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Speakers: Jaume Gomis

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QFT III

lecture 14

Giuseppe Sellaroli

March 6, 2020

Outline

- Review of OPEs with energy-momentum tensor
- Virasoro algebra
- Asymptotic states
- Hilbert space of CFT
- Overview of null states

Review

OPEs with energy-momentum tensor

Last time we have used the fact that for a primary field

$$-[Q_\xi, \phi] = \delta\phi = -\left(h\partial_z \xi + \bar{h}\partial_{\bar{z}}\bar{\xi} + \xi\partial_z + \bar{\xi}\partial_{\bar{z}}\right)\phi$$

to find out the OPEs

$$R(T(z)\phi(w,\bar{w})) \sim \frac{h\,\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial_w \phi(w,\bar{w})}{z-w}$$

$$R(\bar{T}(\bar{z})\phi(w,\bar{w})) \sim \frac{\bar{h}\,\phi(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial_{\bar{w}} \phi(w,\bar{w})}{\bar{z}-\bar{w}}$$

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In fact, these two OPEs can be taken as the definition of ϕ being primary with conformal dimension $(h, \bar{h})$.

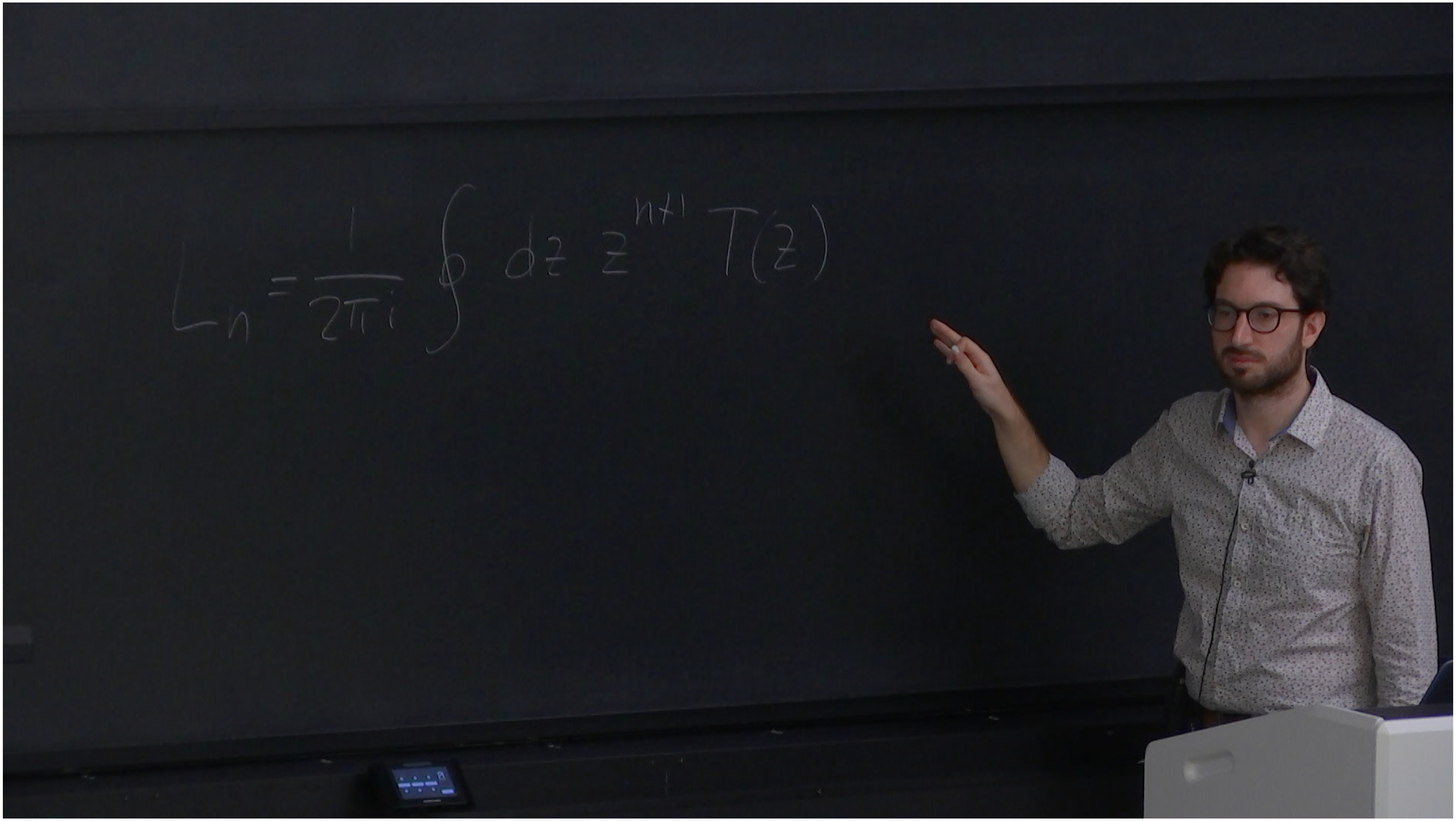
Virasoro algebra

Modes of energy-momentum tensor

Let's write the holomorphic and anti-holomorphic components of the energy-momentum tensor as the Laurent series

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n, \quad L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z)$$

$$\bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} (\bar{z})^{-n-2} \bar{L}_n, \quad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{n+1} \bar{T}(\bar{z})$$


$L_n = \frac{1}{2\pi i} \oint dz\, z^{n+1} T(z)$

The Virasoro algebra

In general $\xi(z) = \sum_n a_n z^{n+1}$ so that

$$
\begin{aligned}
Q_\xi &= \sum_{n\in\mathbb{Z}} \left(\frac{a_n}{2\pi i} \oint dz\, z^{n+1} T(z) + \frac{\bar{a}_n}{2\pi i} \oint d\bar{z}\, \bar{z}^{n+1} \bar{T}(\bar{z}) \right) \\
&= \sum_{n\in\mathbb{Z}} (a_n L_n + \bar{a}_n \bar{L}_n)
\end{aligned}
$$

Charge associated to conformal transformation

Recall that the charge associated to the infinitesimal conformal transformation

$$z \longrightarrow z + \varepsilon\xi(z), \quad \bar{z} \longrightarrow \bar{z} + \varepsilon\bar{\xi}(\bar{z})$$

is

$$Q_{\xi} = \frac{1}{2\pi i}\oint dz\, \xi(z) T(z) + \frac{1}{2\pi i}\oint d\bar{z}\, \bar{\xi}(\bar{z}) \bar{T}(\bar{z})$$

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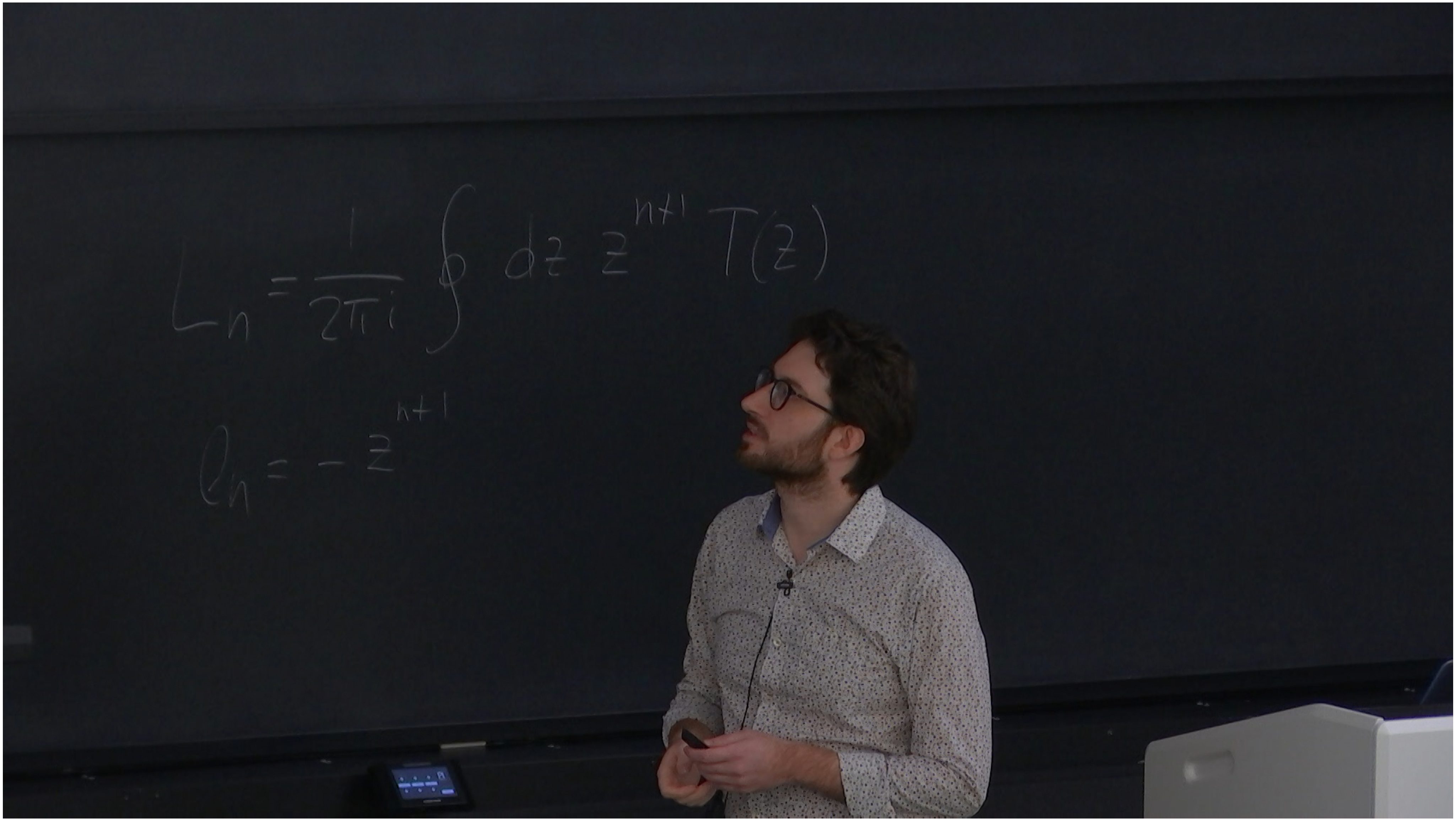
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The Virasoro algebra

We expect the $L_n, \bar{L}_n$ to be a representation of the conformal algebra generators on the CFT Hilbert space.

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However, since we have to allow projective representations we have to use a central extension of the Witt algebra, known as Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}$$
$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}$$
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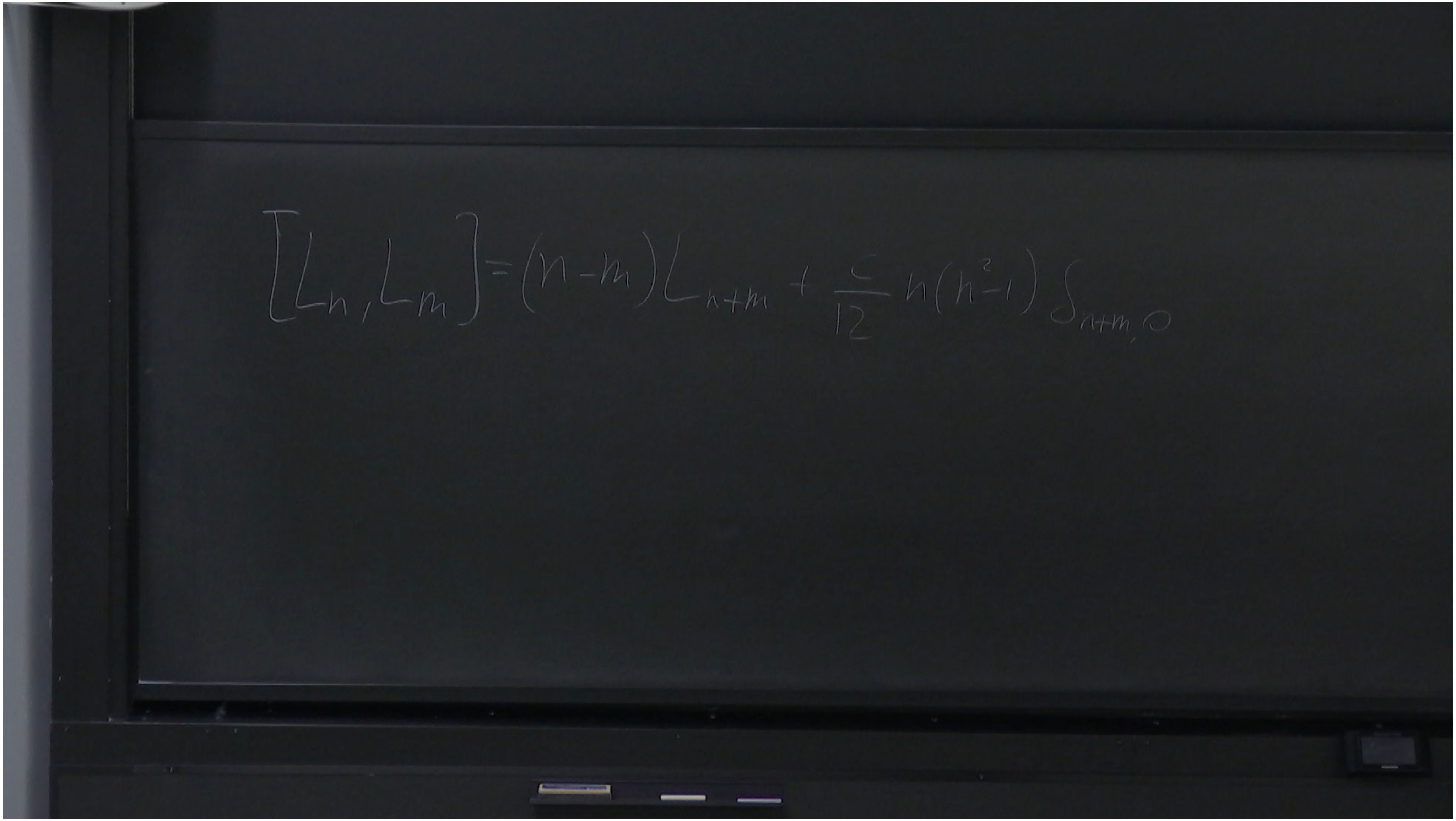
c is called the central charge or conformal anomaly. Note that the commutation relations with $n, m \in \{-1, 0, 1\}$ are not affected.

OPE of T with itself

Making use of the Virasoro algebra commutation relations, we can see that

$$R(T(z)T(w)) \sim \frac{c/2}{(z-w)^2} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}$$

$$R(\bar{T}(\bar{z})T(w)) \sim 0$$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$$R\big(T(z)T(w)\big) \sim \frac{c/2}{(z-w)^4} + \ldots$$

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$T(z)$ is almost a primary field with $(h, \bar{h}) = (2, 0)$!

Action of Virasoro algebra on primaries

The (adjoint) action of the Virasoro algebra on primary fields is given by

$$[L_n, \phi(z,\bar{z})] = h(n+1)z^n\phi(z,\bar{z}) + z^{n+1}\partial_z\phi(z,\bar{z})$$
$$[L_n, \phi(z,\bar{z})] = \bar{h}(n+1)\bar{z}^n\phi(z,\bar{z}) + \bar{z}^{n+1}\partial_{\bar{z}}\phi(z,\bar{z})$$

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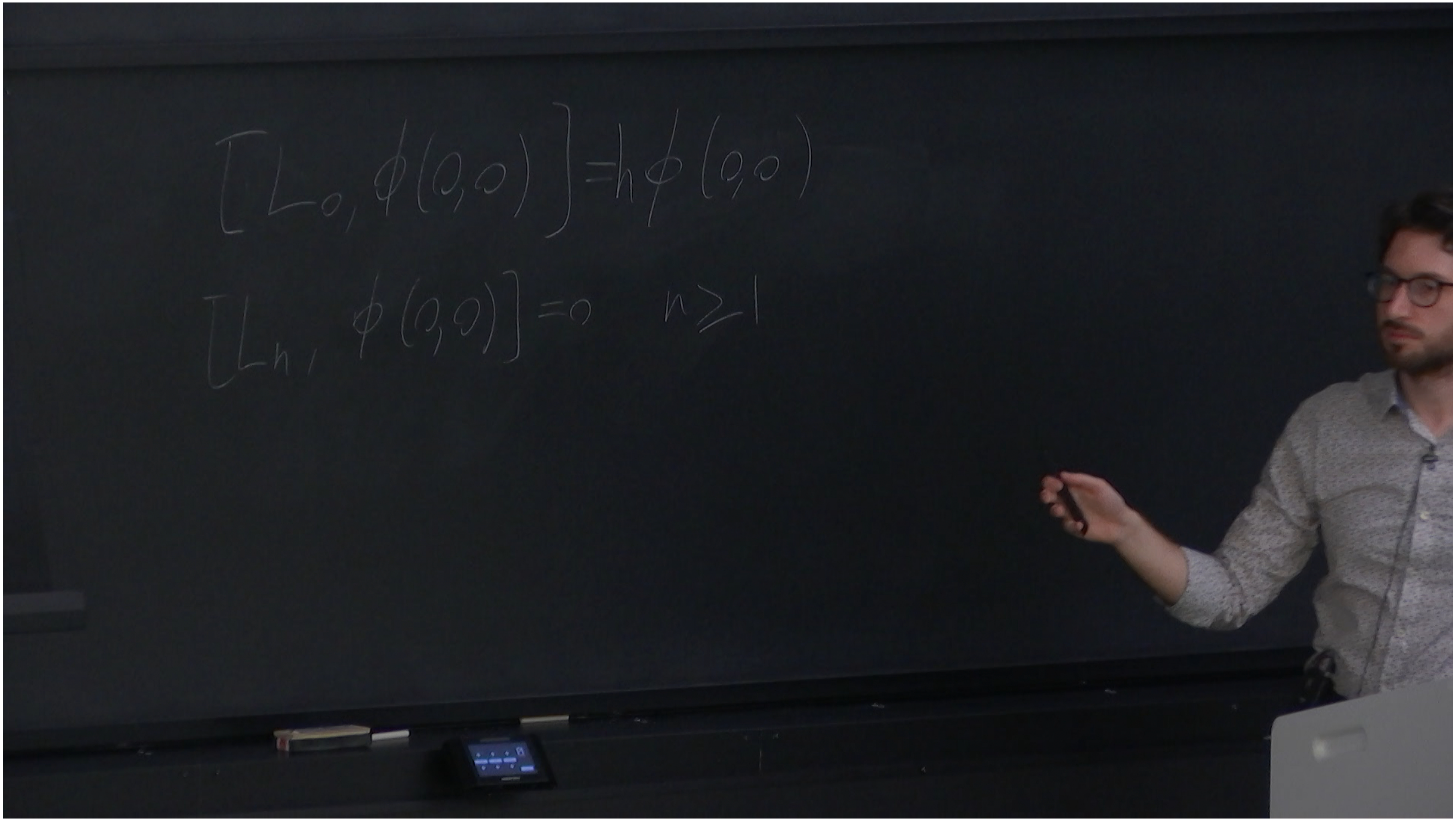
Note that

$$[L_0, \phi(0,0)] = h\phi(0,0)$$
$$[\bar{L}_0, \phi(0,0)] = \bar{h}\phi(0,0)$$
$$[L_n, \phi(0,0)] = [\bar{L}_n, \phi(0,0)] = 0, \quad n \geq 1$$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We can use this fact to define asymptotic in-states associated to primary operators:

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We need a notion of adjoint (and of inner product) first, though!

Hermitian conjugate

If we were working in Minkowski space we would expect hermitian conjugation to leave space-time coordinates invariant. However, we are using a wick-rotated coordinate $x = it$, so we expect

$$x \longrightarrow -x \quad \Longrightarrow \quad z = e^{x+iy} \longrightarrow e^{-x+iy} = 1/\bar{z}$$

This leads us to define

$$[\phi(z,\bar{z})]^\dagger = f(z,\bar{z})\phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right)$$

with f to be decided.

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\begin{aligned}
\| |\phi_{\text{in}}\rangle \|^2 &= \langle \phi_{\text{out}} \mid \phi_{\text{in}} \rangle \\
&= \lim_{z,\bar{z}\to 0} f(z,\bar{z}) \left\langle 0 \middle| \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right) \phi(0,0) \middle| 0 \right\rangle \\
&= \lim_{w,\bar{w}\to\infty} f\left(\frac{1}{w}, \frac{1}{\bar{w}}\right) \langle 0 \mid \phi(\bar{w}, w) \phi(0,0) \mid 0 \rangle
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so we choose

$$
f(z,\bar{z}) = z^{-2\bar{h}} \bar{z}^{-2h}
$$

$$
\Downarrow
$$

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[\phi(z,\bar{z})]^{\dagger} = z^{-2\bar{h}} \bar{z}^{-2h} \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right)
$$

What does this mean for the Virasoro algebra?

One can prove that this choice of hermitian conjugation implies that

$$L_n^\dagger = L_{-n}, \quad \bar{L}_n^\dagger = \bar{L}_{-n}$$

if we require $T(z)$, $\bar{T}(\bar{z})$ to be self-adjoint.

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What does $i(L_0 - \bar{L}_0)$ generate?

Hilbert space of CFT

Vacuum state

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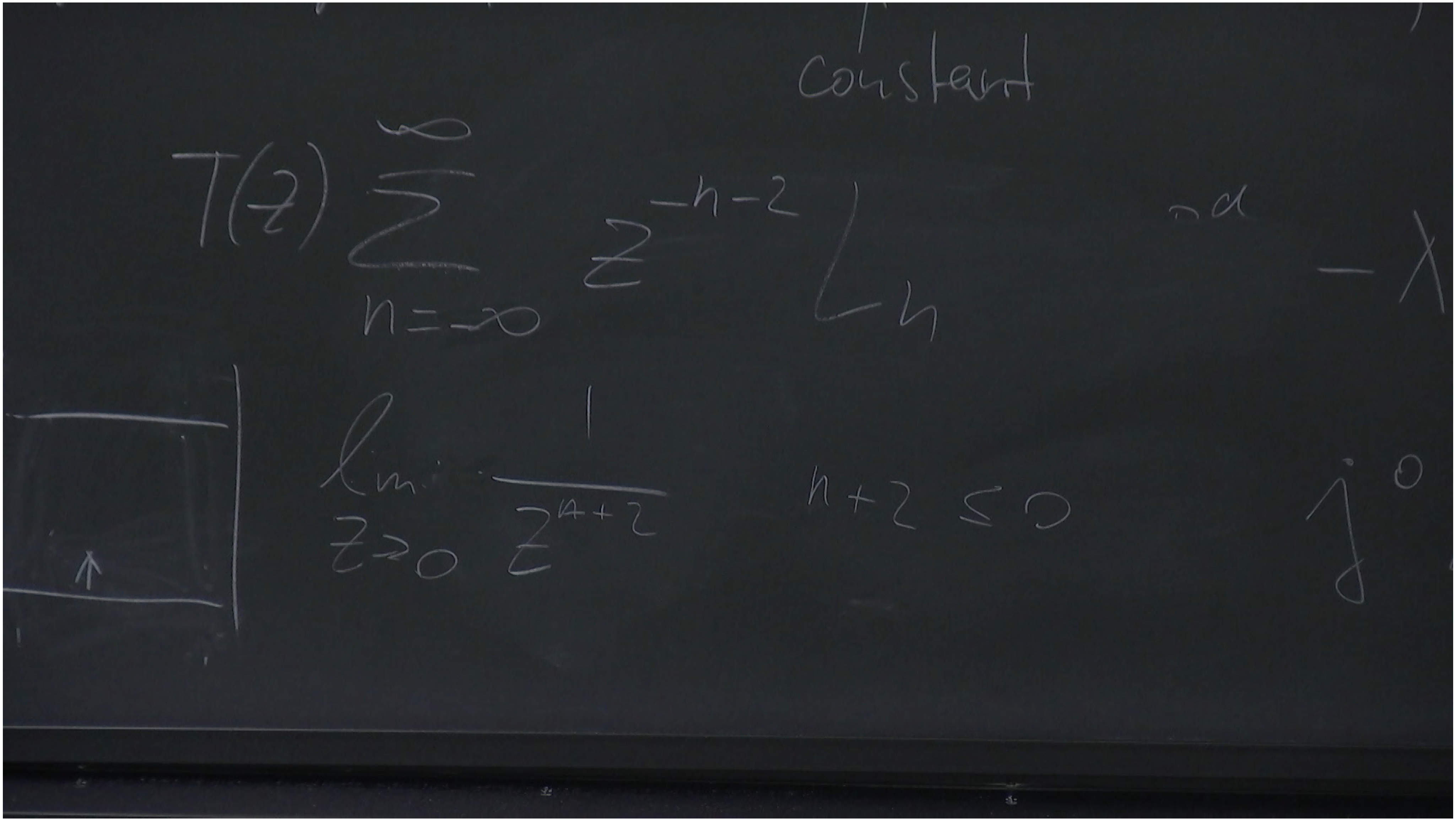
constant
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lim z→0 1/z^{n+2}
n+2 ≤ 0

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Highest weight states

In the following we will use the notation

$$|h, \bar{h}\rangle = \phi(0,0)|0\rangle$$

for the in-state of a primary operator of dimension $(h, \bar{h})$.

We have

$$L_n|h, \bar{h}\rangle = [L_n, \phi(0,0)]|0\rangle + \phi(0,0)L_n|0\rangle$$

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$$L_n|h, \bar{h}\rangle =$$

Highest weight states

Note that

$$[L_0, L_n] = -nL_n, \quad [\bar{L}_0, \bar{L}_n] = -n\bar{L}_n$$

which means that L_n and $\bar{L}_n$ act as ladder operators:

$$L_0 L_n |h, \bar{h}\rangle = (h - n) L_n |h, \bar{h}\rangle$$
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The state $|h, \bar{h}\rangle$ is an eigenvector of L_0, $\bar{L}_0$ and is annihilated by all lowering operators. We call this kind of state a highest weight state.

What is the physical motivation for requiring the in-states to be highest weight states?

Descendant states

All other states can be obtained from the highest weight states by repeated application of the lowering operators. Note that this is considerably more complicated that an harmonic oscillator, since there are infinitely many lowering operators!

Unitarity

Unitarity

Is the Hilbert space we constructed actually an Hilbert space? As things are, we may get vectors with negative norm! We say the CFT is unitary if all states have non-negative norm.

Note first of all that

$$\begin{aligned} \|L_{-1}|h,\bar{h}\rangle\|^2 &= \langle h,\bar{h}\,|\,L_1 L_{-1}\,|\,h,\bar{h}\rangle \\ &= \langle h,\bar{h}\,|\,[L_1,L_{-1}]\,|\,h,\bar{h}\rangle \\ &= 2\langle h,\bar{h}\,|\,L_0\,|\,h,\bar{h}\rangle \\ &= 2h\langle h,\bar{h}\,|\,h,\bar{h}\rangle \end{aligned}$$

So $h \geq 0$ is a necessary condition for unitarity. What about $\bar{h}$?

Unitarity

We can also constrain the central charge c with the unitarity requirement. In fact

$$\begin{aligned} \|L_{-2}|0\rangle\|^2 &= \langle 0\,|\,L_2 L_{-2}\,|\,0\rangle \\ &= \langle 0\,|\,[L_2, L_{-2}]\,|\,0\rangle \end{aligned}$$

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so we also require $c \geq 0$.

Unitarity

We can keep playing this game to try to find more and more requirements for c and h. At the end we get:

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Theorem

The CFT is unitary if and only if one of the following happens:

- $c \geq 1$ *and* $h \geq 0$
- *there are integers* $m \geq 2$, $1 \leq p \leq q < m$ *such that*

$$c = 1 - \frac{6}{4m(m+1)}$$

$$h = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}$$

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Note: when we talk about h we mean individually for each primary operator. Also, the same applies to $\bar{h}$.

Null states