Title: PSI 2019/2020 - Chern-Simons Theory Part 1 - Lecture 10

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Collection: PSI 2019/2020 - Chern-Simons Theory (Part 1)

Date: March 04, 2020 - 11:30 AM

URL: http://pirsa.org/20030018

Last Time

- Axiomatic TFT
- Wanted to show that Hilbert space

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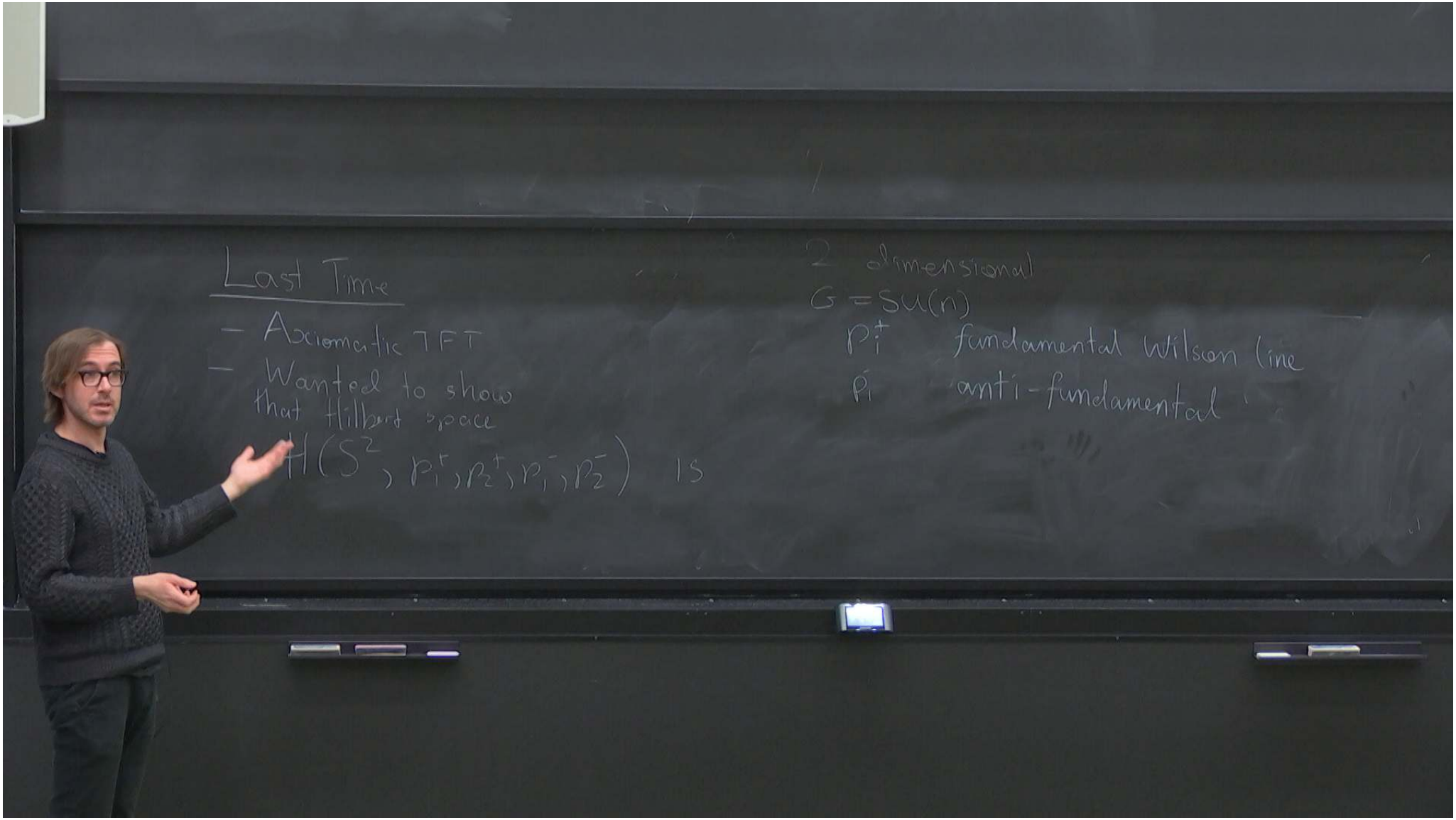
Last Time
- Axiomatic TFT
- Wanted to show that Hilbert space
$\mathcal{H}(S^2, P_1^+, P_2^+, P_1^-, P_2^-)$ is
2 dimensional
$G = SU(n)$
P_i^+ fundamental Wilson line
P_i^- anti-fundamental

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$P_i^\dagger$ fundamental Wilson line

P_i anti-fundamental

$\mathcal{H}(S, p_1^+, p_2^+, p_1^-, p_2^-)$ is

How to compute the Hilbert space?

Ansatz (any field theory)

- On a manifold $M \times (-\varepsilon, \varepsilon)$ study the space of solns to EOM

Call this $EOM(M)$

Claim, because we are studying a PDE of variational origin, $EOM(m)$ has a natural closed 2-form ω, so that there are coords p_i, q_i

$$\omega = \sum dq_i \wedge dp_i$$

[Note: ω^{-1} = Peirls bracket]

Any such φ is

$$\varphi = \varphi(0) + t\varphi'(0)$$

The space is 2d, coordinates

$$p = \varphi'(0)$$

$$q = \varphi(0)$$

$$\omega = dp \wedge dq$$

The Hilbert space is functions of q – correct answer.

Hilb

Example

CS theory on S^2, no Wilson lines.

EOM says $F(A)=0$

In general, if we have a manifold M and a gauge field on M w $F(A)=0$

if all loops in M can be contracted to a point (M is simply connected)

then, $F(A)=0$

$\Rightarrow A$ is gauge trivial

S^2 has this property

So sol'ns to EOM = a point

$G = U(1)$

CS action $\int \frac{k}{4\pi i} A dA$

on $S^2 \times \mathbb{R}$

Add Wilson lines at $p_i \in S^2$ of charge ℓ_i

$$\int_{S^2 \times \mathbb{R}} \frac{k}{4\pi i} A dA + \sum_i \int_{p_i \times \mathbb{R}} \ell_i A$$

$S^2 = \mathbb{C} \cup \{\infty\}$

$$A = \sum \frac{c_i}{k} \frac{1}{z - p_i} dz$$

away from ∞

$$d\left(\frac{1}{z} dz\right) = 2\pi i \, \delta_{z=0}$$

((auchy's formula)

Near ∞,

$$A \sim \sum \frac{c_i}{k} \frac{dz}{z}$$

1st order pole at ∞

Can we remove this by a gauge transform?

gauge transform?

the singularity at c_0.

This happens if $\sum l_i/k$ is an integer

EOM tell us that

$$F(A) = \sum \frac{l_i}{k} 2\pi_i \delta_{p_i}$$

Claim that we can solve this if and only if $\sum \frac{l_i}{k}$ is an integer.

Then, there is 1 solution, so the Hilbert space is $\mathbb{C}$

Otherwise, Hilbert space is 0.

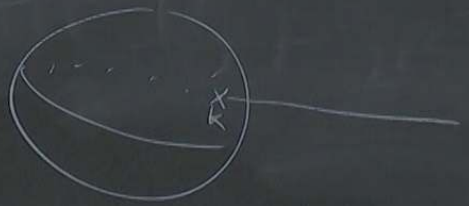
Note

For U(1) CS at level k,

Wilson line of charge k

= Wilson line of charge 0

We're seeing why State at end of



Wilson (

is an 'int

charge 0

Wilson line of charge k,
is an interface between charge 0 and charge k lines

Wilson lines are coupled Q.M systems

Consider a gauge theory with a Wilson line - fundamental
$G = SU(n)$

Treat the gauge field as a background, classical field

Wilson line is defined by parallel transport:

$\psi_0 \in \mathbb{C}^n$

have the equation

$$dt \frac{\partial}{\partial t} \psi(t) + (\gamma^* A)\psi(t) = 0$$

$\gamma : [0,1] \longrightarrow \mathbb{R}^n$ is the path

$\gamma^* A = dt\, m(t)$, $m(t)$ is $n \times n$ Hermitian matrix.

$$\frac{\partial}{\partial t}\psi(t) + m(t)\psi(t) = 0$$

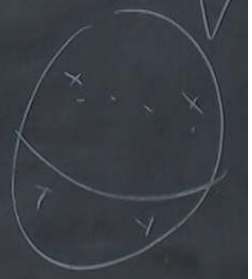
This is the Schrödinger equation!

Time-dependent Hamiltonian which is $\gamma^* A$

Wilson line

= quantum particle
travelling along γ
Hamiltonian $\gamma^* A$.

The Hilbert space of a Wilson line is then, obviously, $\mathbb{C}^n$ (or whatever rep. of G we use to define it)



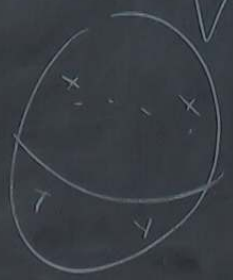
Study

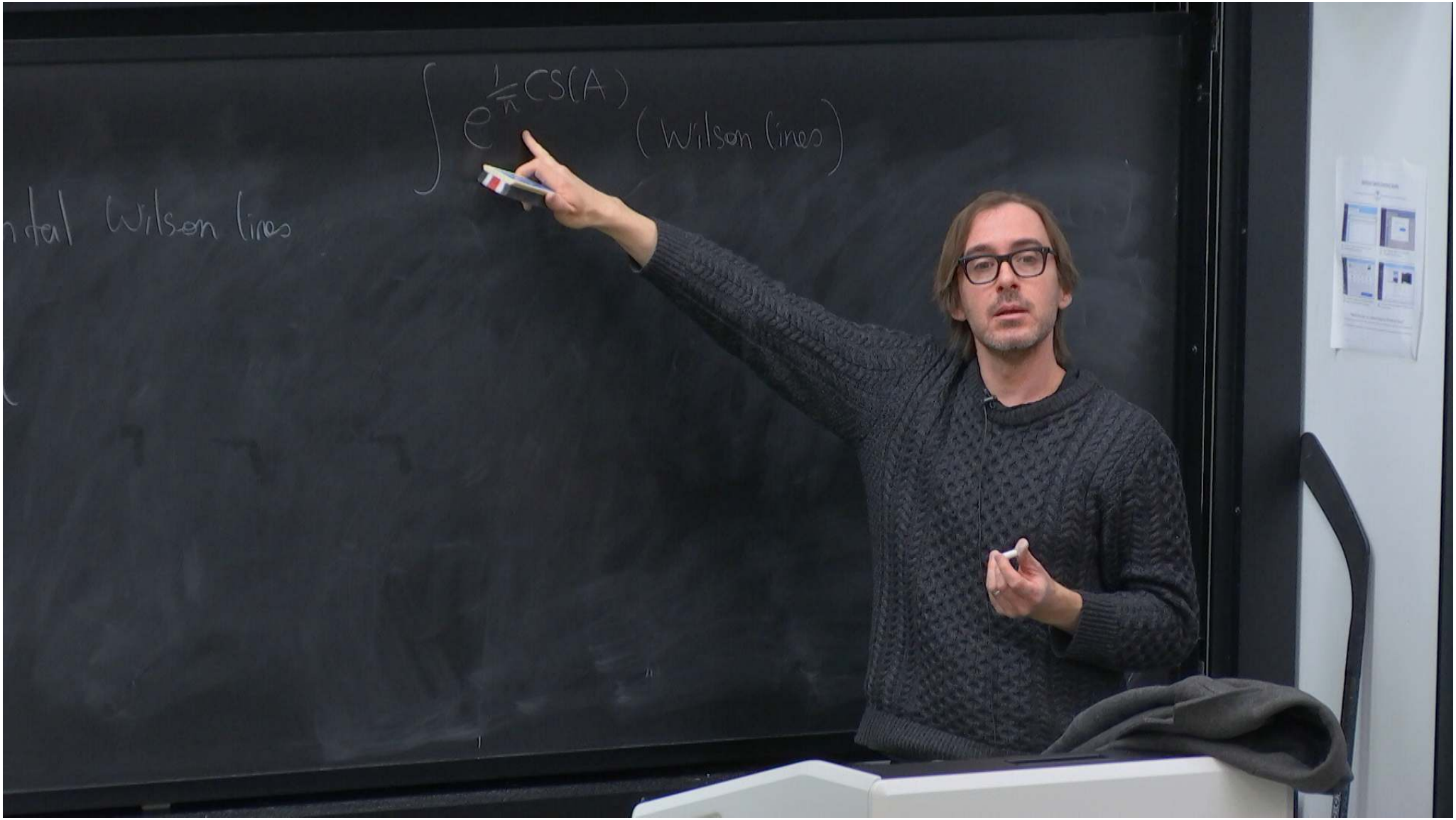
S^2, with 2 fundamental Wilson lines

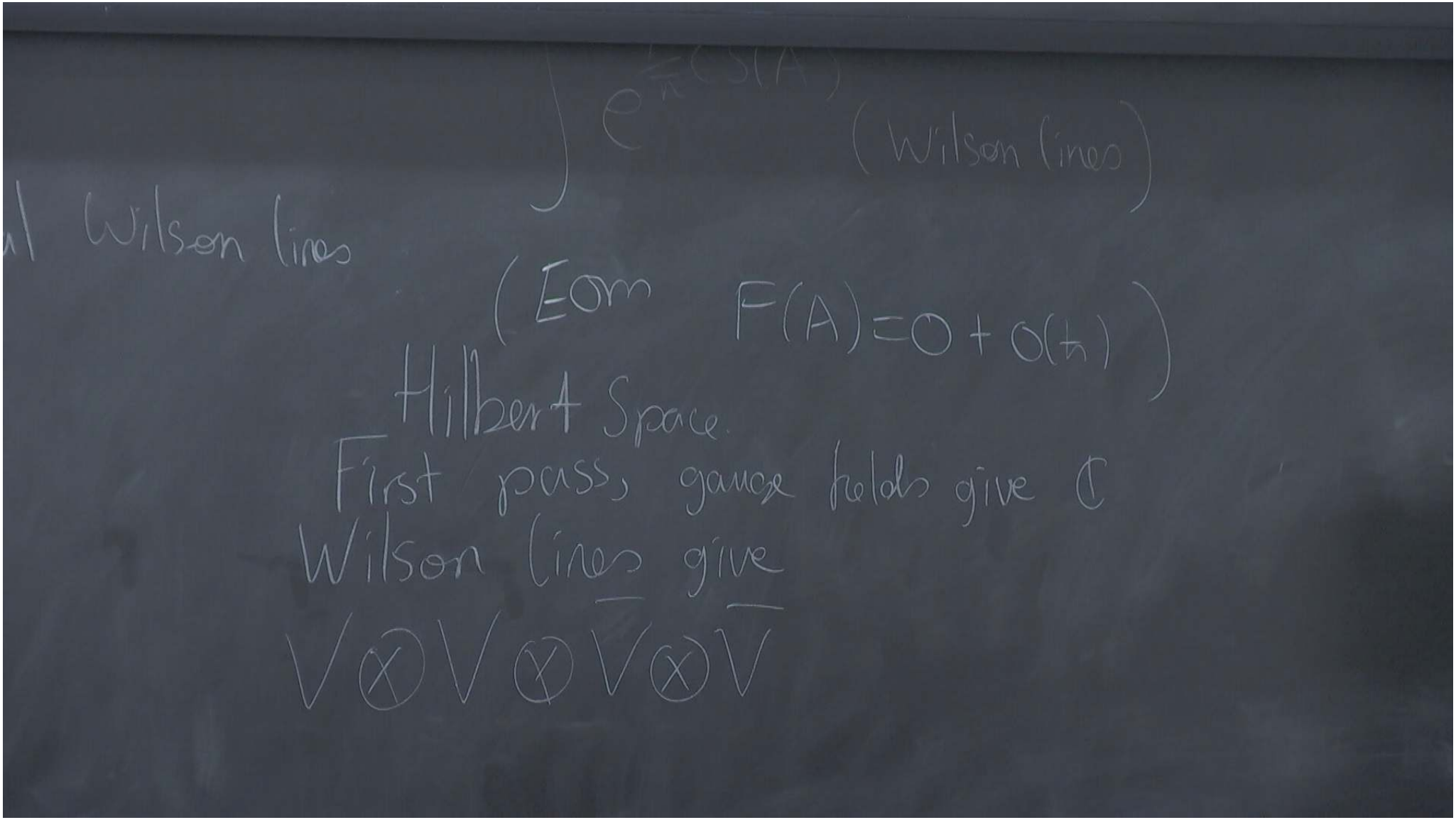
$V = \mathbb{C}^n$

2 anti-fundamental

$V^* = \overline{V} = \overline{\mathbb{C}}^n$



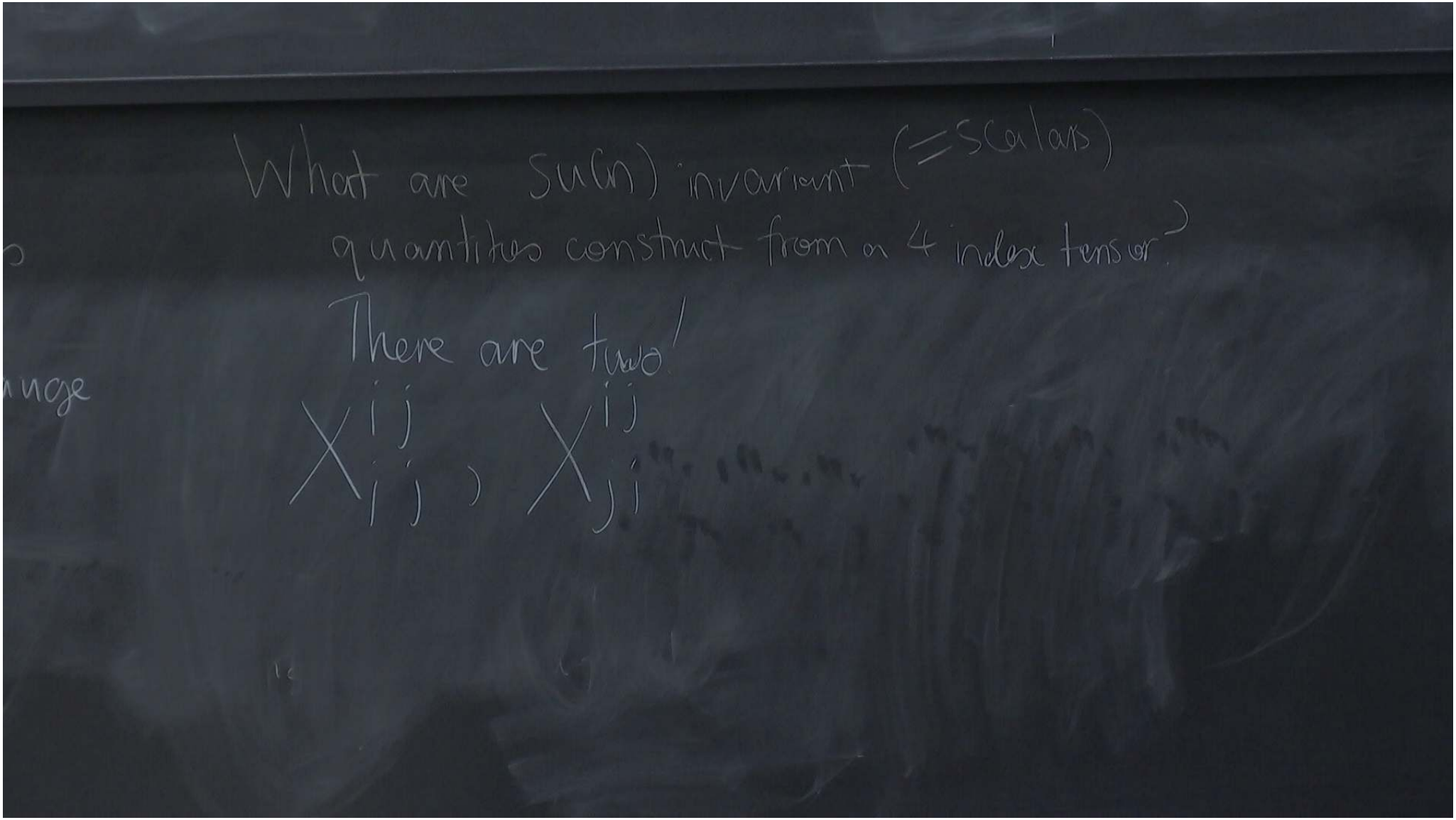

$\int e^{\frac{1}{\hbar}CS(A)}$ (Wilson lines)
ntal Wilson lines

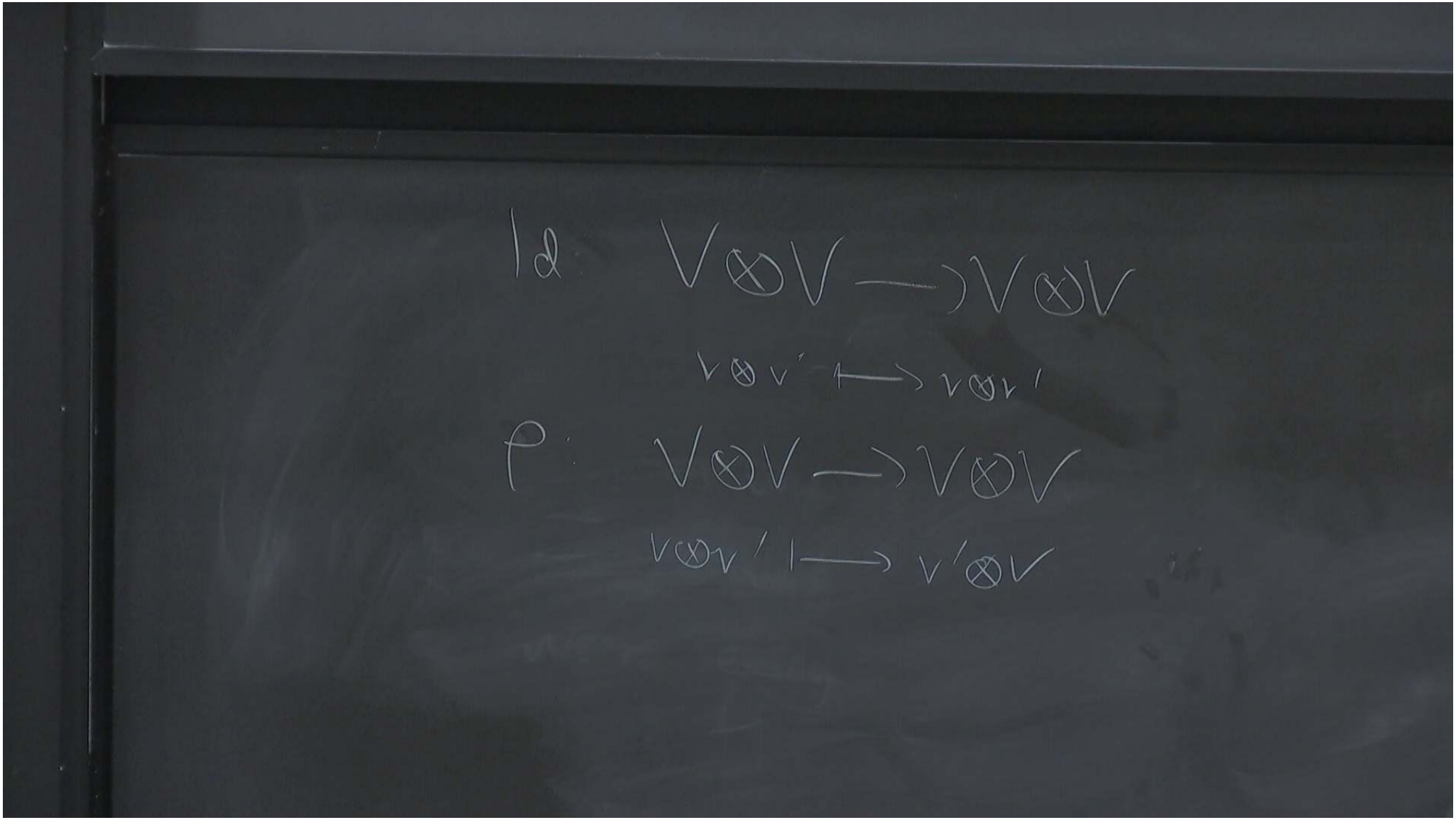

$\int e^{\frac{i}{\hbar}S(A)}$
(Wilson lines)
l Wilson lines
(Eom $F(A)=0+O(\hbar)$)
Hilbert Space.
First pass, gauge holds give $\mathbb{C}$
Wilson lines give
$V \otimes V \otimes \bar{V} \otimes \bar{V}$

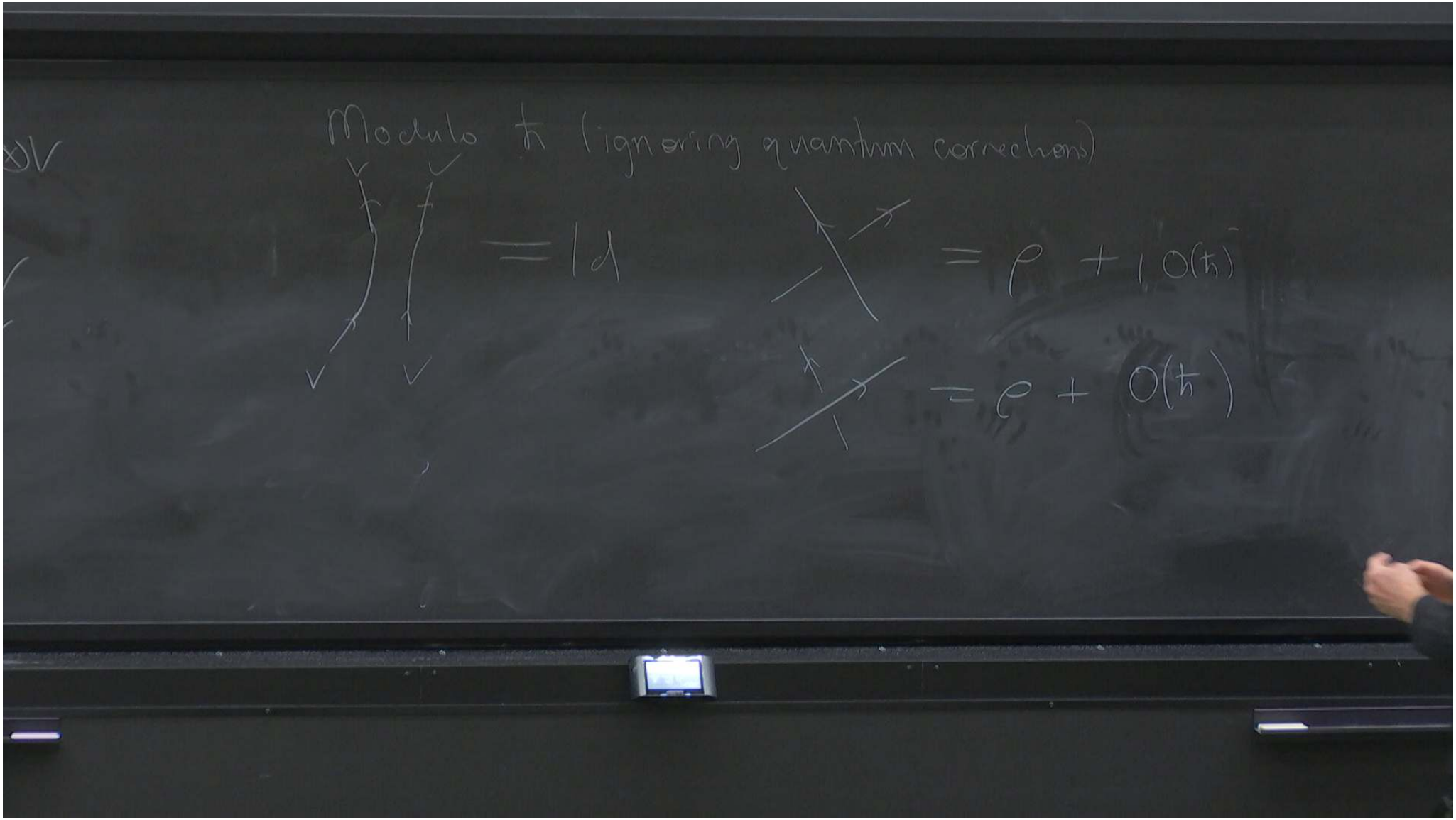
But, $A=0$ has constant gauge symmetries $SU(n)$. We need to project onto gauge invariant states; these are

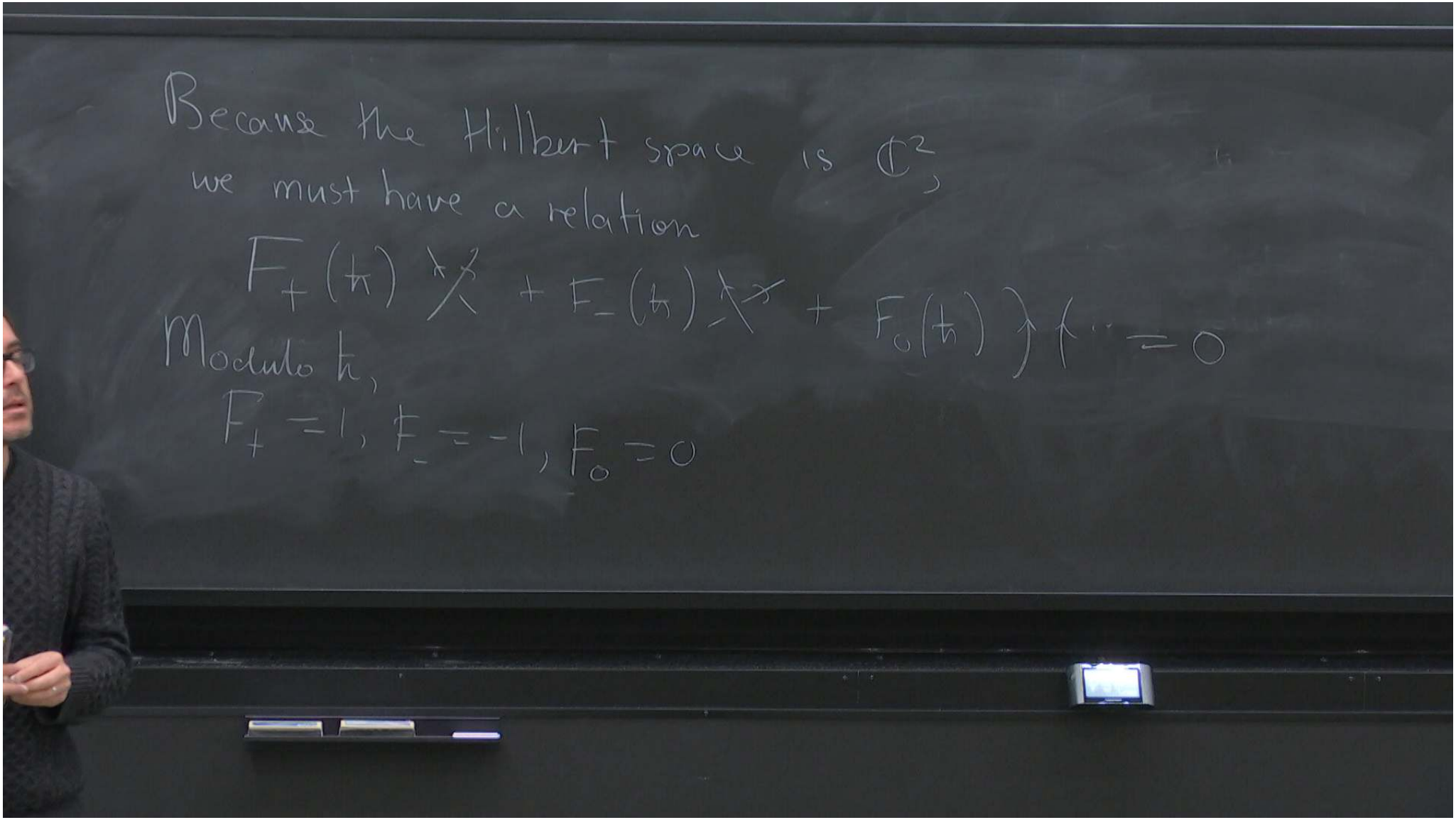
$$(V \otimes V \otimes \bar{V} \otimes \bar{V})^{SU(n)}$$

$$\chi^{k\ell}_{ij}$$


What are SU(n) invariant (=scalars)
quantities construct from a 4 index tensor?
There are two!
X^{ij}_{ij} , X^{ij}_{ji}


Id: V⊗V → V⊗V
v⊗v' ⟼ v⊗v'
P: V⊗V → V⊗V
v⊗v' ⟼ v'⊗v


Modulo ħ (ignoring quantum corrections)
= Id
= ρ + O(ħ)
= ρ + O(ħ)


Because the Hilbert space is $\mathbb{C}^2$,
we must have a relation
$F_+(\hbar)$ + $F_-(\hbar)$ + $F_0(\hbar)$ $= 0$
Modulo $\hbar$,
$F_+ = 1$, $F_- = -1$, $F_0 = 0$

Because the Hilbert space is $\mathbb{C}^2$, on $(S^2, p_1^+, p_2^+, p_1^-, p_2^-)$ $SU(n)$ we must have a relation

$$F_+(\hbar)\, X^+ + F_-(\hbar)\, X^- + F_0(\hbar)\, \} \cdots = 0$$

Modulo $\hbar$,

$$F_+ = 1,\ F_- = -1,\ F_0 = 0$$