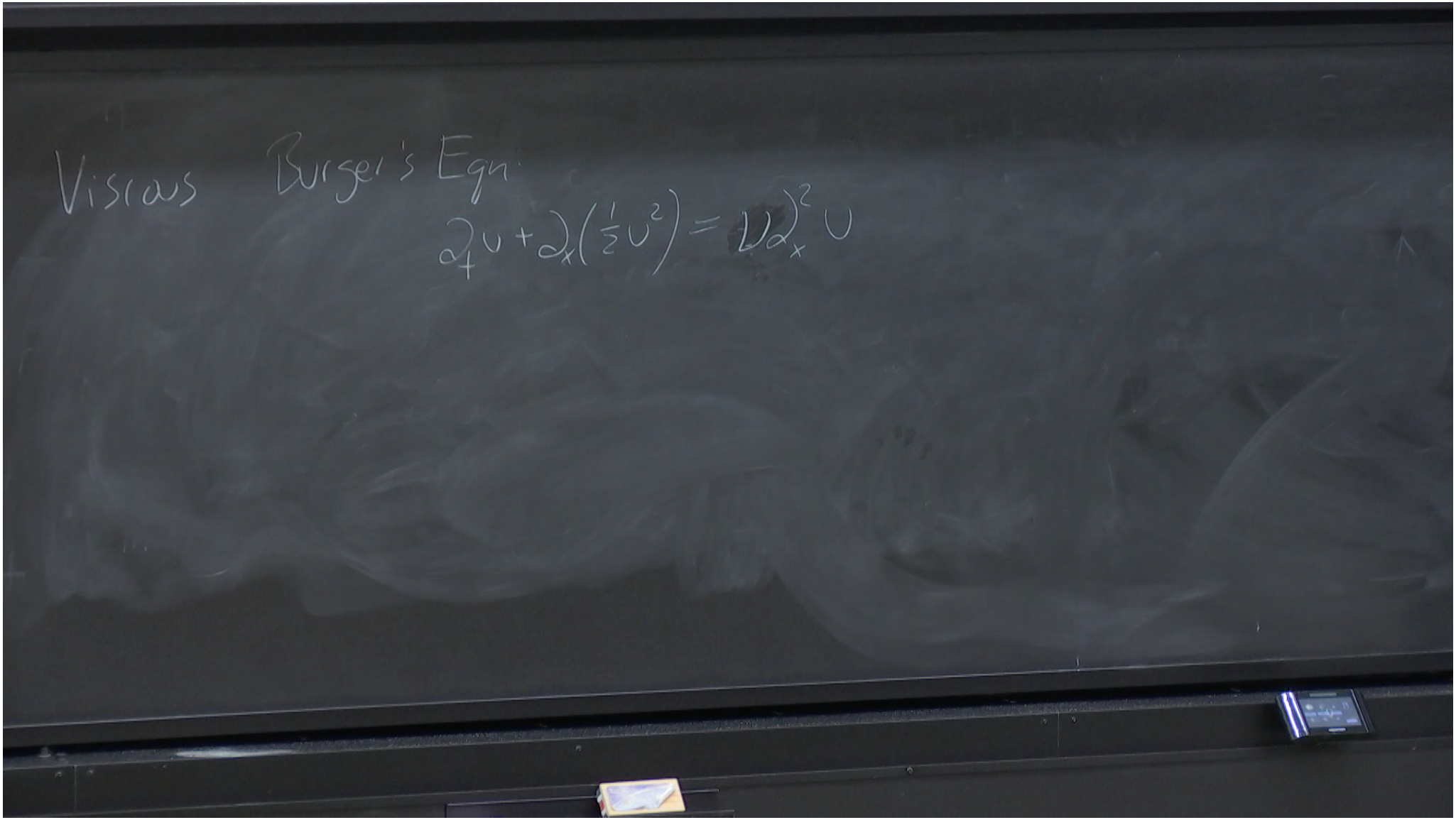
(vector in general)

F_{i+} i^{th} spatial direction
in general non-linear func

Burger's Eqn $\partial_t u + \partial_x\left(\frac{1}{2}u^2\right) = 0$

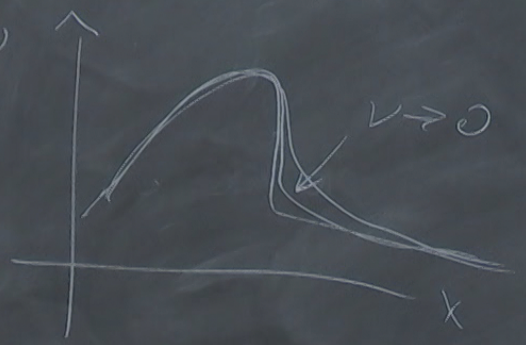
$$\frac{d}{dt}[u(x(t),t)] = \partial_t u + x'\partial_x u$$
$$= u\partial_x u + x'\partial_x u$$
$$= 0 \quad \text{if} \quad x'(t) = u(x(t),t)$$


t=tcross
t=0
x


Viscous Burger's Eqn:
$\partial_t u + \partial_x(\frac{1}{2}u^2) = \nu\partial_x^2 u$

Viscous Burger's Eqn:

$$\partial_t u + \partial_x\left(\frac{1}{2}u^2\right) = \nu \partial_x^2 u$$



$x'(t) = v(x(t), t)$

Let $\phi(x,t)$ be C^1 function with compact support

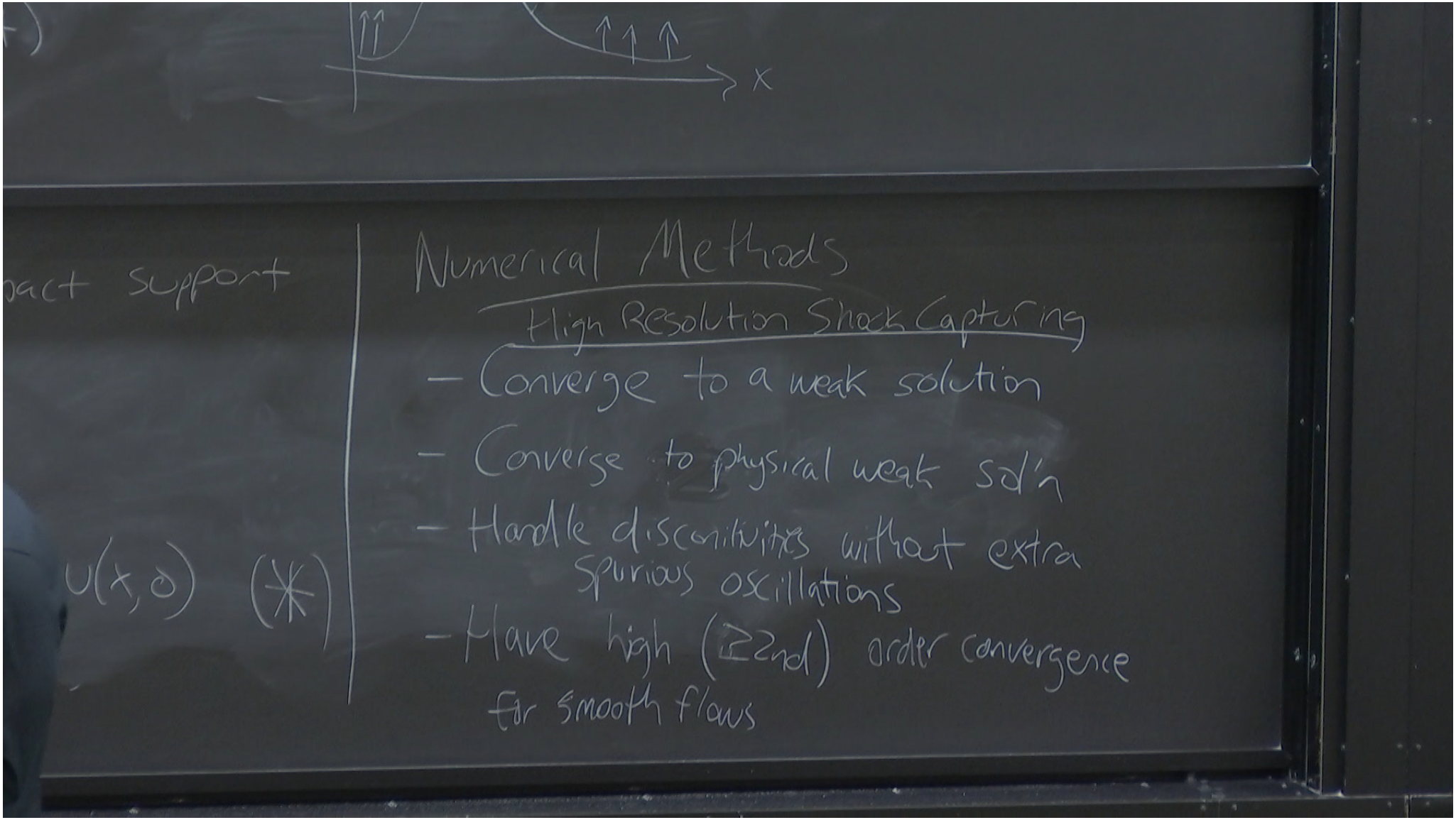
zero outside $x \in [x_1, x_2]$ $\quad t \in [t_1, t_2]$

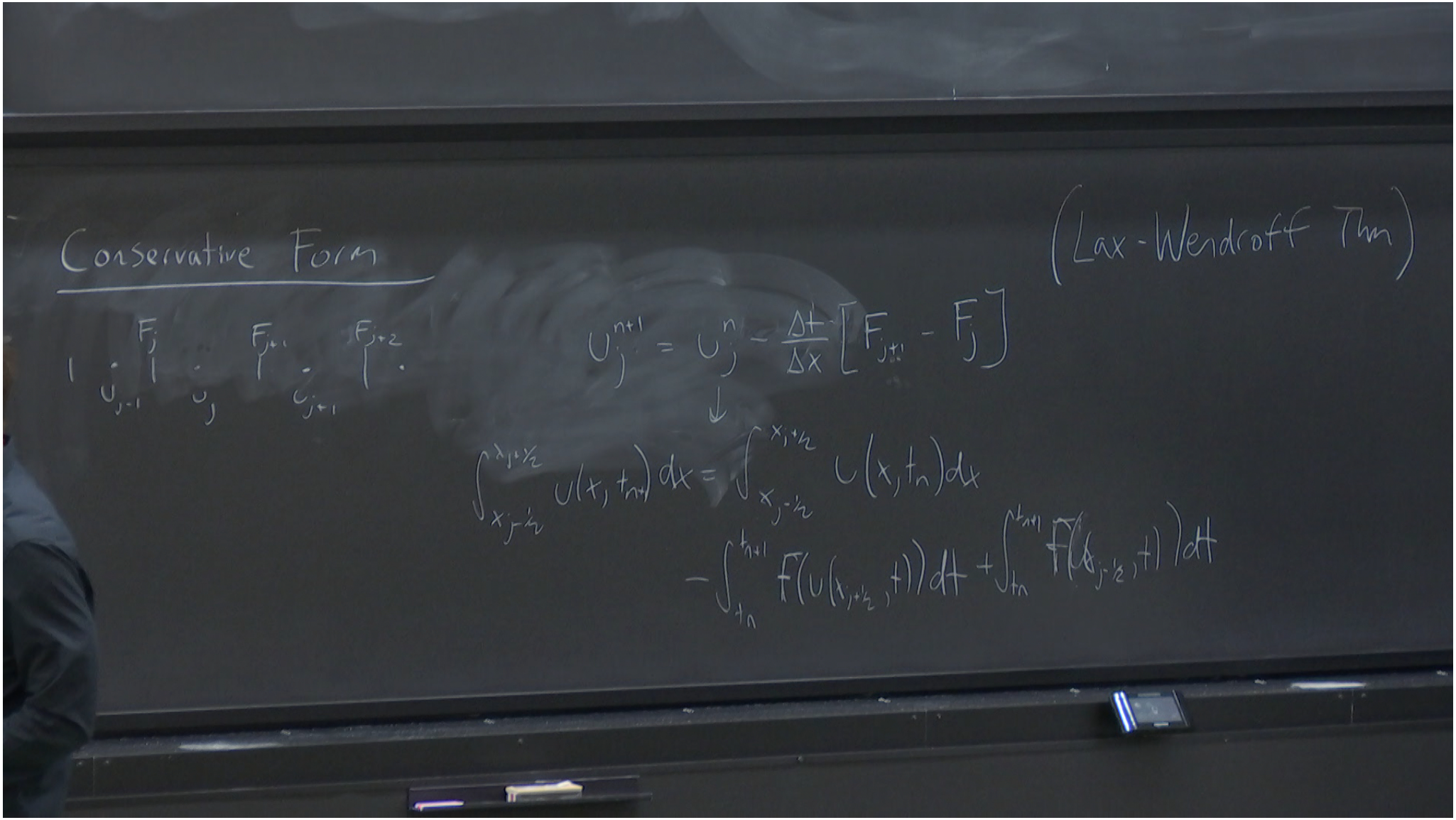
$$0 = \int_0^\infty dt \int_{-\infty}^\infty dx \; \phi [\partial_t u + \partial_x F]$$

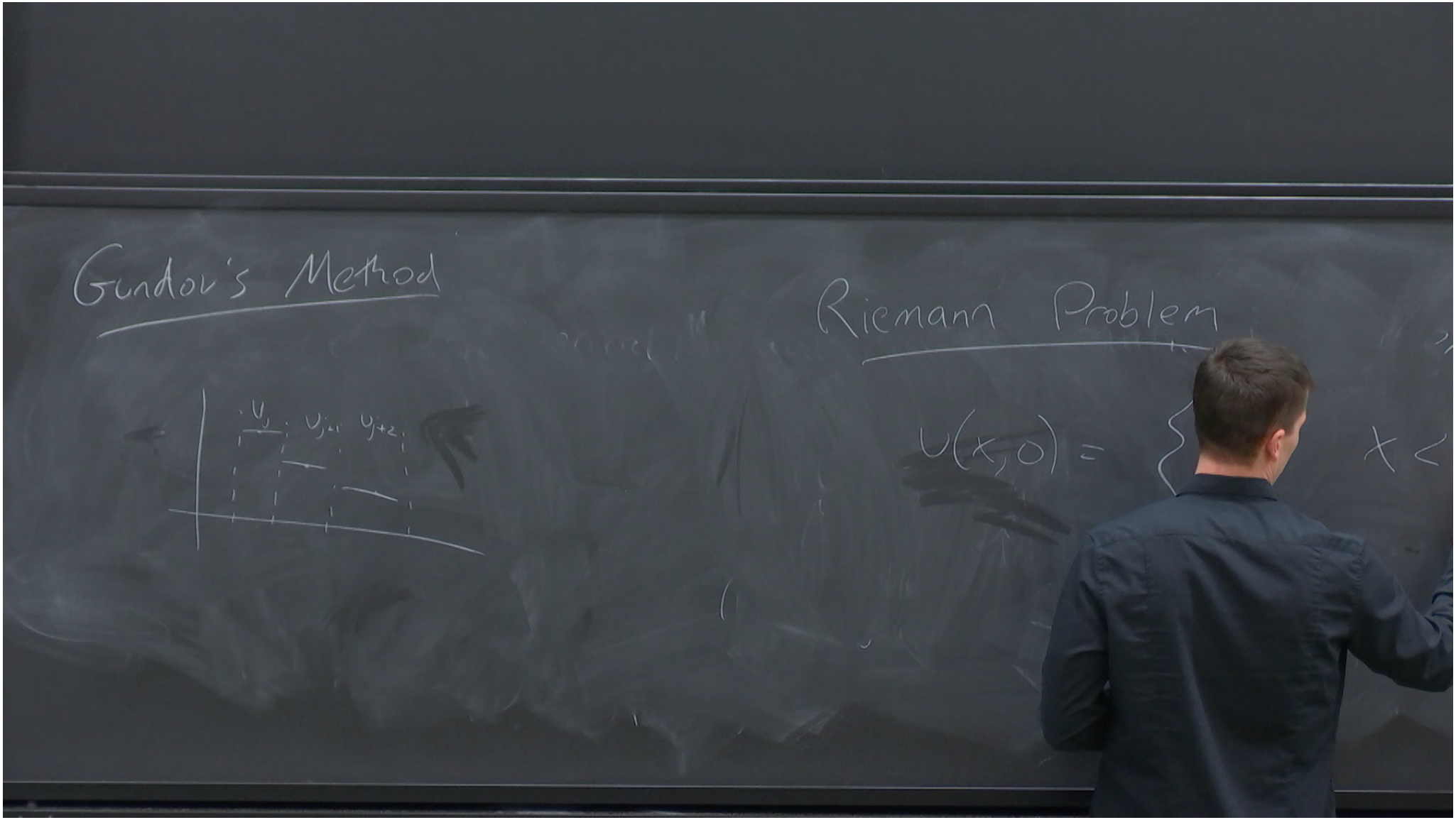
$$\Rightarrow \int_0^\infty dt \int_{-\infty}^\infty dx \left[(\partial_t \phi) u + (\partial_x \phi) F \right] = \int_{-\infty}^\infty dx \; \phi(x,0) \, u(x,0) \quad (*)$$

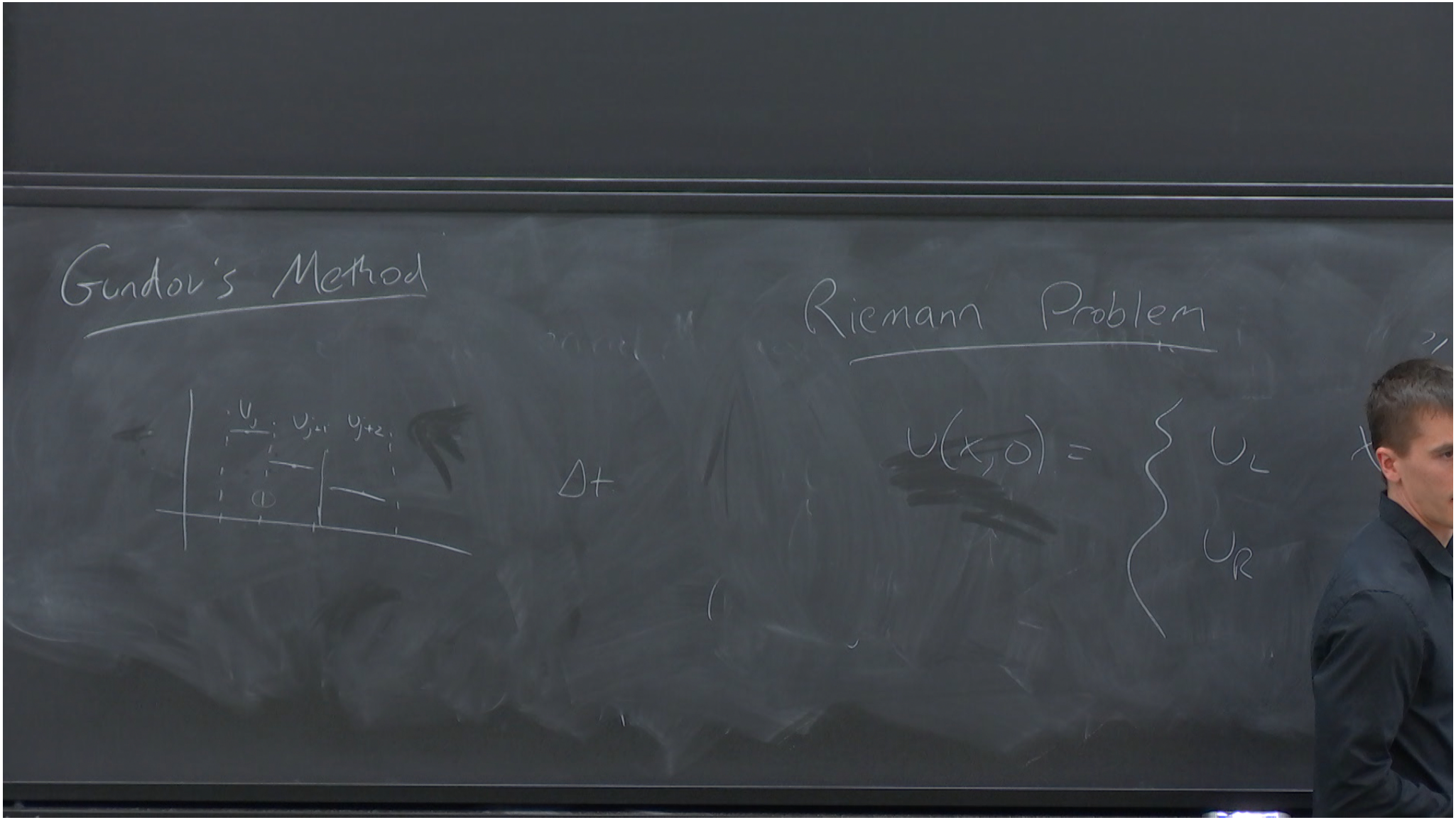
Integral form:

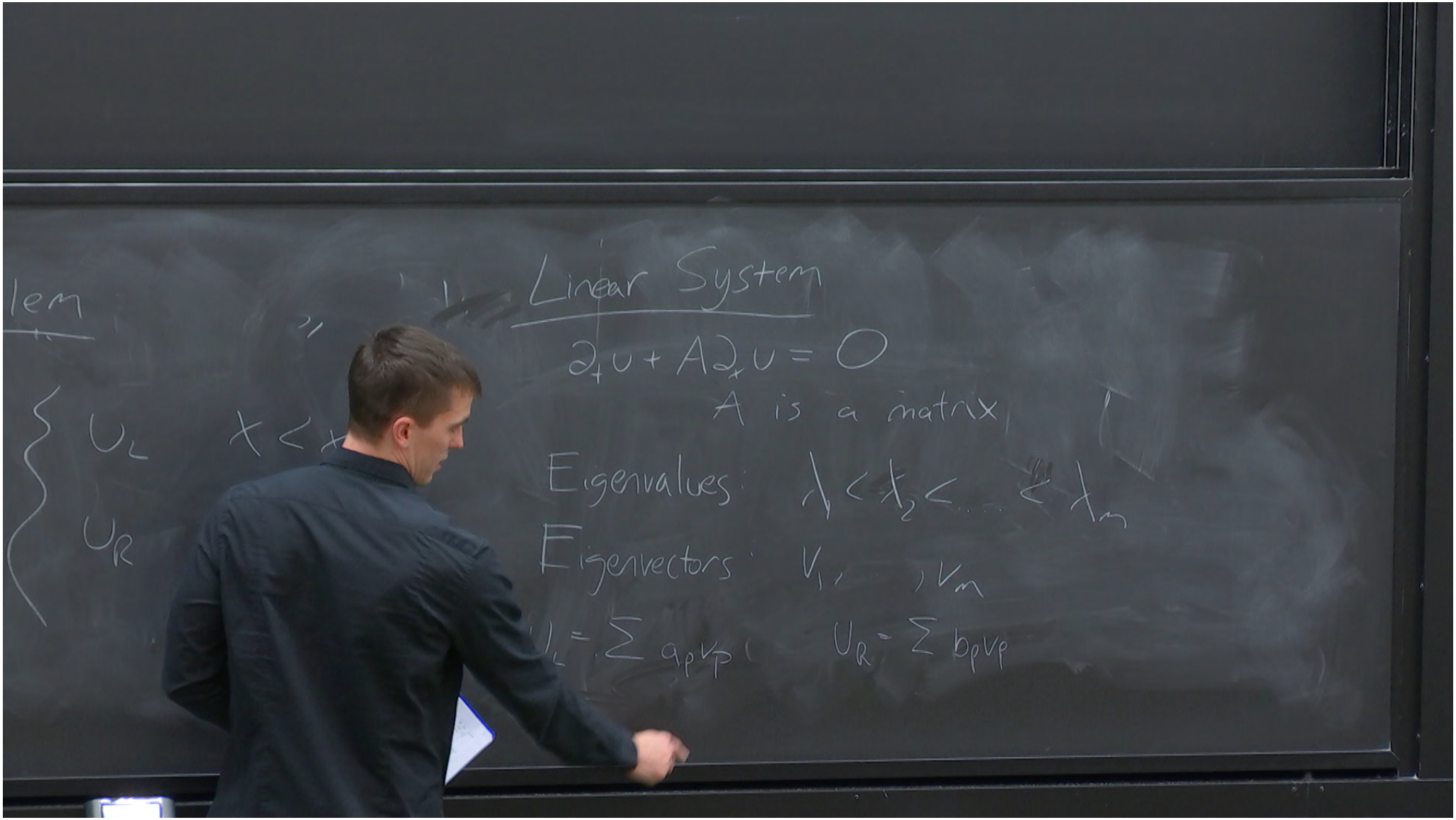
$$\int_{x_1}^{x_2} u(x,t_2)\,dx = \int_{x_1}^{x_2} u(x,t_1)\,dx + \int_{t_1}^{t_2} F(u(x_1,t))\,dt - \int_{t_1}^{t_2} F(u(x_2,t))\,dt$$

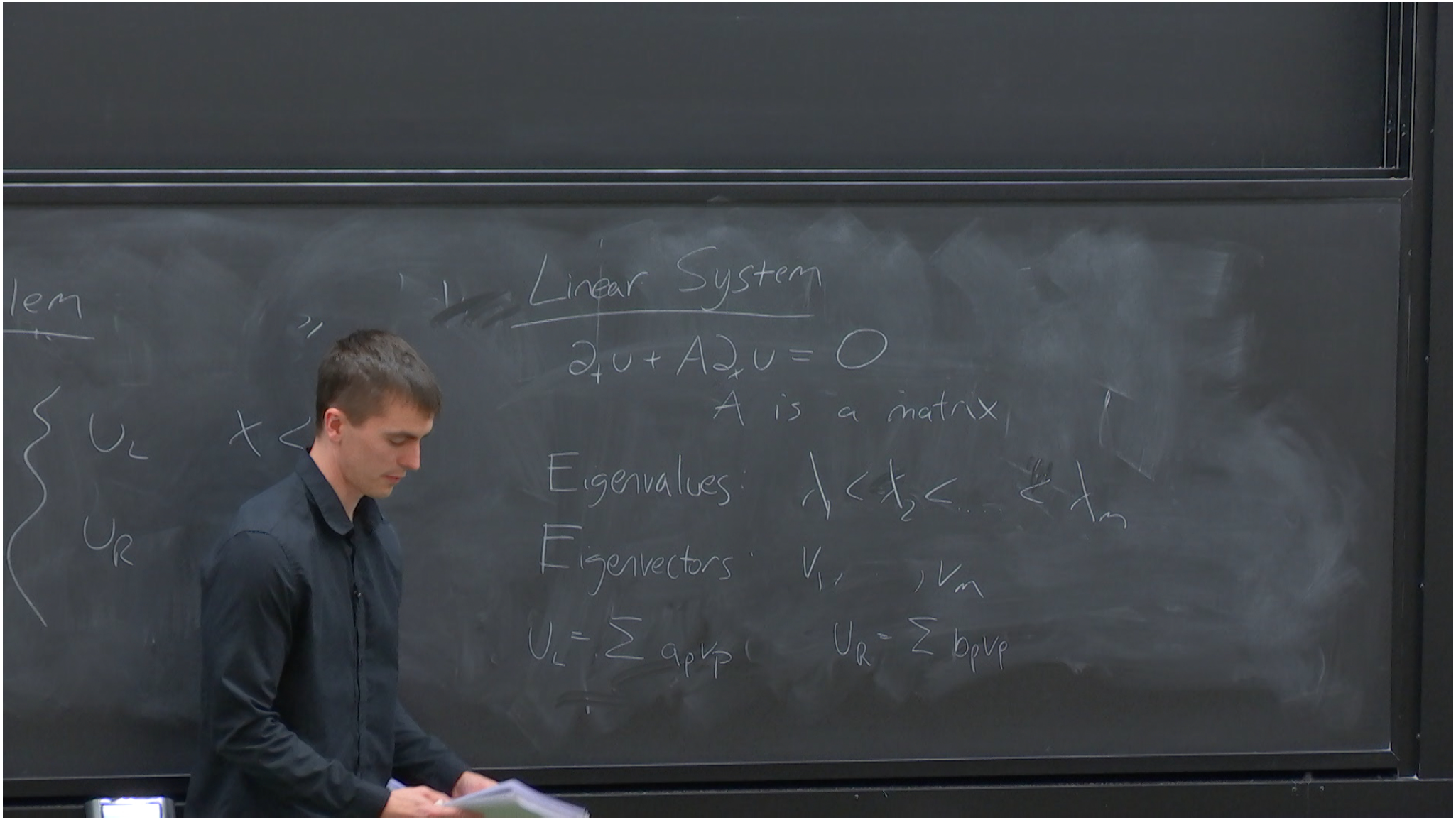

x
pact support
u(x,0) (*)
Numerical Methods
High Resolution Shock Capturing
- Converge to a weak solution
- Converge to physical weak sol'n
- Handle discontinuities without extra spurious oscillations
- Have high (≥2nd) order convergence for smooth flows


Conservative Form
(Lax-Wendroff Thm)
$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left[F_{j+1/2} - F_{j-1/2}\right]$
$\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_{n+1})\,dx = \int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_n)\,dx$
$-\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2},t))\,dt + \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2},t))\,dt$


Godunov's Method
U_j U_{j+1} U_{j+2}
Riemann Problem
$U(x,0) =$
$x <$

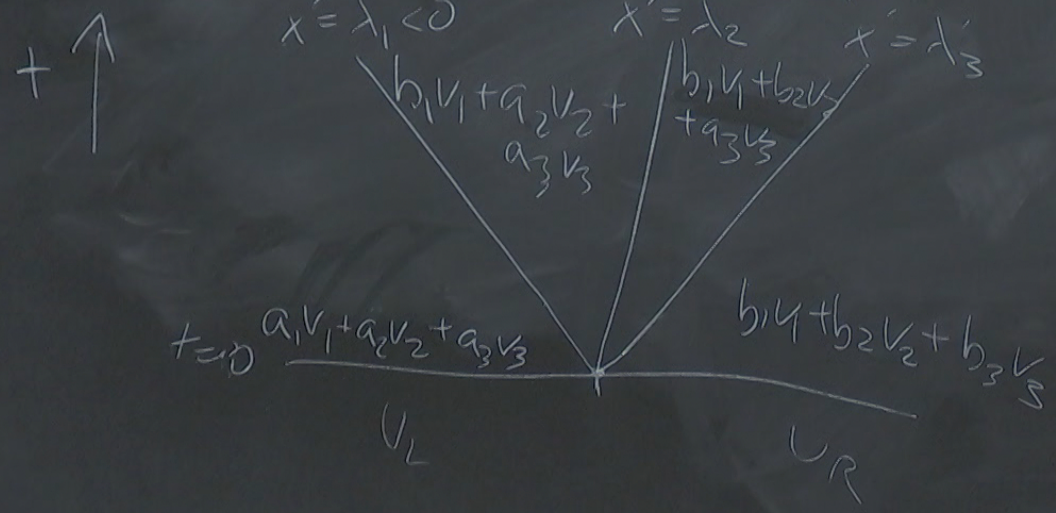

Godunov's Method
Riemann Problem
U_j U_{j+1} U_{j+2}
Δt
$U(x,0) = \begin{cases} U_L \\ U_R \end{cases}$


Linear System
$\partial_t u + A \partial_x u = 0$
A is a matrix
Eigenvalues: $\lambda_1 < \lambda_2 < \dots < \lambda_m$
Eigenvectors: $v_1, \dots, v_m$
$u_L = \sum a_p v_p$
$u_R = \sum b_p v_p$


Linear System
$\partial_t u + A \partial_x u = 0$
A is a matrix
Eigenvalues: $\lambda_1 < \lambda_2 < \ldots < \lambda_m$
Eigenvectors: $v_1, \ldots, v_m$
$U_L = \sum a_p v_p$
$U_R = \sum b_p v_p$

$$U(x,t) = \sum_{p=1}^{\bar{p}} b_p v_p + \sum_{p=\bar{p}+1}^{n} a_p v_p$$

with $\bar{p}$ $\max p$ s.t. $x - \lambda_p t > 0$



Roe's method (Approximate Riemann Solver)

$\partial_t U + \partial_x F = 0 \Rightarrow \partial_t U + A \partial_x U = 0$

$A(U_L, U_R)$ is a matrix

Roe's conditions on A:

(i) A is diagonalizable w/ real eigenvalues → hyperbolic

(ii) $A(U_L, U_R) \longrightarrow DF(\bar{U})$ $\quad U_L, U_R \to \bar{U}$

(iii) $A(U_L, U_R)(U_R - U_L) = F(U_R) - F(U_L)$

$$A(U_L, U_R) = DF\Big|_{U_{avg}}$$

HLL.

$$U_{avg} = \frac{1}{2}(U_L + U_R)$$

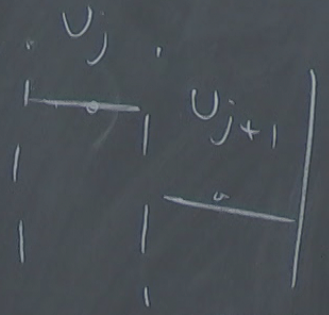
$$F^{Roe}(U_L, U_R) = \frac{1}{2}\left[F(U_L) + F(U_R)\right] - \sum_p |\lambda_p| \Delta_p v_p$$

$$U_R - U_L = \sum_p \Delta_p v_p$$

$DF|_{U_{avg}}$

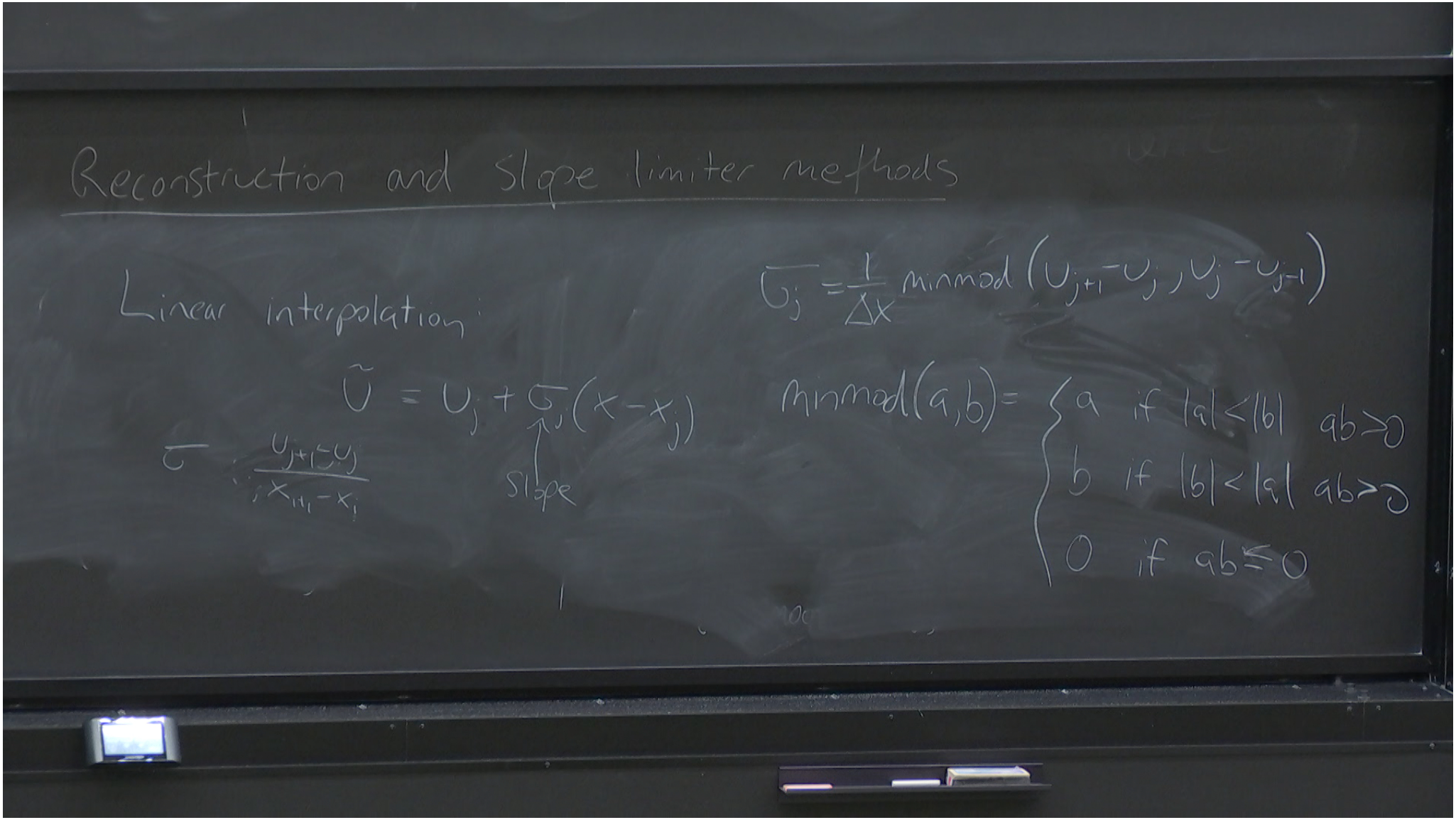
HLL

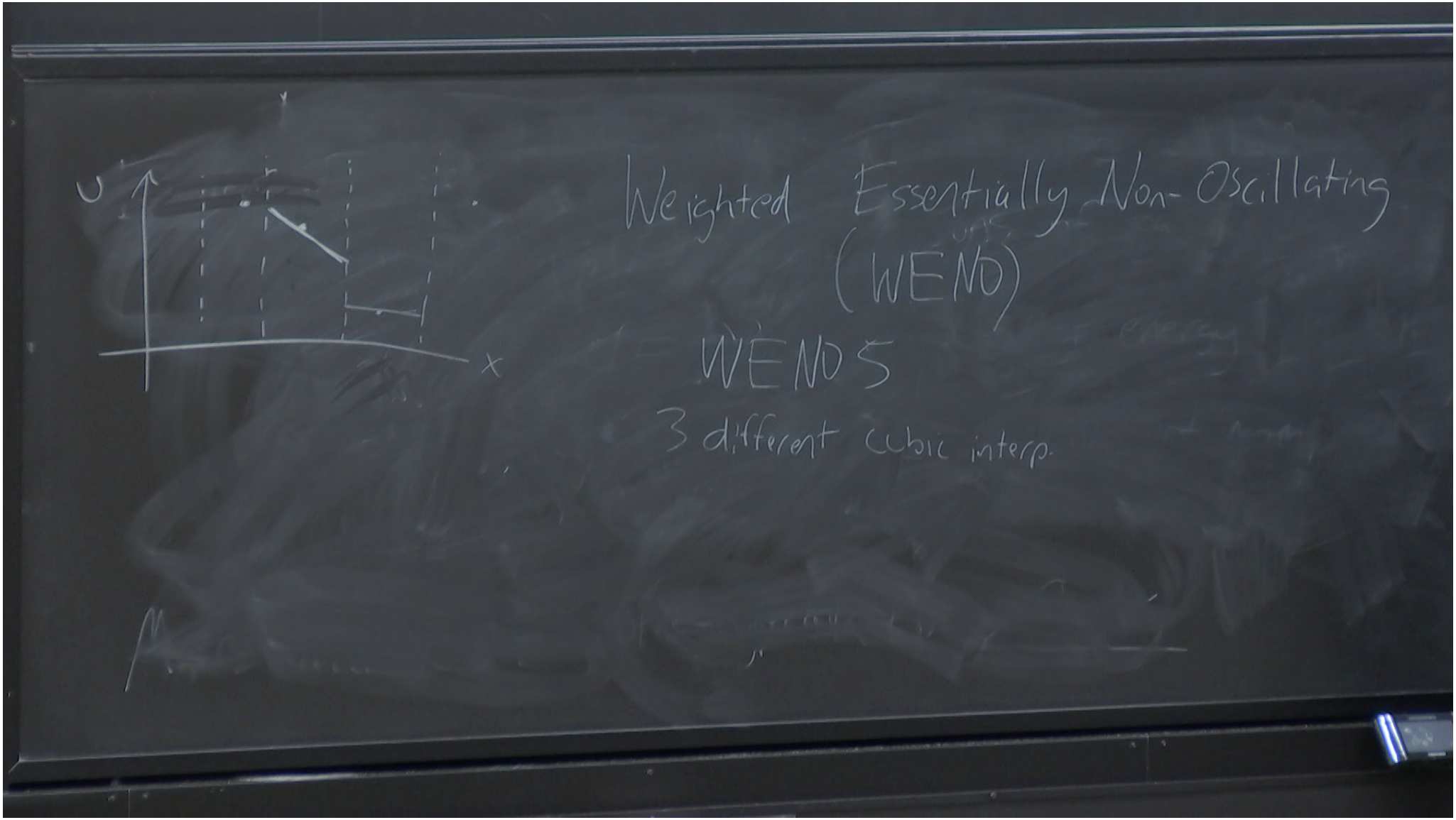
$U_{avg} = \frac{1}{2}(U_L + U_R)$



$$\frac{1}{2}\left[F(U_L) + F(U_R)\right] - \sum_p |\lambda_p| \Delta_p v_p$$

$$U_R - U_L = \sum_p \Delta_p v_p$$


Reconstruction and slope limiter methods
Linear interpolation:
$\tilde{U} = U_j + \sigma_j (x - x_j)$
$\sigma = \frac{U_{j+1} - U_j}{x_{j+1} - x_j}$
slope
$\sigma_j = \frac{1}{\Delta x} \text{minmod}(U_{j+1} - U_j, U_j - U_{j-1})$
$\text{minmod}(a,b) = \begin{cases} a & \text{if } |a| < |b| \quad ab > 0 \\ b & \text{if } |b| < |a| \quad ab > 0 \\ 0 & \text{if } ab \leq 0 \end{cases}$


u
x
Weighted Essentially Non-Oscillating
(WENO)
WENO5
3 different cubic interp.

General Relativistic Hydrodynamics

$\nabla_a(\rho_0 u^a) = 0$ Cons. of rest mass

$\nabla_a T^{ab} = 0$ Cons. of Stress-energy

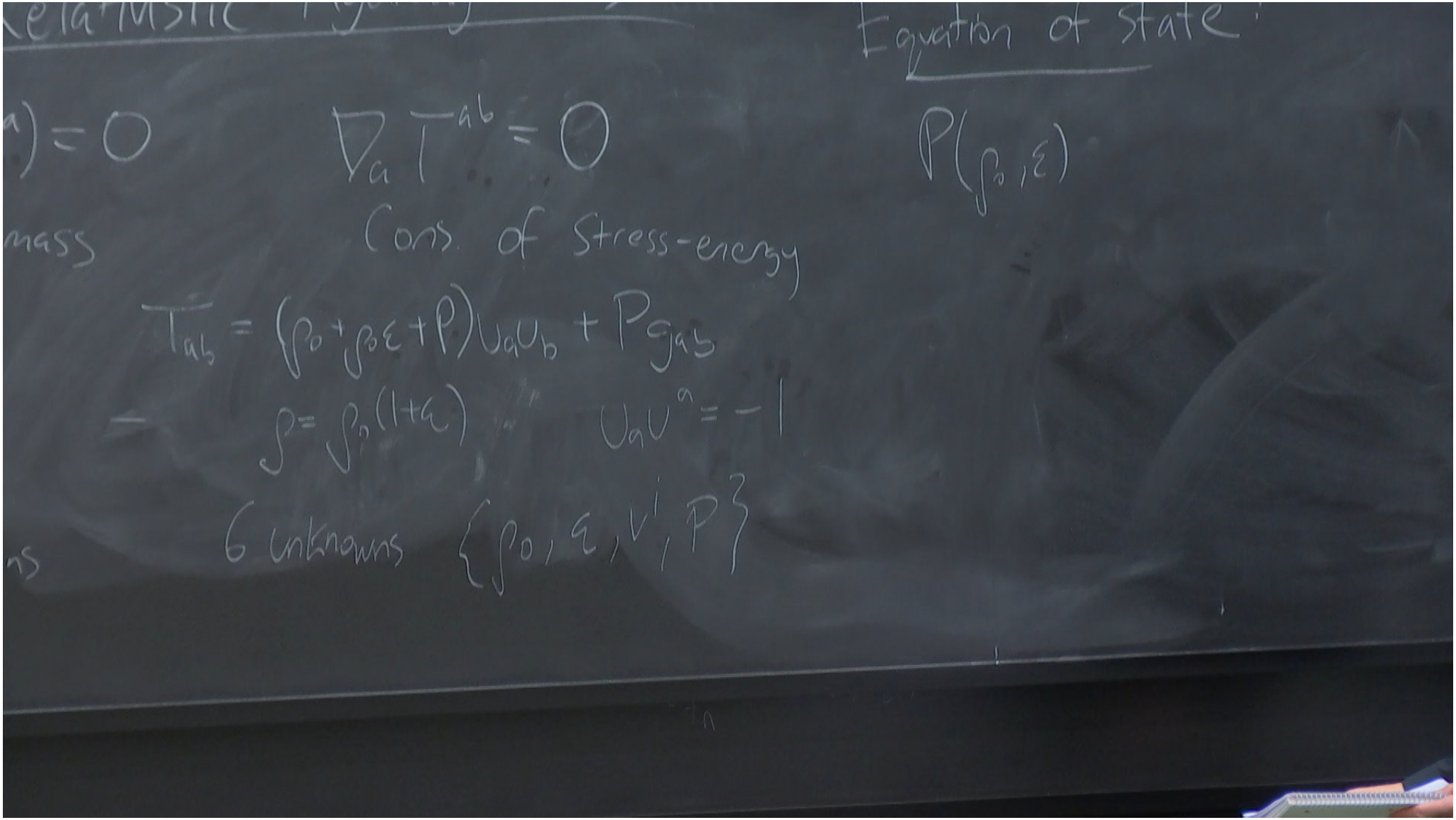
$T_{ab} = (\rho_0 + \rho\varepsilon + P) u_a u_b + P g_{ab}$

$\rho = \rho_0(1+\varepsilon)$

$u_a u^a = -1$

5 Eqns

6 unknowns $\{\rho_0, \varepsilon, v^i, P\}$


Equation of state:
$P(\rho_0, \varepsilon)$
$\nabla_a T^{ab} = 0$
Cons. of Stress-energy
$T_{ab} = (\rho_0 + \rho_0\varepsilon + P)u_a u_b + P g_{ab}$
$\rho = \rho_0(1+\varepsilon)$
$u_a u^a = -1$
6 unknowns $\{\rho_0, \varepsilon, v^i, P\}$

al Relativistic Hydrodynamics

$\nabla_a (\rho_0 u^a) = 0$

of rest mass

$\nabla_a T^{ab} = 0$

Cons. of Stress-energy

$T_{ab} = (\rho_0 + \rho_0 \varepsilon + P) u_a u_b + P g_{ab}$

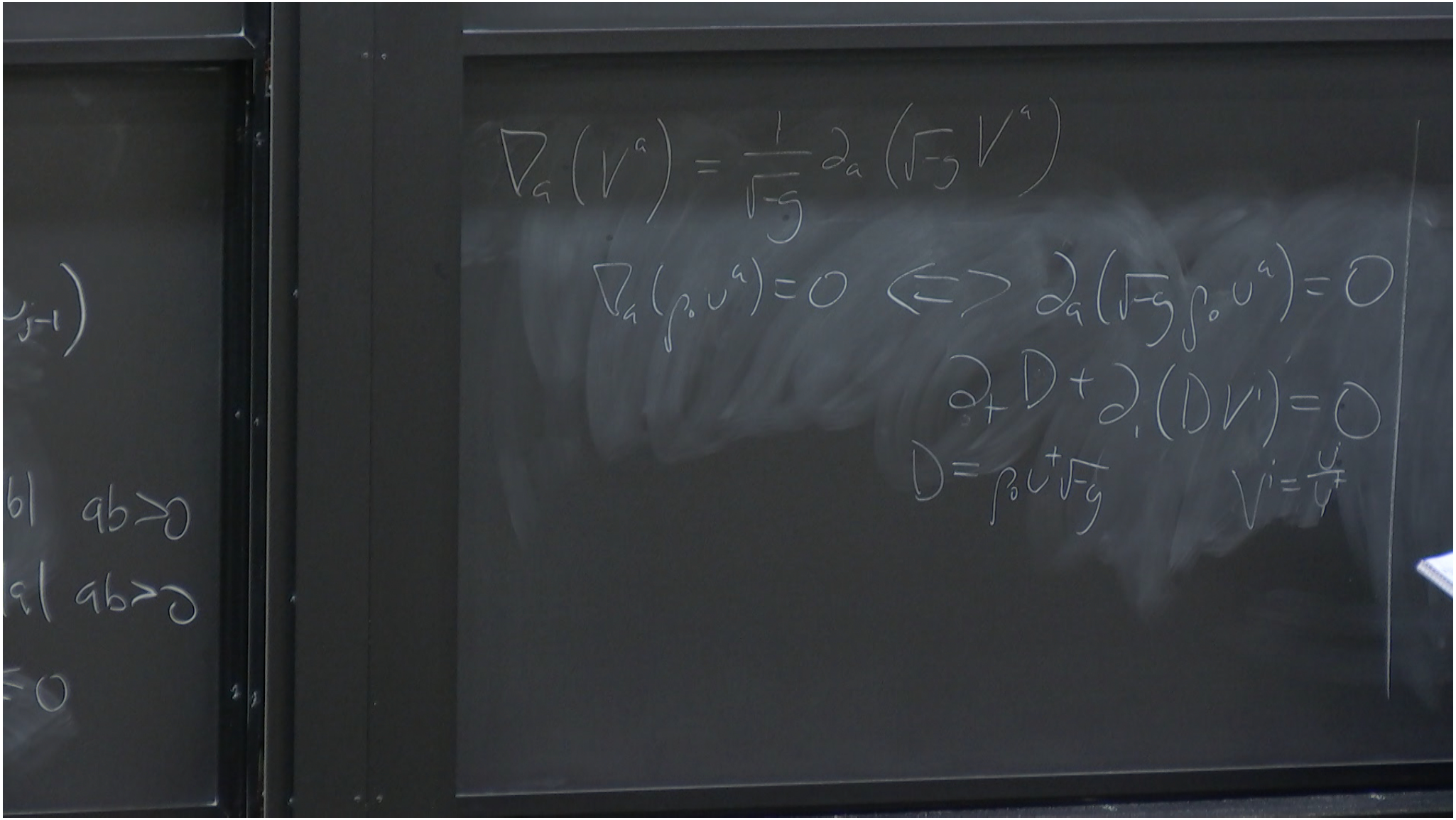
$\rho = \rho_0 (1 + \varepsilon)$ $\quad u_a u^a = -1$

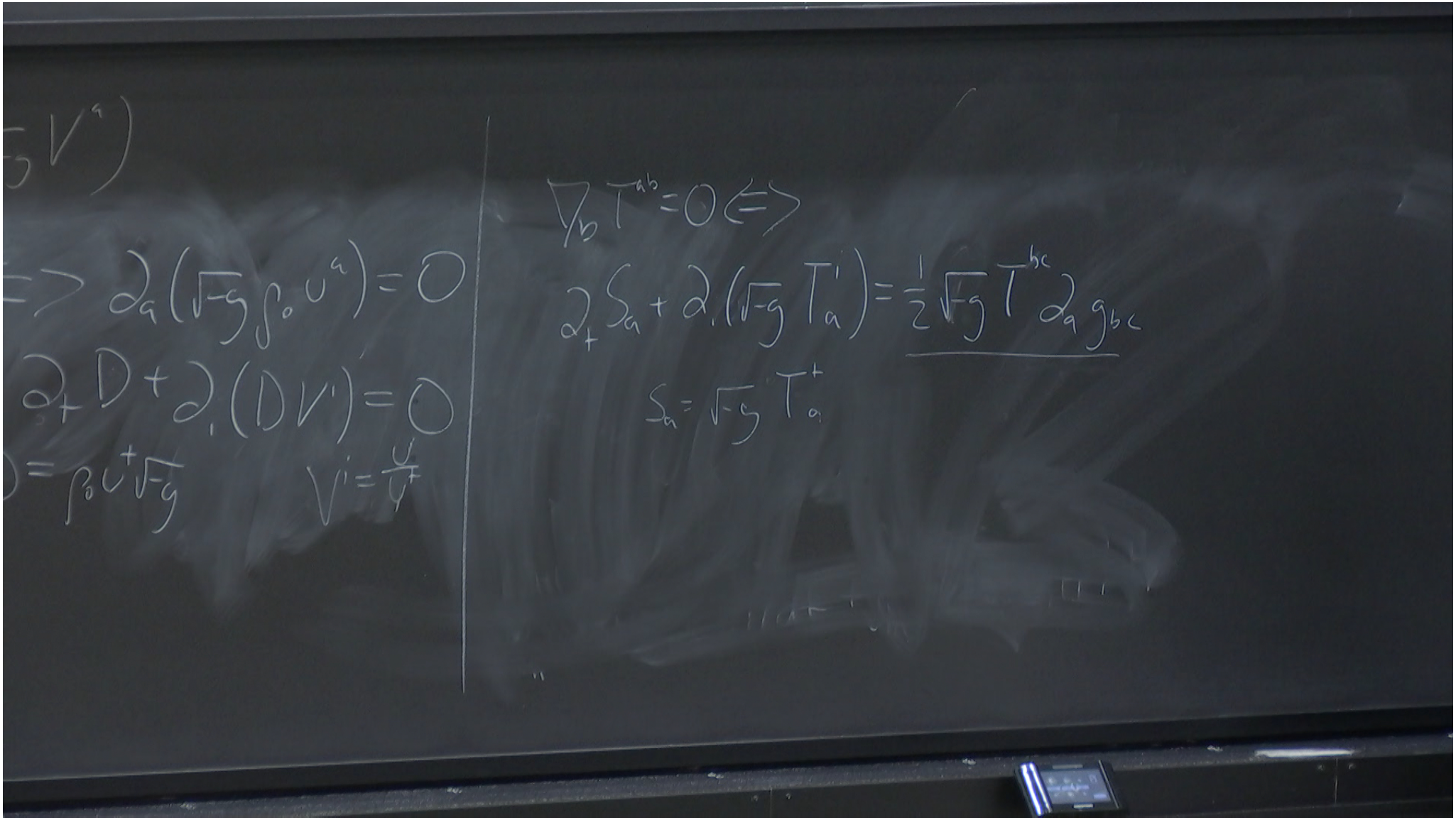
5 Eqns

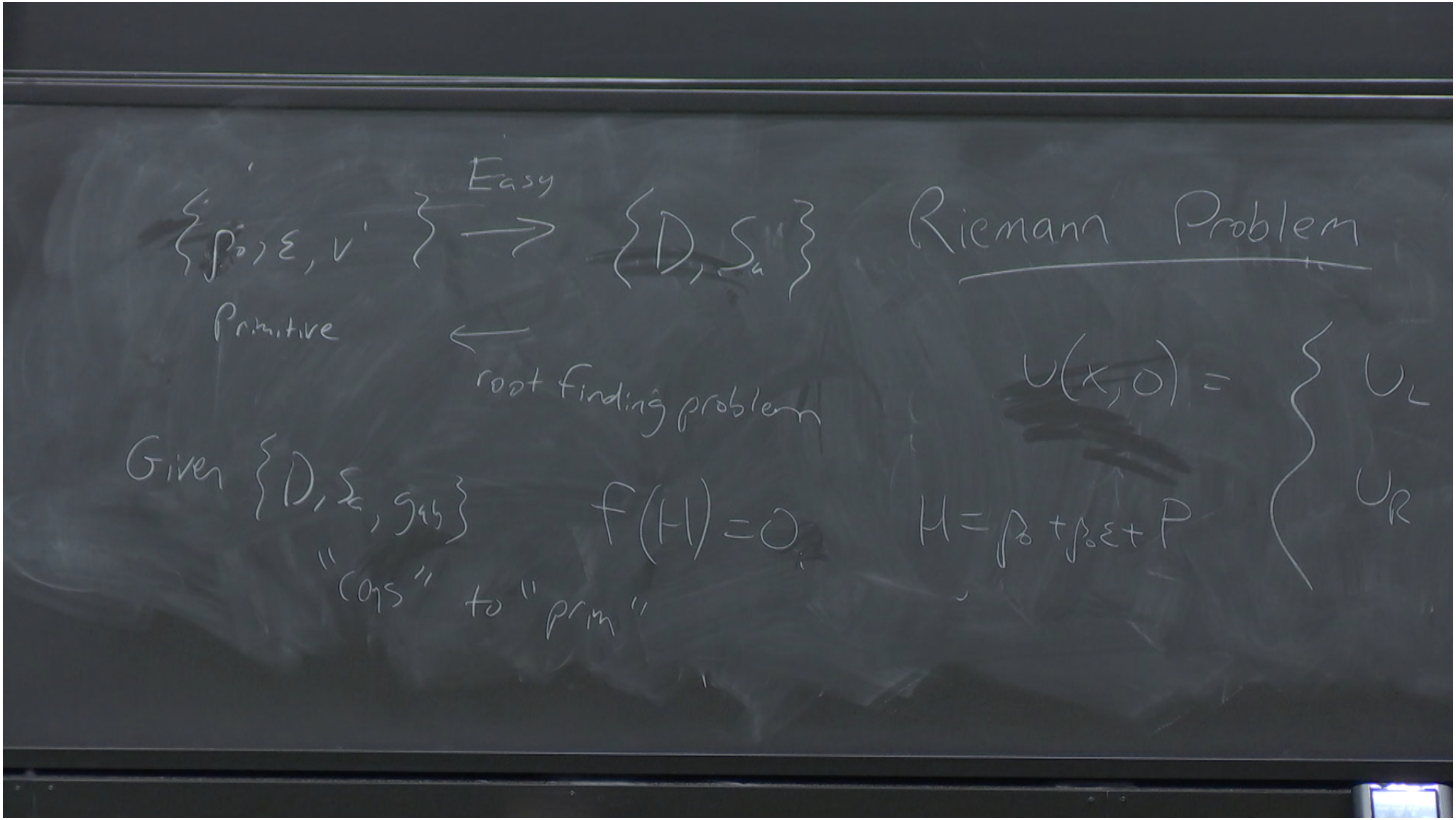
6 unknowns $\{\rho_0, \varepsilon, v^i, P\}$

Equation of state:

$P(\rho_0, \varepsilon)$


$\nabla_a(V^a) = \frac{1}{\sqrt{-g}} \partial_a(\sqrt{-g} V^a)$
$\nabla_a(\rho_0 u^a) = 0 \iff \partial_a(\sqrt{-g}\rho_0 u^a) = 0$
$\partial_t D + \partial_i(D V^i) = 0$
$D = \rho_0 u^t \sqrt{-g}$
$V^i = \frac{u^i}{u^t}$


$\sqrt{-g}\,V^a)$
$\Rightarrow \partial_a(\sqrt{-g}\,\rho_0 u^a) = 0$
$\partial_t D + \partial_i(DV^i) = 0$
$= \rho_0 u^t \sqrt{-g}$
$V^i = \frac{u^i}{u^t}$
$\nabla_b T^{ab} = 0 \Leftrightarrow$
$\partial_t S_a + \partial_i(\sqrt{-g}\,T^i_a) = \frac{1}{2}\sqrt{-g}\,T^{bc}\partial_a g_{bc}$
$S_a = \sqrt{-g}\,T^t_a$


Easy
Primitive
root finding problem
Given
"cons" to "prim"
Riemann Problem
H = β0 + β1ε + P