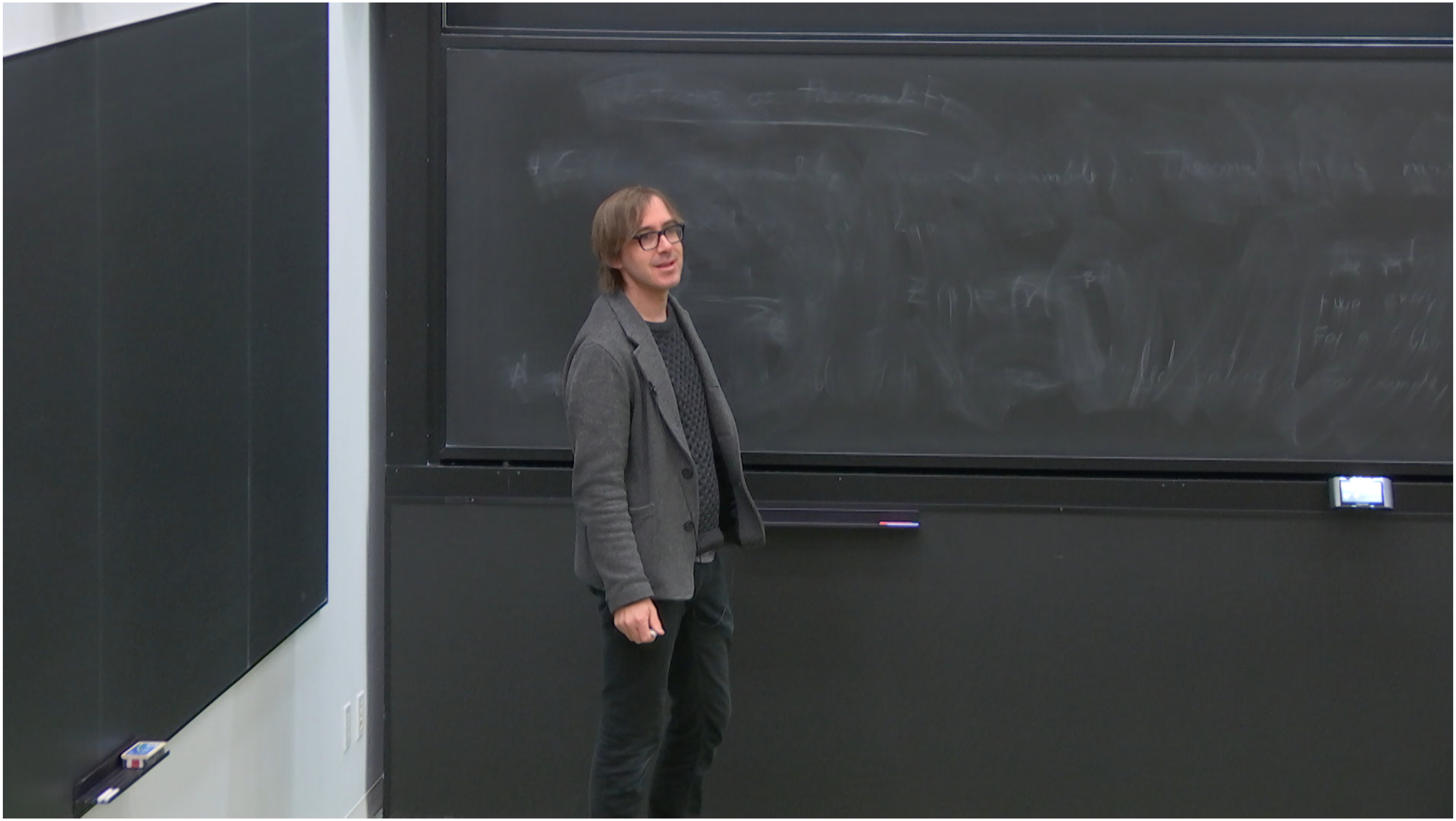
Title: PSI 2019/2020 - Chern-Simons Theory Part 1 - Lecture 12

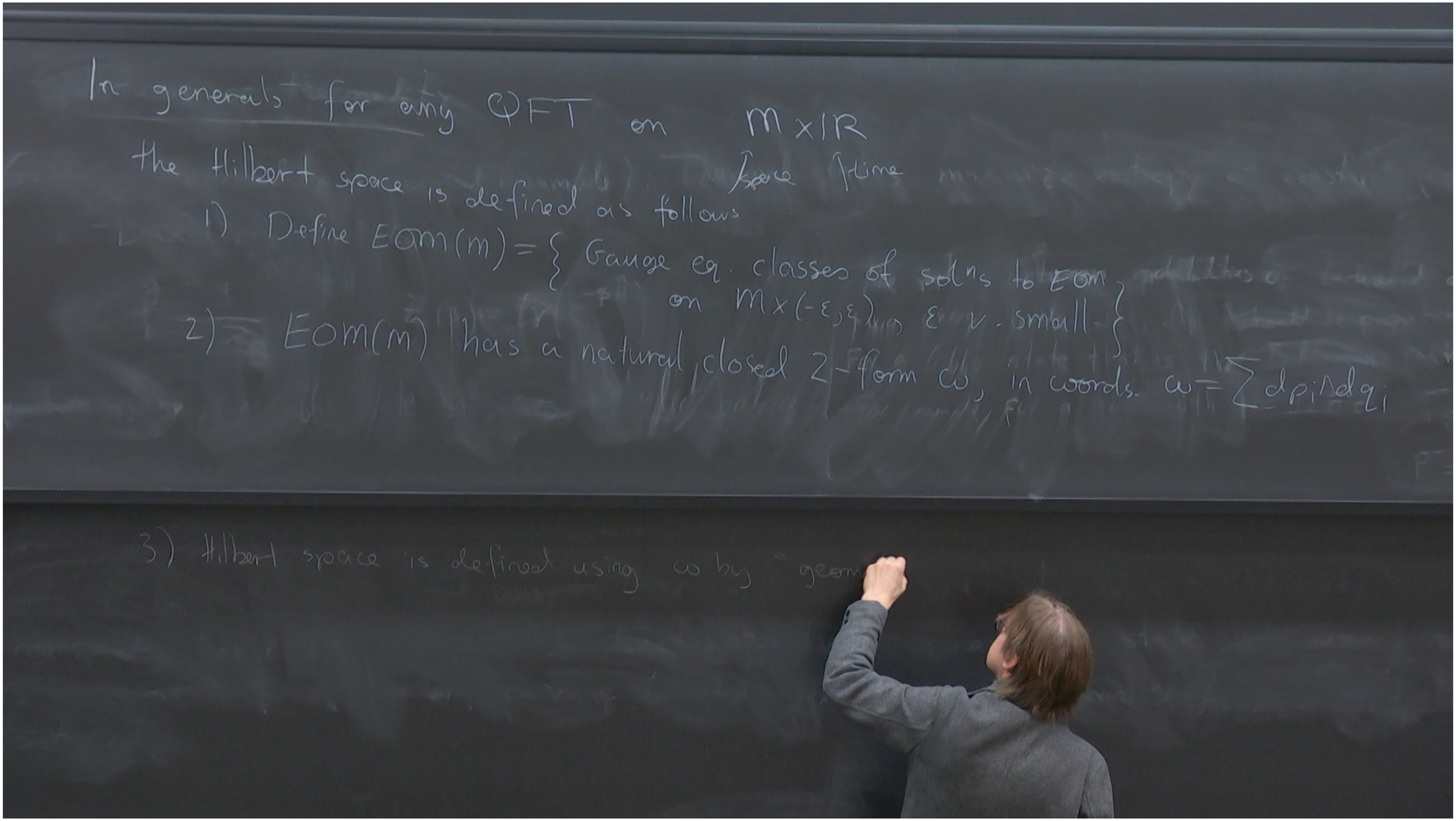
Speakers: Kevin Costello

Collection: PSI 2019/2020 - Chern-Simons Theory (Part 1)

Date: March 11, 2020 - 10:15 AM

URL: http://pirsa.org/20030020




In general, for any QFT on M × ℝ
space time
the Hilbert space is defined as follows
1) Define EOM(M) = { Gauge eq. classes of solns to EOM on M × (-ε, ε), ε v. small }
2) EOM(M) has a natural closed 2-form ω, in words. ω = Σ dpi ∧ dqi
3) Hilbert space is defined using ω by "geom

In general, for any QFT on $M \times \mathbb{R}$ (space, time)

the Hilbert space is defined as follows

1) Define $EOM(M) = \{$ Gauge eq. classes of sol^{n}s to EOM on $M \times (-\varepsilon, \varepsilon)$, ε v. small $\}$

2) $EOM(M)$ has a natural closed 2-form ω, in coords. $\omega = \sum dp_i \wedge dq_i$

3) Hilbert space is defined using ω by "geometric quantization"

Today: Where does ω come from?

For most theories (slight subtlety for CS)

$\omega = d\alpha$

α is a 1-form, α looks like $\sum p_i dq_i$

α is called "variational 1 form"

Let φ be a field on $M \times (-\varepsilon, \varepsilon)$ satisfies the EOM

$T_\varphi EOM(M) = \{$ 1st order variations $\varphi + \delta\varphi \}$

The cotangent space, $T^*_\varphi EOM(M)$ is the linear dual.

$T^*_\varphi EOM(M) = \{$ linear functionals $T_\varphi EOM(M) \to \mathbb{R} \}$

A 1-form α is an element of $T^*_\varphi \mathrm{EOm}(m)$, $\forall \varphi$

i.e. α is something which takes a field φ, and a 1st order variation $\delta\varphi$, and gives us a number $\alpha(\varphi, \delta\varphi)$ linear in $\delta\varphi$.

Let $\mathcal{L}(\varphi)$ be the Lagrangian.

This is an n-form on $M \times (-\varepsilon, \varepsilon)$

$\dim M = n-1$

$$\mathcal{L}(\varphi + \delta\varphi) = \delta\varphi\,(EL(\varphi)) + \text{Total Derivative}$$

- φ satisfies the EOM, so the first term vanishes
- Write the total derivative as

$d\tilde{\mathcal{L}}(\varphi, \delta\varphi)$ where $\tilde{\mathcal{L}}(\varphi, \delta\varphi)$

write the total derivative as

$d\tilde{\mathcal{L}}(\varphi,\delta\varphi)$ where $\tilde{\mathcal{L}}(\varphi,\delta\varphi)$

is an $(n-1)$ form.

Then

$$\alpha(\varphi,\delta\varphi) = \int \tilde{\mathcal{L}}(\varphi,\delta\varphi)$$

<u>Example</u>

QM M = a point

$\varphi \in C^\infty((-\varepsilon,\varepsilon))$

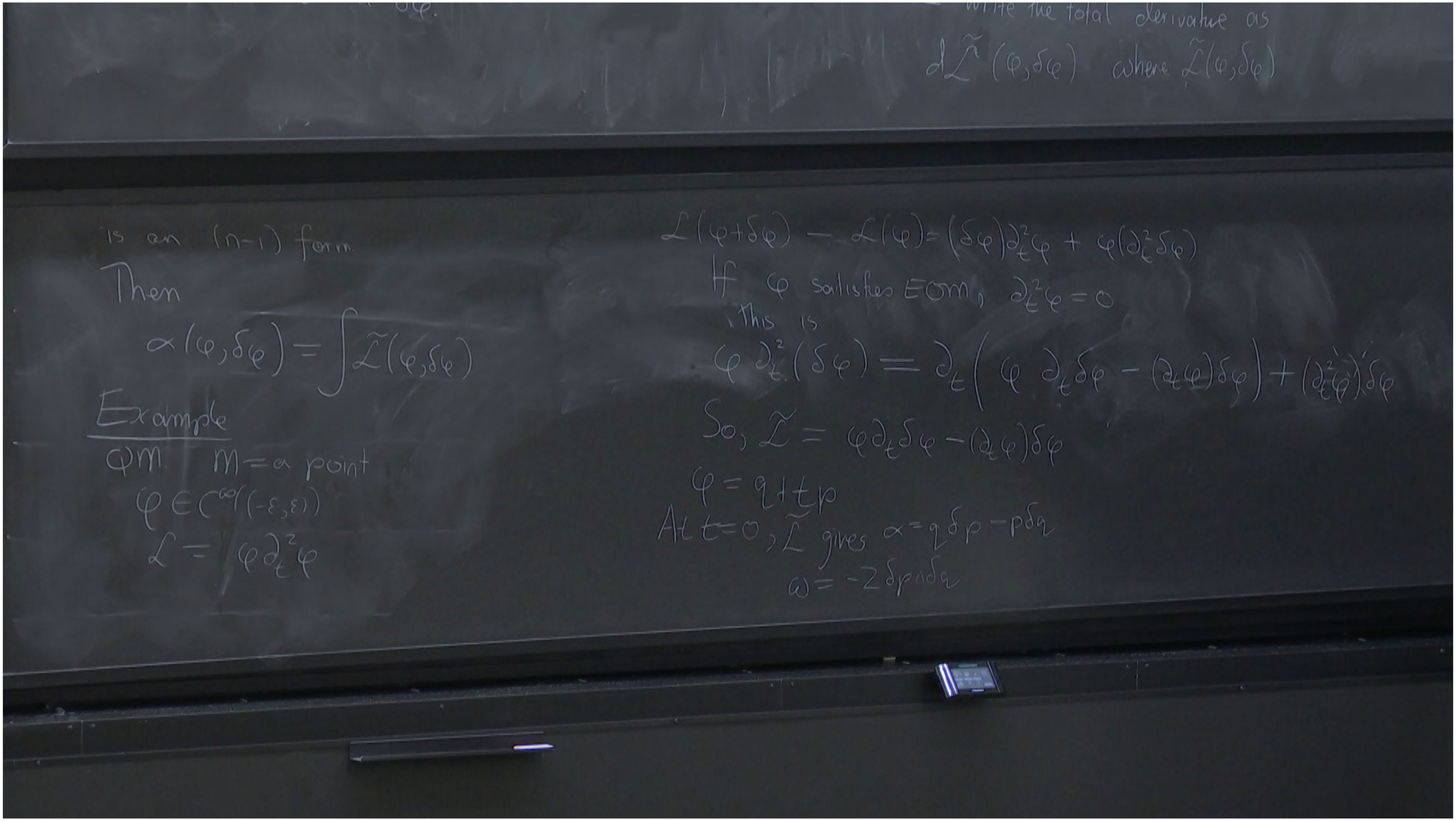
$\mathcal{L} = \varphi\partial_t^2\varphi$

$$\mathcal{L}(\varphi+\delta\varphi) - \mathcal{L}(\varphi) = (\delta\varphi)\partial_t^2\varphi + \varphi(\partial_t^2\delta\varphi)$$

If φ satisfies EOM, $\partial_t^2\varphi = 0$

This is

$$\varphi\,\partial_t^2(\delta\varphi) = \partial_t\Big(\varphi\,\partial_t\delta\varphi - (\partial_t\varphi)\delta\varphi\Big) + (\partial_t^2\varphi)\delta\varphi$$


Write the total derivative as
$d\tilde{\mathcal{L}}(\varphi,\delta\varphi)$ where $\tilde{\mathcal{L}}(\varphi,\delta\varphi)$
is an $(n-1)$ form
Then
$\alpha(\varphi,\delta\varphi)=\int\tilde{\mathcal{L}}(\varphi,\delta\varphi)$
Example
φM M = a point
$\varphi\in C^\infty(-\varepsilon,\varepsilon)$
$\mathcal{L}=\varphi\partial_t^2\varphi$
$\mathcal{L}(\varphi+\delta\varphi)-\mathcal{L}(\varphi)=(\delta\varphi)\partial_t^2\varphi+\varphi(\partial_t^2\delta\varphi)$
If φ satisfies EOM, $\partial_t^2\varphi=0$
This is
$\varphi\partial_t^2(\delta\varphi)=\partial_t(\varphi\partial_t\delta\varphi-(\partial_t\varphi)\delta\varphi)+(\partial_t^2\varphi)\delta\varphi$
So, $\tilde{\mathcal{L}}=\varphi\partial_t\delta\varphi-(\partial_t\varphi)\delta\varphi$
$\varphi=q+tp$
At $t=0$, $\tilde{\mathcal{L}}$ gives $\alpha=q\delta p-p\delta q$
$\omega=-2\delta p\wedge\delta q$

Ex:

Scalar field in dimension n,

$\varphi \in C^\infty(M \times (-\varepsilon, \varepsilon))$

$\mathcal{L}(\varphi) = \varphi \Delta_M \varphi + \varphi \partial_t^2 \varphi$

This is the same!

M has no boundary, so total derivatives in M direction can be discarded.

When we integrate by parts along time,
we get

$$\alpha = \int_M \delta\varphi\, \partial_t \varphi - \varphi\, \partial_t \delta\varphi$$

Example

A fermionic field on $M \times (-\varepsilon, \varepsilon)$

$$\mathcal{L} = \psi \not{\partial} \psi = \psi \Gamma^0 \partial_t \psi + \psi \not{\partial}_M \psi$$

We find

$$\alpha = \int_M \psi \Gamma^0 \delta \psi$$

ψ_i

$\{\psi_i, \psi_j\} = \delta_{ij}$

$d\psi_i \, d\psi_j \, \delta_{ij}$

2 fermions

$\psi^+ = \psi_1 + i\psi_2$

$\psi^- = \psi_1 - i\psi_2$

$d\psi^+ \wedge d\psi^-$

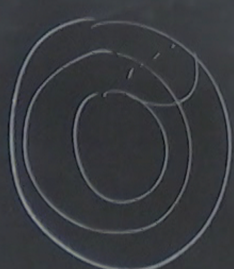
Chern-Simons Theory

$M = T^2$ coords θ_1, θ_2

$G = U(1)$

Then, space of sol^{n}s to the EOM on T^2

$= \left\{ \begin{array}{l} \text{Gauge equivalence} \\ \text{classes of flat } U(1) \text{ bundles} \end{array} \right\}$



$F(A) = 0$

This space is also a torus, $U(1) \times U(1)$

$\exp \int_{\theta_1} A \quad \in U(1)$

$\exp \int_{\theta_2} A \quad \in U(1)$

are coordinates

Choose coordinates,

$\lambda_1 = \int_{\theta_1} A \qquad \lambda_1 \sim \lambda_1 + 1$

$\lambda_2 = \int_{\theta_2} A \qquad \lambda_2 \sim \lambda_2 + 1$

Inserting $A = \lambda_1 \, d\theta_1 + \lambda_2 \, d\theta_2$,

$\delta A = \delta\lambda_1 \, d\theta_1 + \delta\lambda_2 \, d\theta_2$, we find

the 1-form α

is

$$-\int_{T^2} A \, \delta A = -\int_{T^2} (\lambda_1 \delta\lambda_2 - \delta\lambda_1 \lambda_2) \, d\theta_1 \, d\theta_2$$

$=$ up to a constant, $\frac{1}{2}(\lambda_1 \delta\lambda_2 - \delta\lambda_1 \lambda_2)$

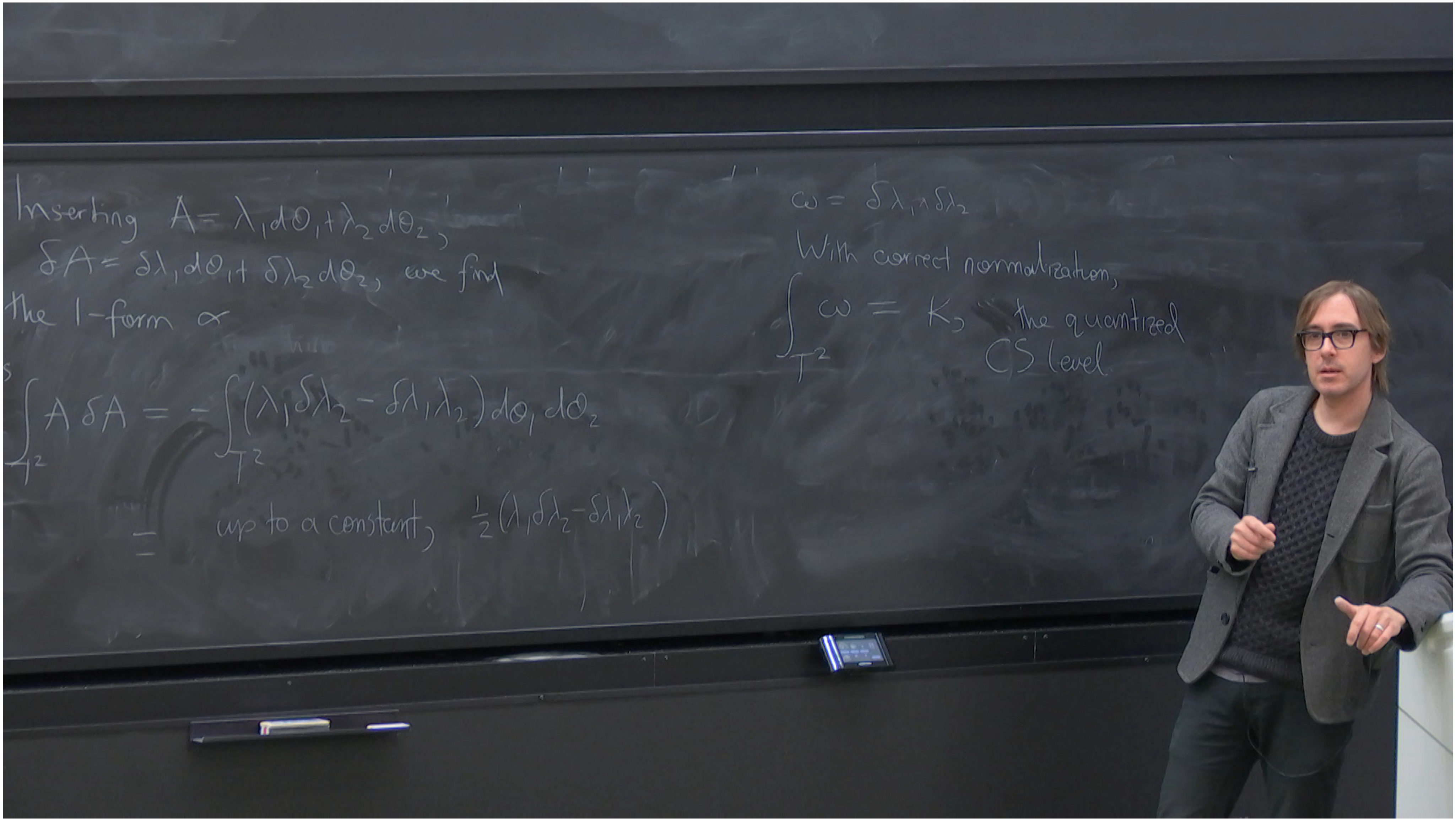
$\omega = \delta\lambda_1 \wedge \delta\lambda_2$

Inserting $A = \lambda_1 d\theta_1 + \lambda_2 d\theta_2$,
$\delta A = \delta\lambda_1 d\theta_1 + \delta\lambda_2 d\theta_2$, we find
the 1-form α

is

$$-\int_{T^2} A\,\delta A = -\int_{T^2} (\lambda_1 \delta\lambda_2 - \delta\lambda_1 \lambda_2)\, d\theta_1\, d\theta_2$$

$=$ up to a constant, $\frac{1}{2}(\lambda_1 \delta\lambda_2 - \delta\lambda_1 \lambda_2)$



Inserting $A = \lambda_1 d\theta_1 + \lambda_2 d\theta_2$,
$\delta A = \delta\lambda_1 d\theta_1 + \delta\lambda_2 d\theta_2$, we find
the 1-form α
$\int_{T^2} A\,\delta A = -\int_{T^2} (\lambda_1 \delta\lambda_2 - \delta\lambda_1 \lambda_2)\, d\theta_1 d\theta_2$
up to a constant, $\frac{1}{2}(\lambda_1 \delta\lambda_2 - \delta\lambda_1 \lambda_2)$
$\omega = \delta\lambda_1 \wedge \delta\lambda_2$
With correct normalization,
$\int_{T^2} \omega = K$, the quantized CS level

$$\omega = \delta\lambda_1 \wedge \delta\lambda_2 .$$

With correct normalization,

$$\int_{T^2} \omega = K, \text{ the quantized CS level.}$$

Hilbert Space

Given a manifold X, and a symplectic form ω,
"geometric quantization"
produces a Hilbert space.

$\omega,$

If $X = \mathbb{R}^{2n}$

$$\omega = \sum dp_i \wedge dq_i$$

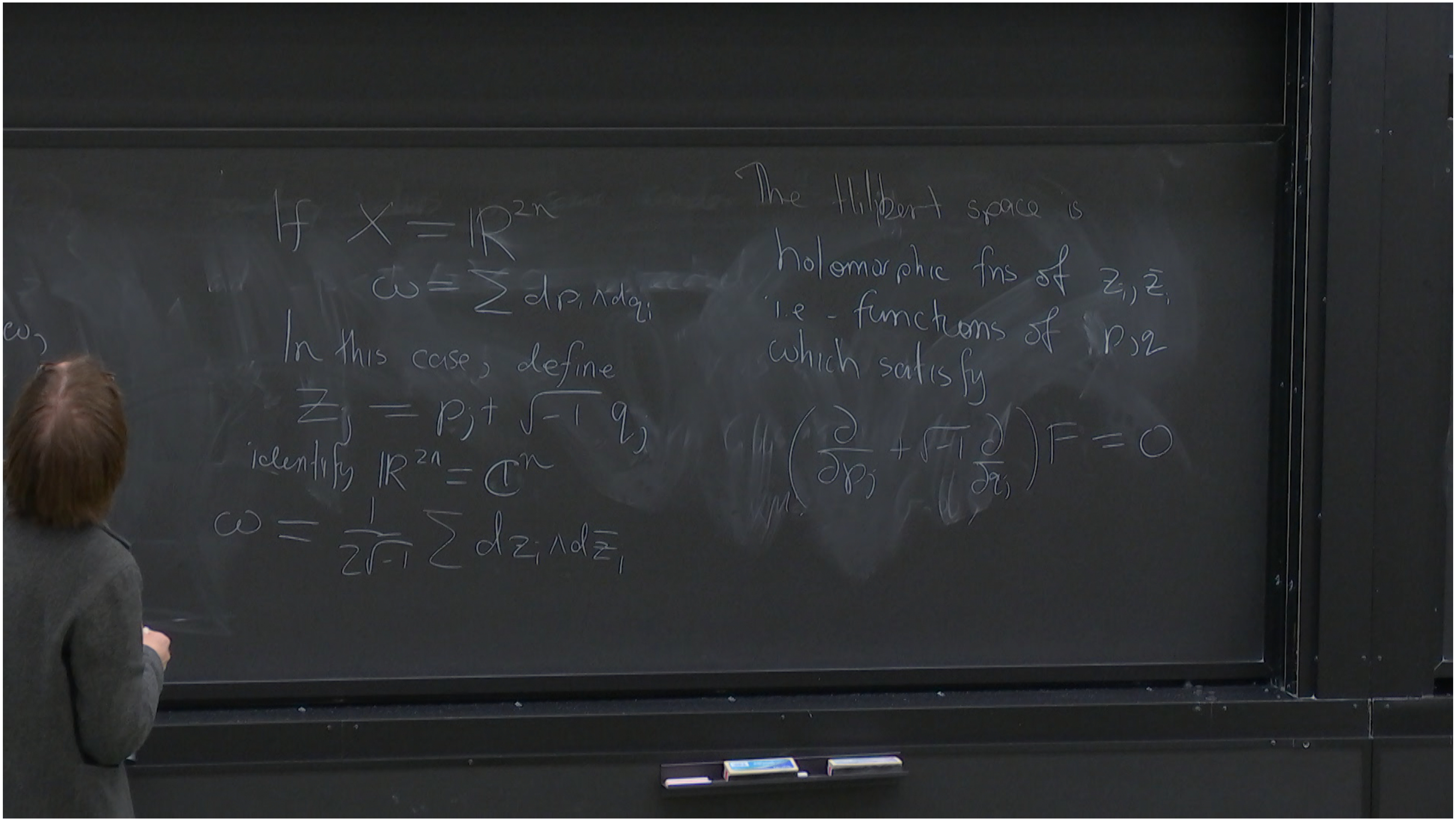
In this case, define

$$z_j = p_j + \sqrt{-1}\, q_j$$

identify $\mathbb{R}^{2n} = \mathbb{C}^n$

$$\omega = \frac{1}{2\sqrt{-1}} \sum dz_i \wedge d\bar{z}_i$$

The Hilbert space is
holomorphic fns of $z_i, \bar{z}_i$
i.e. functions of p, q
which satisfy


If $X = \mathbb{R}^{2n}$
$\omega = \sum dp_i \wedge dq_i$
In this case, define
$z_j = p_j + \sqrt{-1}\, q_j$
identify $\mathbb{R}^{2n} = \mathbb{C}^n$
$\omega = \frac{1}{2\sqrt{-1}} \sum dz_i \wedge d\bar{z}_i$
The Hilbert space is
holomorphic fns of $z_i, \bar{z}_i$
i.e. functions of p, q
which satisfy
$\left(\frac{\partial}{\partial p_j} + \sqrt{-1}\frac{\partial}{\partial q_j}\right) F = 0$

What happens globally?

If we have X and if globally $\omega = d\alpha$

α a 1-form, then we do the same thing.

- Choose a complex str. on X,

given by coords $z_i, \bar{z}_i$, locally

So that coordinate changes between patche,

$$\omega = \frac{?}{2\sqrt{-1}} \sum dz_i \wedge d\bar{z}_i$$

is holomorphic

Then, also assume

$$\omega = \sum g^{ij} dz_i \wedge d\bar{z}_j$$

The Hilbert space is hol. functions on X

When $\omega \neq d\alpha$ globally, but only locally on X,

(e.g. $X = T^2$, $\omega = d\lambda_1 \wedge d\lambda_2$)

then this only works locally, and gluing rules are changed...