Title: Entanglement entropy of highly excited eigenstates of many-body lattice Hamiltonians

Speakers: Marcos Rigol

Series: Colloquium

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Abstract:

The average entanglement entropy of subsystems of random pure states is (nearly) maximal. In this talk, we discuss the average entanglement entropy of subsystems of highly excited eigenstates of integrable and nonintegrable many-body lattice Hamiltonians. For translationally invariant quadratic models (or spin models mappable to them) we prove that, when the subsystem size is not a vanishing fraction of the entire system, the average eigenstate entanglement entropy exhibits a leading volume-law term that is different from that of random pure states. We argue that such a leading term is likely universal for translationally invariant (noninteracting and interacting) integrable models. For random pure states with a fixed particle number (random canonical states) away from half filling and normally distributed real coefficients, we prove that the deviation from the maximal value grows with the square root of the system's volume when the size of the subsystem is one half of that of the system. We then show that the average entanglement entropy of highly excited eigenstates of a particle number conserving quantum chaotic model is the same as that of random canonical states.

Entanglement entropy of highly excited eigenstates of many-body lattice Hamiltonians

Marcos Rigol

Department of Physics
The Pennsylvania State University

Perimeter Institute for Theoretical Physics

March 4, 2020

Outline

1. Introduction
   - Experiments with ultracold gases in one dimension
   - Classical and quantum: integrability vs chaos
2. Entanglement entropy
   - Entanglement entropies and experiments
   - Integrability vs nonintegrability
   - Quadratic fermionic Hamiltonians
   - Quantum chaotic Hamiltonians
   - Quantum chaos and eigenstate thermalization
3. Summary

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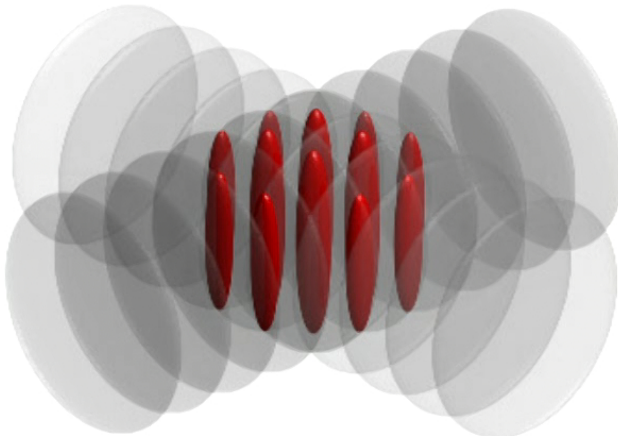
Experiments in the 1D regime



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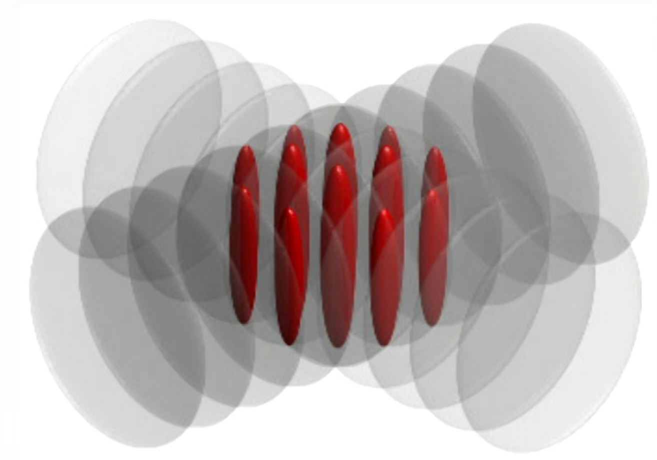

Effective one-dimensional δ potential

M. Olshanii, PRL **81**, 938 (1998).

$$U_{1D}(x) = g_{1D}\delta(x)$$

where

$$g_{1D} = \frac{2\hbar a_s \omega_\perp}{1 - C a_s \sqrt{\frac{m\omega_\perp}{2\hbar}}}$$

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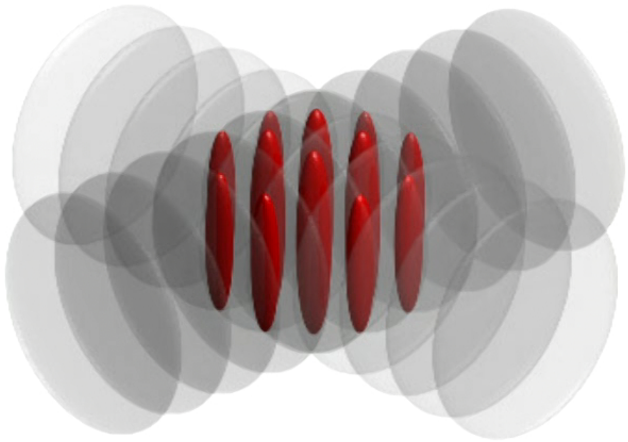

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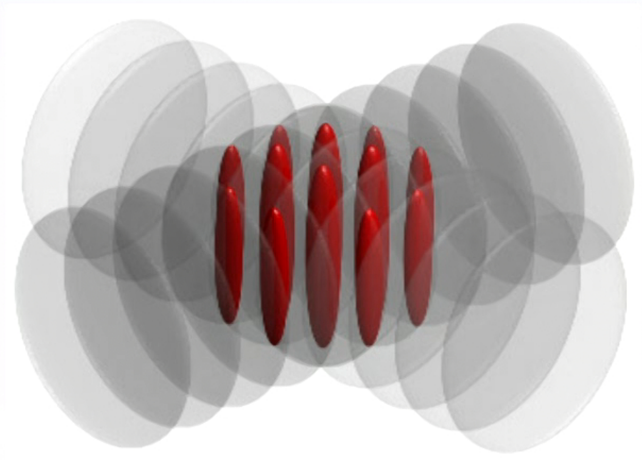

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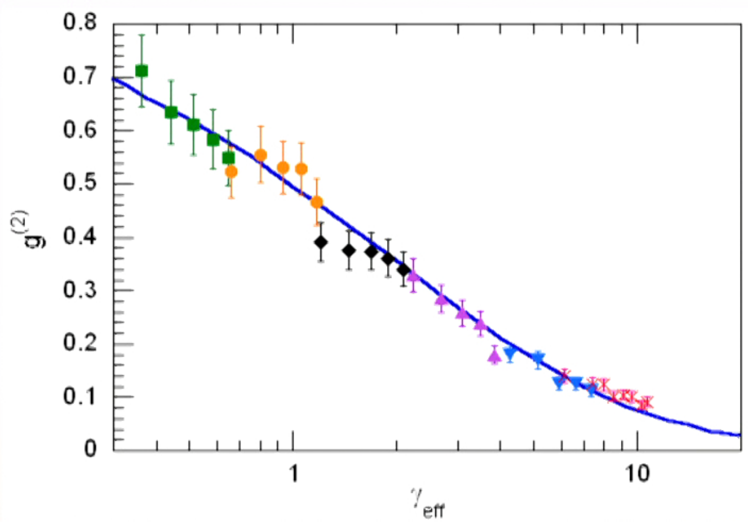

$g^{(2)}(x) = \frac{\langle \hat{\Psi}^{\dagger 2}(x)\Psi^2(x)\rangle}{n_{1D}^2(x)}$ and $\gamma = \frac{m g_{1D}}{\hbar^2 n_{1D}}$ $\Leftrightarrow$

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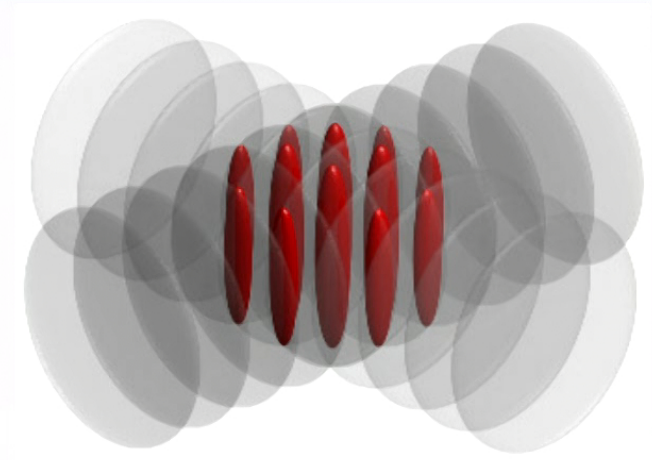

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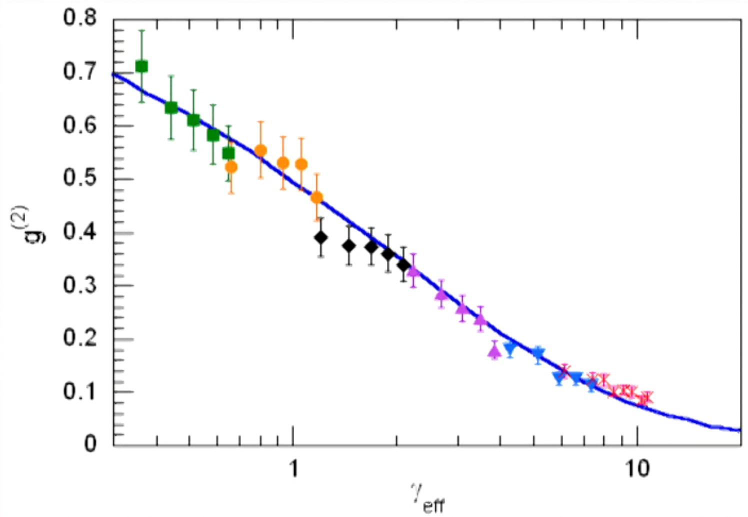

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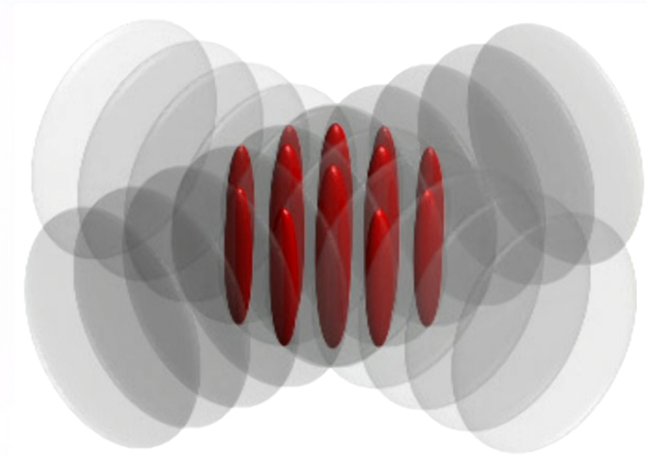

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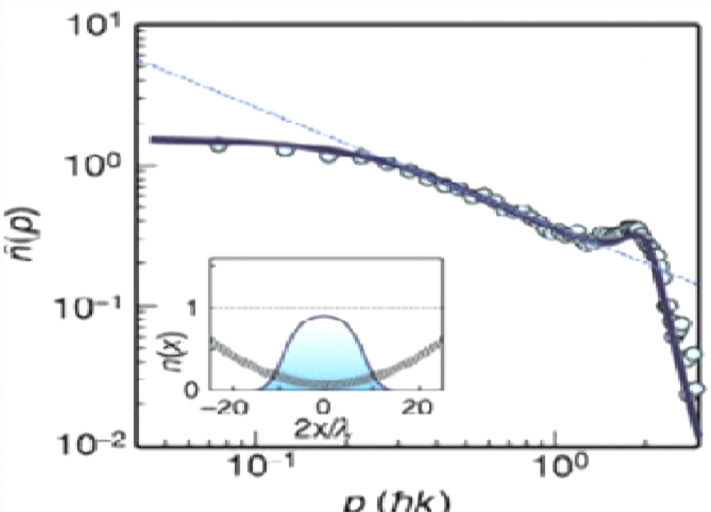
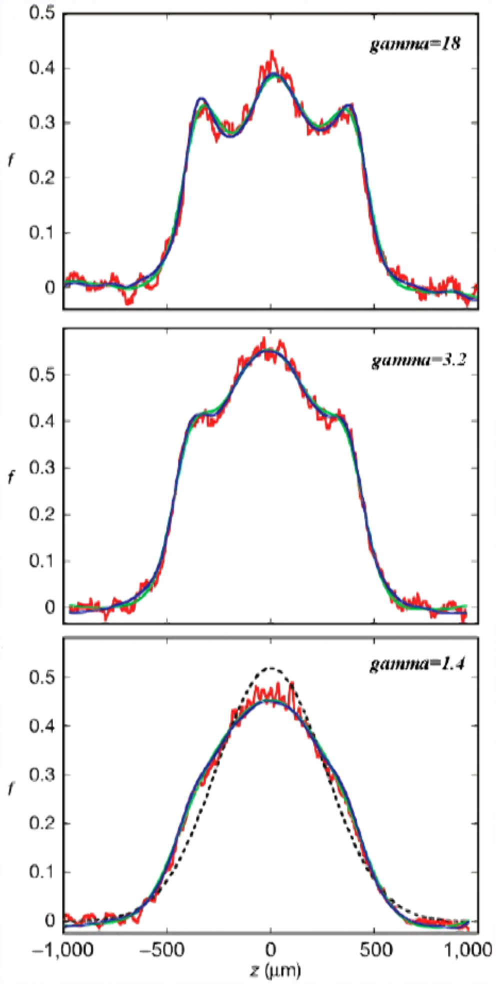
Lieb, Schulz, and Mattis '61 ($U/J = \infty$)

B. Paredes *et al.*,
Nature (London) **429**, 277 (2004).

$n(p)$: Momentum distribution ⇔
$n(x)$: Density distribution ⇔



Quantum Newton's cradle



T. Kinoshita, T. Wenger, and D. S. Weiss, Nature **440**, 900 (2006).

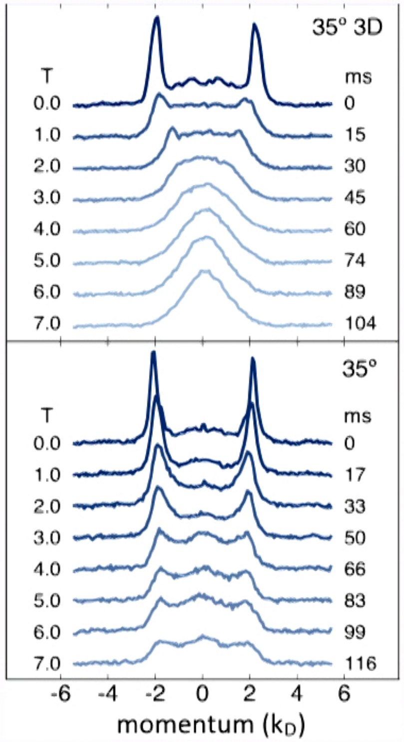
$$\gamma = \frac{m g_{1D}}{\hbar^2 n_{1D}}$$

g_{1D}: Contact interaction strength
n_{1D}: One-dimensional density

If $\gamma \gg 1$ the system is in the strongly correlated Tonks-Girardeau regime

If $\gamma \ll 1$ the system is in the weakly interacting regime

Dipolar quantum Newton's cradle (dysprosium atoms)



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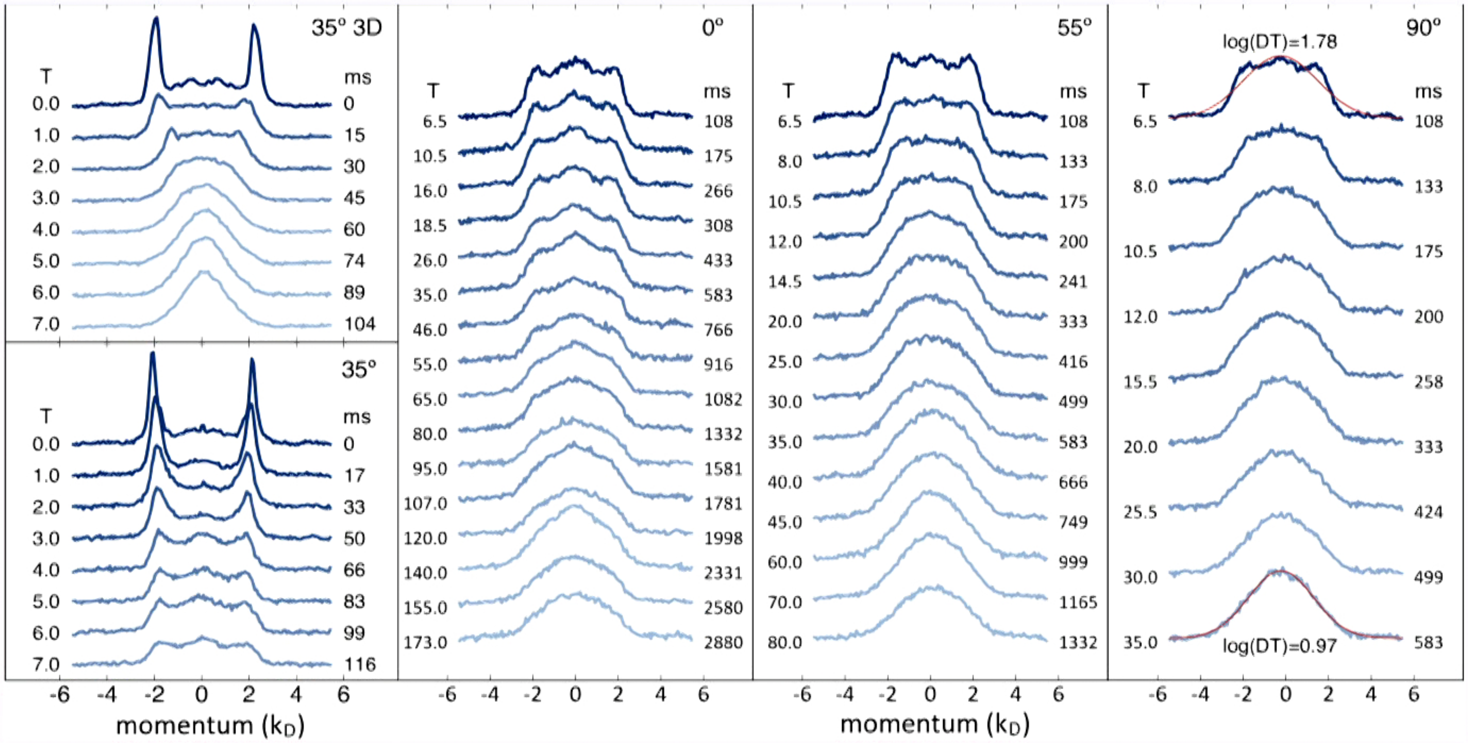

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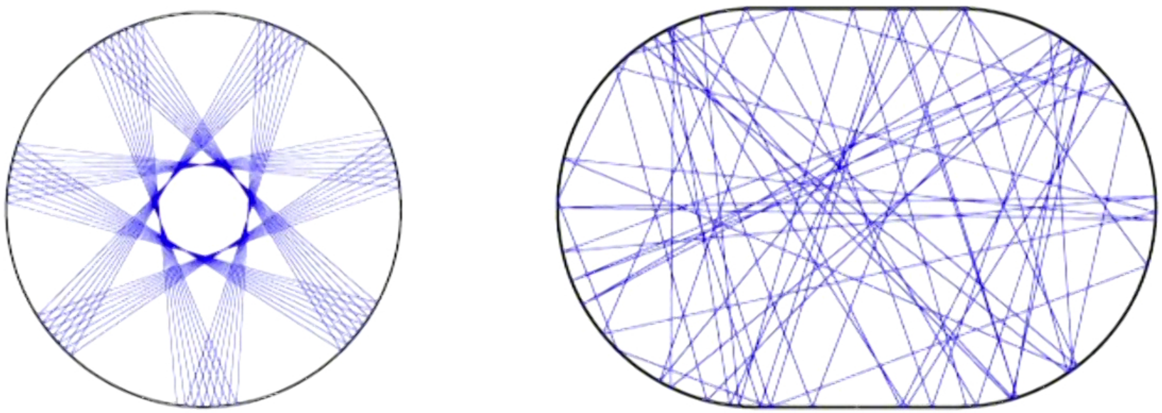
Classical integrability and chaos

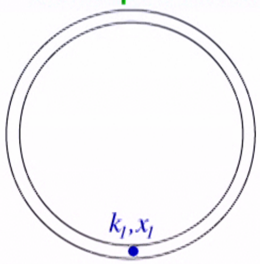
Particle trajectories in a circular cavity and a Bunimovich stadium (scholarpedia)



- Integrability: A system is said to be integrable if it has as many functionally independent constants of motion in involution as degrees of freedom
- Chaos: exponential sensitivity of the trajectories to perturbations

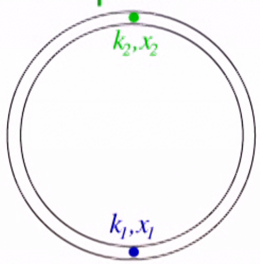
Scattering without diffraction (Quantum integrability)

One particle



Momentum	Energy	Wavefunction
k_1	$\varepsilon(k_1) = \frac{(k_1)^2}{2}$	$\Psi(x_1) = e^{ik_1 x_1}$

Two particles



$$K = k_1 + k_2 \qquad E = \varepsilon(k_1) + \varepsilon(k_2)$$

$$\Psi(x_1, x_2) \rightarrow \sum_P A(P)\, e^{i(k_{P1}x_1 + k_{P2}x_2)}$$

$$= A(12)\, e^{i(k_1 x_1 + k_2 x_2)} + A(21)\, e^{i(k_2 x_1 + k_1 x_2)}$$

B. Sutherland, *Beautiful Models* (World Scientific, Singapore, 2004).

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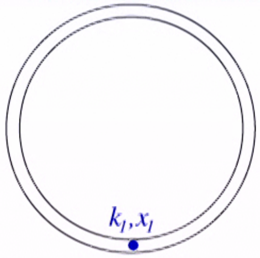

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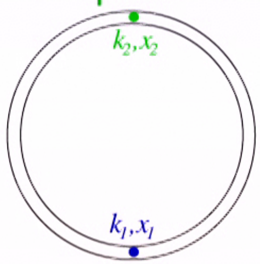

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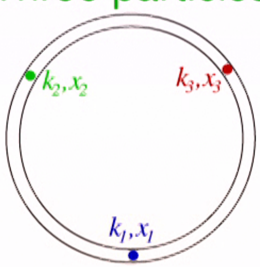
Three particles



$$K = k_1 + k_2 + k_3 \qquad E = \varepsilon(k_1) + \varepsilon(k_2) + \varepsilon(k_3)$$

$$\Psi(x_1, x_2, x_3) \rightarrow \sum_P A(P)\, e^{i(k_{P1}x_1 + k_{P2}x_2 + k_{P3}x_3)}$$

$+$ ~~diffractive scattering~~

B. Sutherland, *Beautiful Models* (World Scientific, Singapore, 2004).

Random matrix theory (Quantum chaos)

- Wigner (1955) & Dyson (1962): Statistical properties of the spectra of complex quantum systems (in a narrow energy window) can be predicted from the statistical properties of the spectra of random matrices (with the appropriate symmetries). It was used with great success to understand the spectra of complex nuclei.

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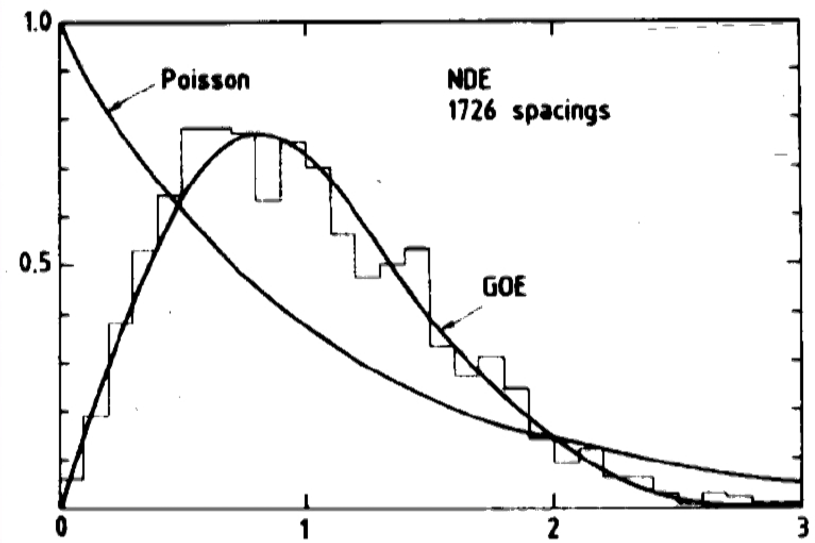
Distribution of level spacings for the "Nuclear Data Ensemble"



T. Guhr *et al.*, Physics Reports **299**, 189 (1998).

Semi-classical limit: Statistics of energy levels

- Berry-Tabor conjecture (1977): The statistics of level spacings of quantum systems whose classical counterpart is integrable is described by a Poisson distribution. (Energy eigenvalues behave like a sequence of independent random variables.)

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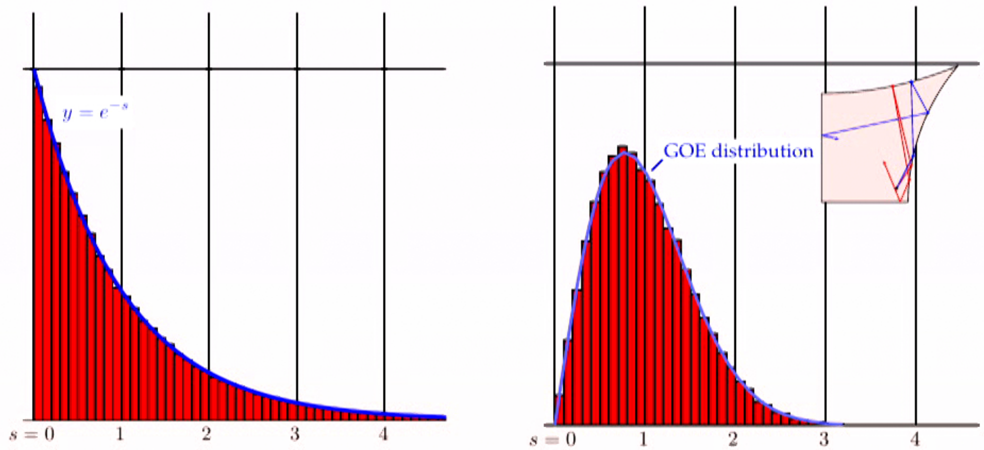
Distribution of level spacings: rectangular and chaotic cavities



Z. Rudnik, Notices AMS **55**, 32 (2008).

Integrability to quantum chaos transition

Spinless fermions (hard-core bosons, spin-1/2) in one dimension

$$\hat{H} = \sum_{i=1}^{L} \left\{ -t \left(\hat{f}_i^\dagger \hat{f}_{i+1} + \text{H.c.} \right) + V \hat{n}_i \hat{n}_{i+1} - t' \left(\hat{f}_i^\dagger \hat{f}_{i+2} + \text{H.c.} \right) + V' \hat{n}_i \hat{n}_{i+2} \right\}$$

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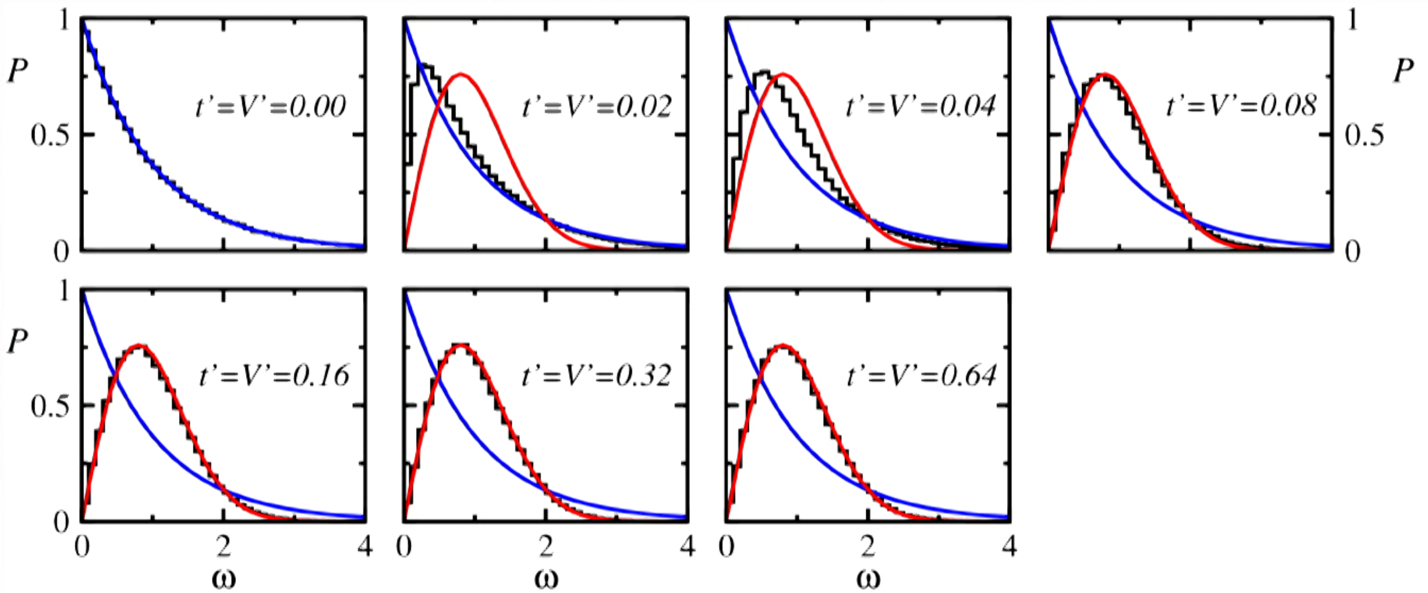
Level spacing distribution ($N_f = L/3$)



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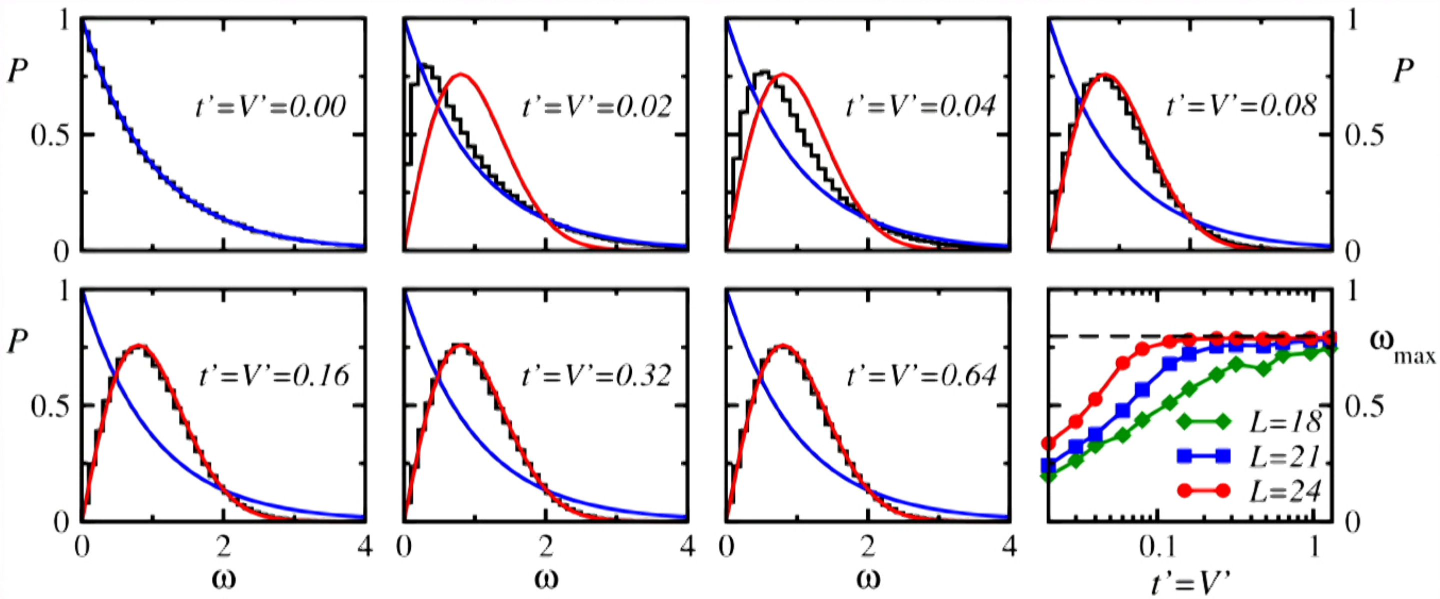

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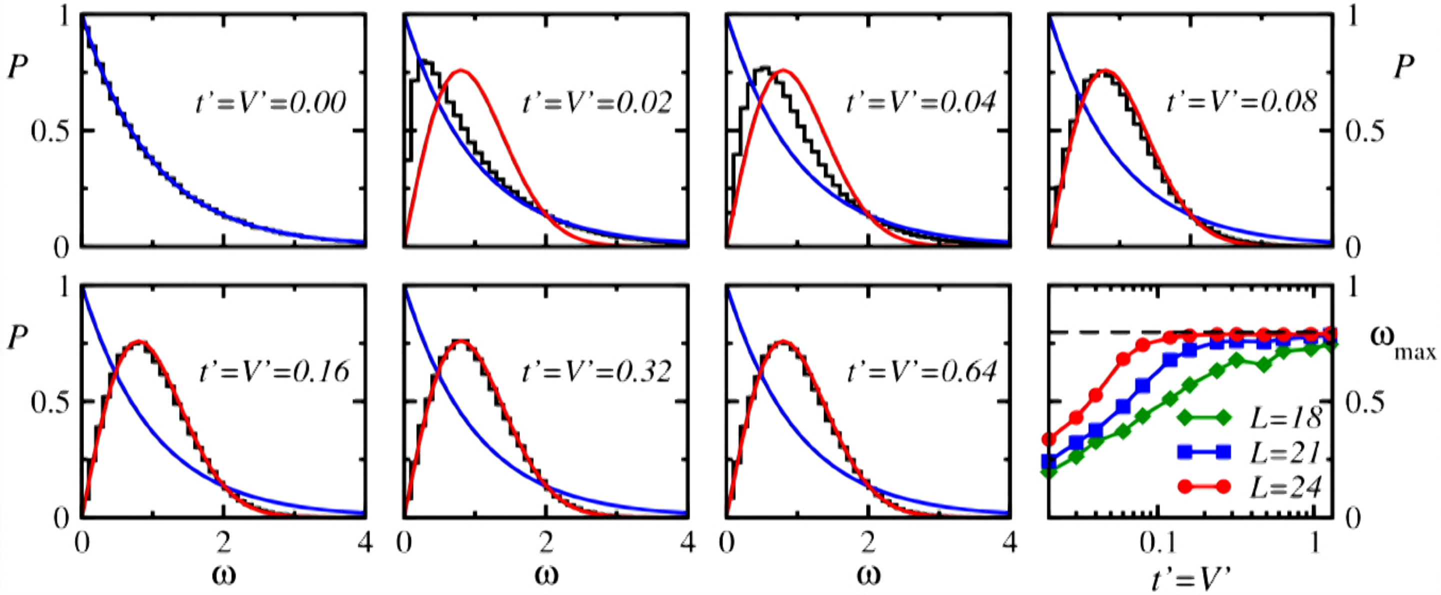

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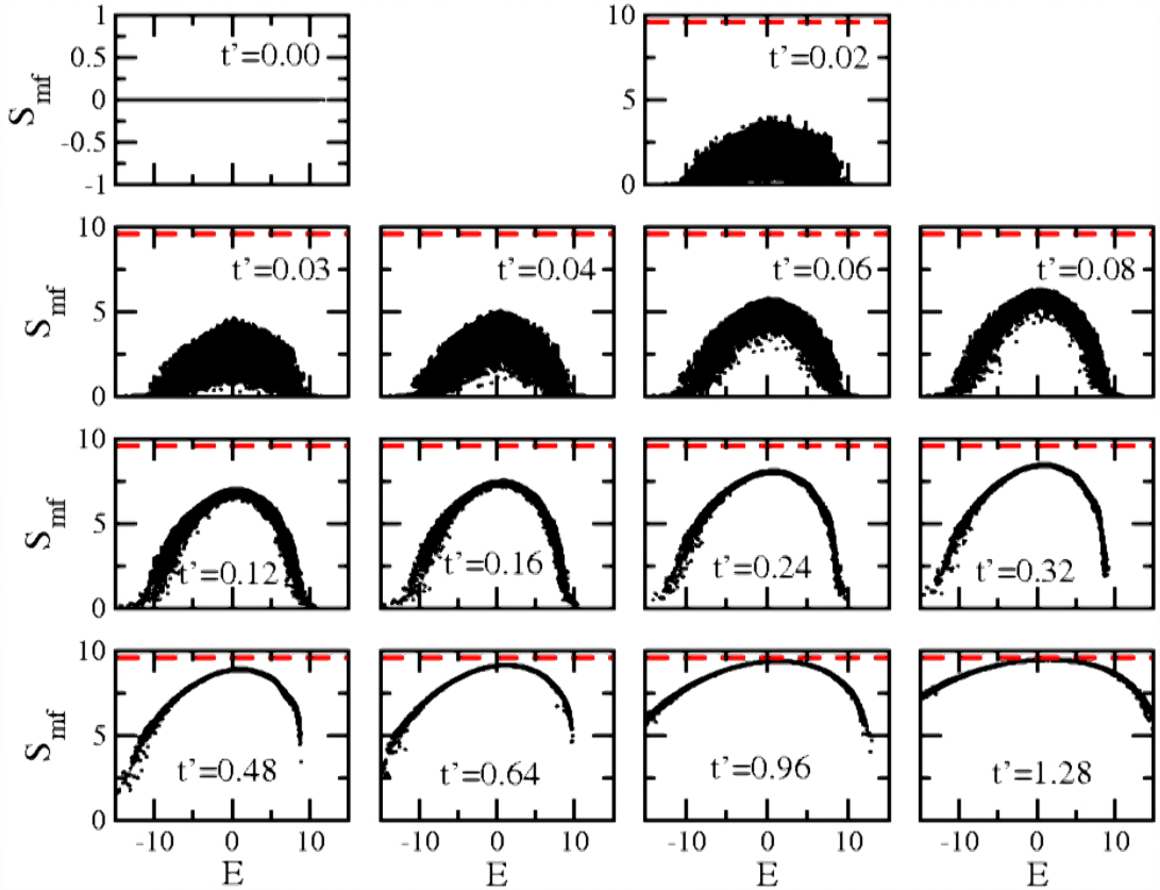
Information entropy ($S_j = -\sum_{k=1}^{D} |c_j^k|^2 \ln |c_j^k|^2$)



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von Neumann and Renyi entanglement entropies

Let $|m\rangle$ be a state ket of a many-body system. The reduced density matrix

$\hat{\rho}_A(m) = \mathrm{Tr}_{\bar{A}}(|m\rangle\langle m|)$, A is the subsystem of interest and $\bar{A}$ is its complement

The von Neumann and n-Renyi entanglement entropies are defined as:

$$S_{\mathrm{vN}}(m) = -\mathrm{Tr}[\hat{\rho}_A(m)\ln\hat{\rho}_A(m)] \quad \text{and} \quad S_n(m) = \frac{1}{1-n}\ln\mathrm{Tr}[\hat{\rho}_A(m)^n]$$

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Ground-state entanglement entropies (area laws, quantum phase transitions)

Srednicki'93; Osterloh, Amico, Falci & Fazio'02; Osborne & Nielsen'02;
Vidal, Latorre, Rico & Kitaev'03; Calabrese & Cardy'04, . . . , Eisert, Cramer & Plenio, RMP'10

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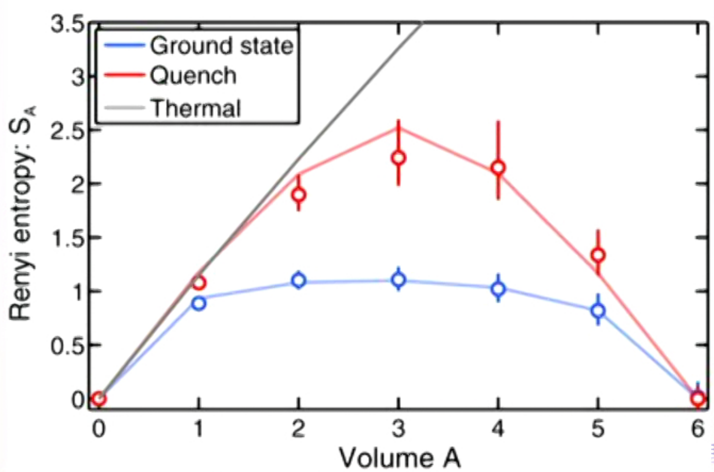
S_2 has been measured in experiments with ultracold quantum gases

Kaufman et al. (Greiner's group), Science **353**, 794 (2016).



Outline

Typicality (uniform distribution in the unit sphere)

Average (vN) entanglement entropy of subsystems of random pure states

$$\bar{S}_{\rm ran} \simeq \ln \mathcal{D}_{\rm A} - (1/2)\mathcal{D}_{\rm A}^2/\mathcal{D},$$

for $1 \ll \mathcal{D}_{\rm A} \leq \sqrt{\mathcal{D}}$.
D. N. Page, PRL **71**, 1291 (1993).

$\bar{S}_{\rm ran}$ for typical pure states exhibits a volume law ($\ln \mathcal{D}_{\rm A} \propto V_{\rm A}$ and $\ln \mathcal{D} \propto V$).
Typical pure states are (nearly) maximally entangled
(the correction is exponentially small for $V_{\rm A} < V/2$. It is $1/2$ for $V_{\rm A} = V/2$).

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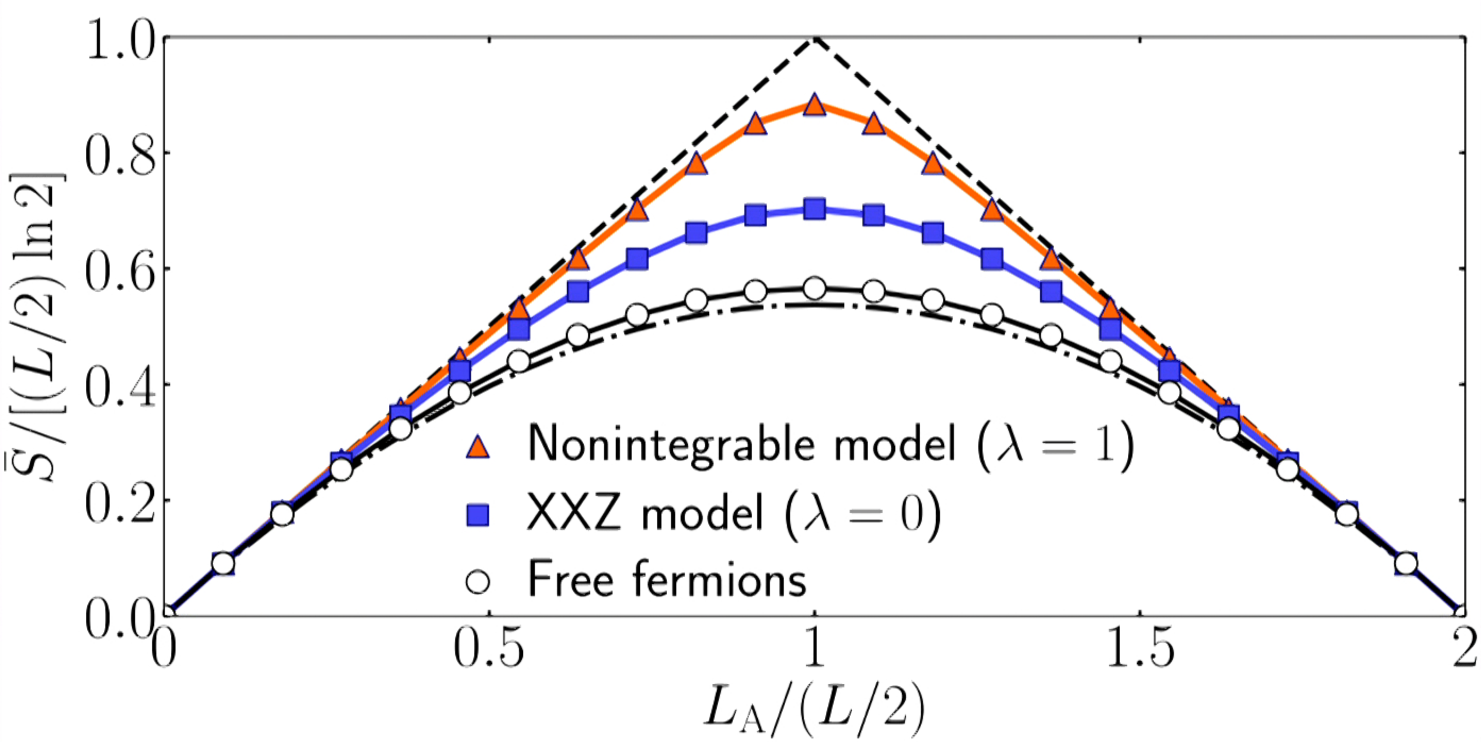
Spin-1/2 XXZ chain with next nearest neighbors interactions:

$$\hat{H} = \sum_{i=1}^{L} \left[\frac{1}{2}\left(\hat{S}_i^+ \hat{S}_{i+1}^- + \text{H.c.} \right) + \Delta \hat{S}_i^z \hat{S}_{i+1}^z \right] + \lambda \sum_{i=1}^{L} \left[\frac{1}{2}\left(\hat{S}_i^+ \hat{S}_{i+2}^- + \text{H.c.} \right) + \frac{1}{2} \hat{S}_i^z \hat{S}_{i+2}^z \right].$$

$\Delta = 0,\ \lambda = 0$: Mappable to free fermions
$\Delta \neq 0,\ \lambda = 0$: Interacting integrable model
$\Delta \neq 0,\ \lambda \neq 0$: Nonintegrable model

Integrability vs nonintegrability and entanglement

Average entanglement entropy ($L = 22$, $S^z_{\text{total}} = 0$, $L_A/L \in [0, 1]$)



T. LeBlond, K. Mallayya, L. Vidmar, and MR, PRE **100**, 062134 (2019).

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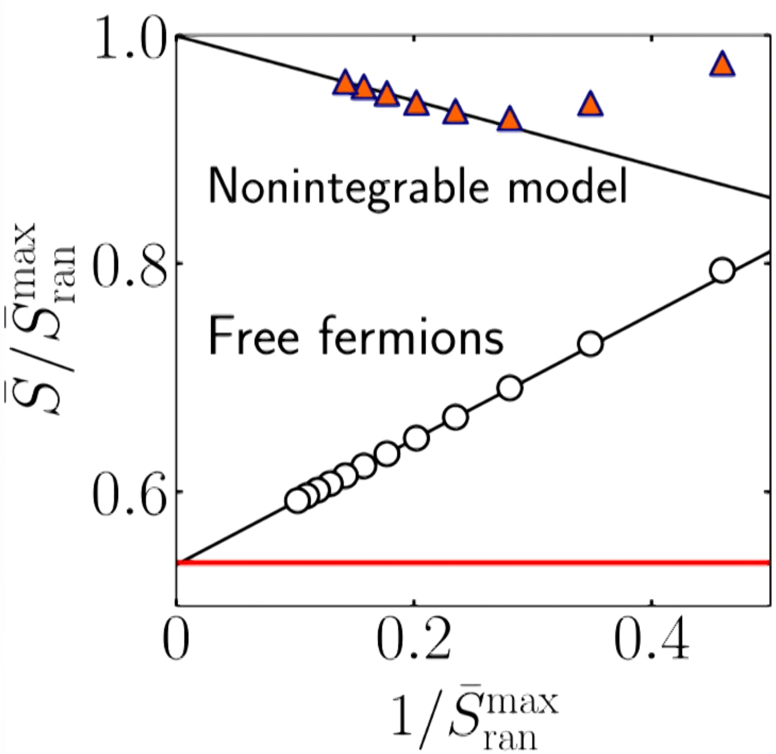

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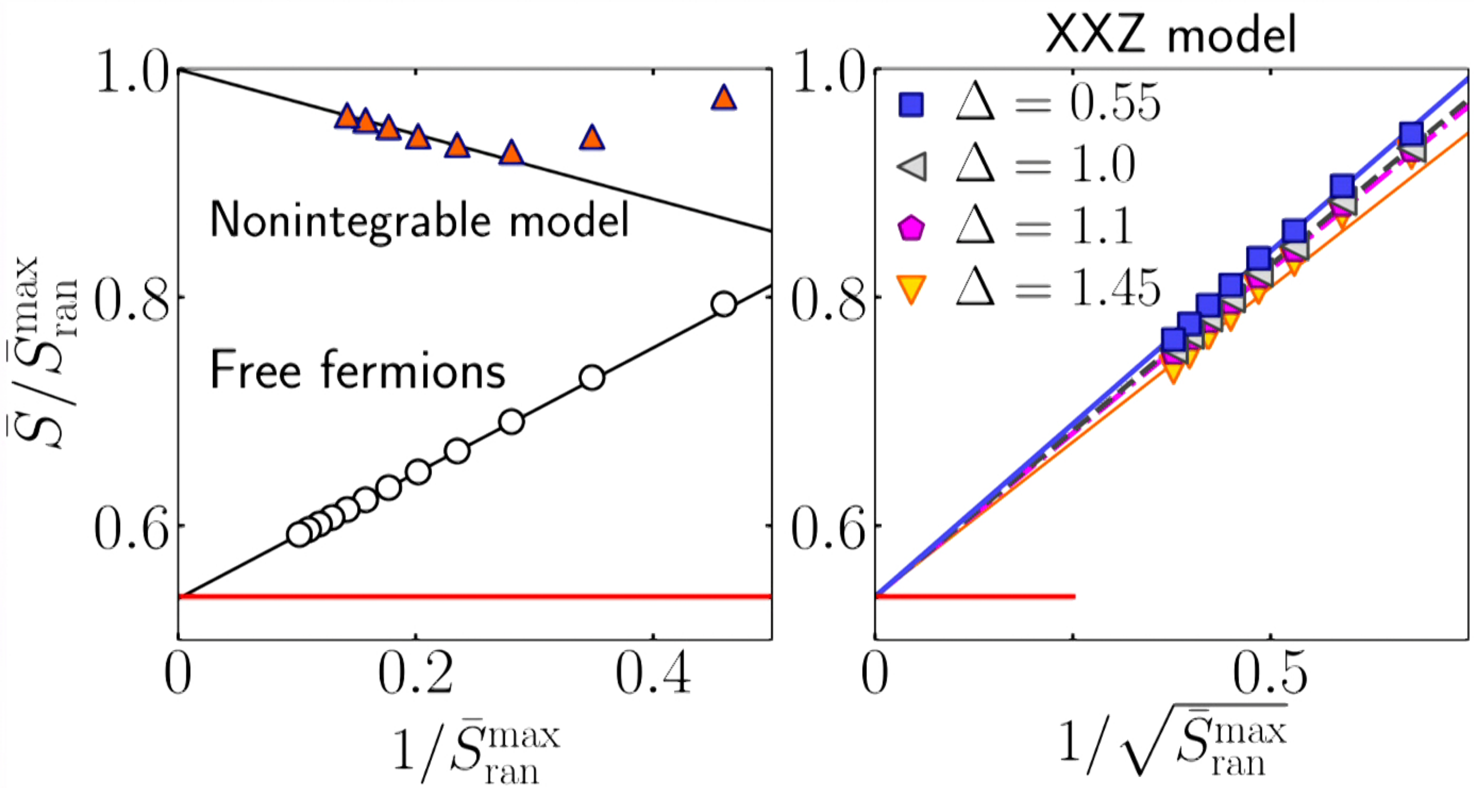
Integrability vs nonintegrability and entanglement

Average entanglement entropy ($L = 22$, $S^z_{\text{total}} = 0$, $L_A = L/2$)



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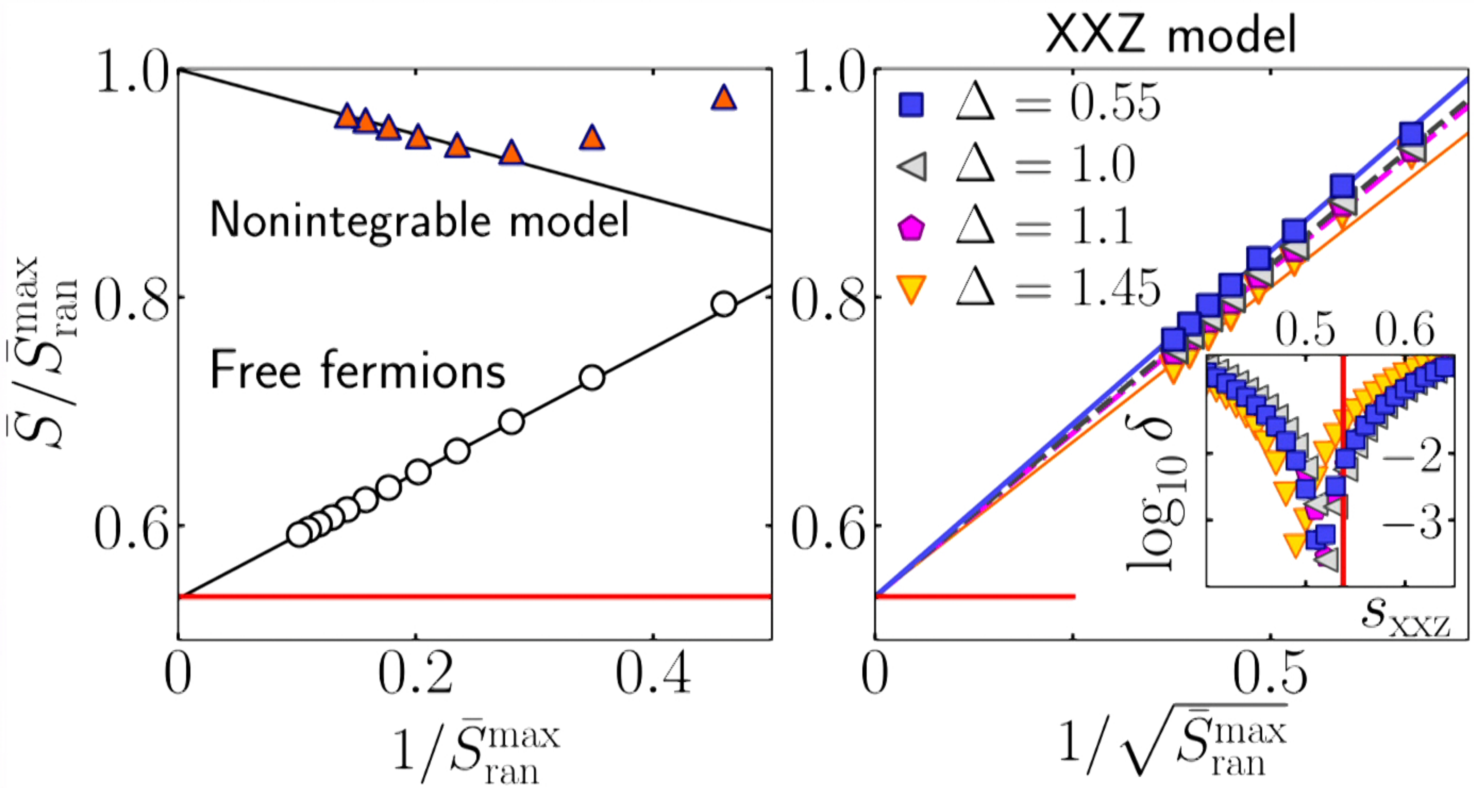

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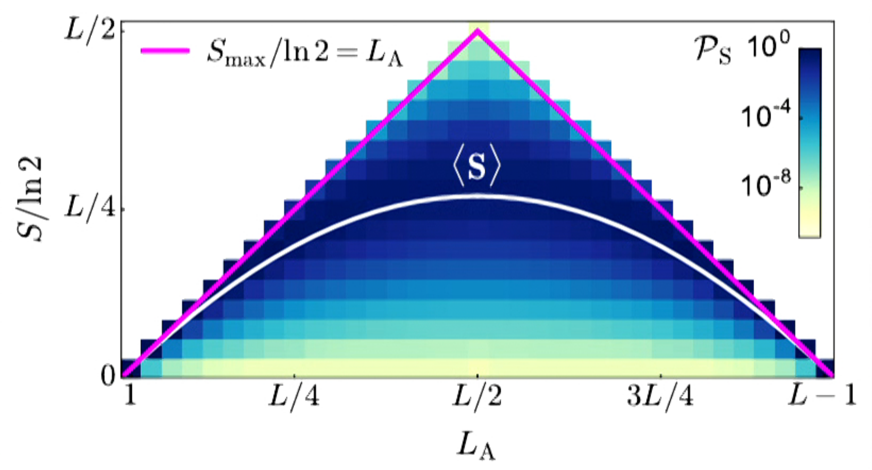
S_{vN} in eigenstates of quadratic fermionic Hamiltonians

Free fermions in 1D ($\hat{H} = -\sum_{\langle i,j \rangle}^{L} (\hat{f}_i^\dagger \hat{f}_j + \mathrm{H.c.})$, $L = 36$ sites)



L. Vidmar, L. F. Hackl, E. Bianchi, and MR, PRL **119**, 020601 (2017).
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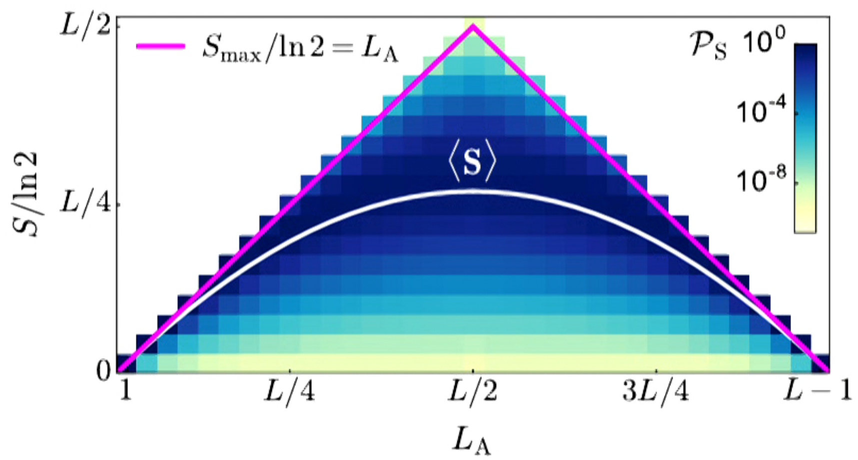

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Translational invariant quadratic Hamiltonian

$$\hat{H} = -\sum_{i,j=1}^{V}(t_{ij}\hat{f}_i^\dagger \hat{f}_j + \Delta_{ij}\hat{f}_i^\dagger \hat{f}_j^\dagger + \Delta_{ij}^*\hat{f}_j\hat{f}_i), \quad \Delta_{ij} = -\Delta_{ji} \text{ and } t_{ij} = t_{ji}^*$$

It is diagonalizable via a Bogoliubov transformation: $\hat{f}_i = \sum_{l=1}^{V}(\alpha_{il}\hat{c}_l + \beta_{il}\hat{c}_l^\dagger)$.
It is straightforward find all many-body eigenstates $|m\rangle$.

S_{vN} in eigenstates of quadratic fermionic Hamiltonians

The $(2V \times 2V)$ one-body correlation matrix fully characterizes the state

$$F = \left(\begin{array}{c|c} \langle m|\hat{f}_i^\dagger \hat{f}_j - \hat{f}_j \hat{f}_i^\dagger|m\rangle & \langle m|\hat{f}_i^\dagger \hat{f}_j^\dagger - \hat{f}_j^\dagger \hat{f}_i^\dagger|m\rangle \\ \hline \langle m|\hat{f}_i \hat{f}_j - \hat{f}_j \hat{f}_i|m\rangle & \langle m|\hat{f}_i \hat{f}_j^\dagger - \hat{f}_j^\dagger \hat{f}_i|m\rangle \end{array} \right)$$

I. Peschel, J. Phys. A **36**, L205 (2003); I. Peschel & V. Eisler, J. Phys. A **42**, 504003 (2009).

The entanglement entropy

$$S_{\mathrm{vN}} = -\mathrm{Tr}\left\{ \left(\frac{\mathbb{1} + [F]_A}{2} \right) \ln \left(\frac{\mathbb{1} + [F]_A}{2} \right) \right\},$$

where $[F]_A$ is the $2V_\mathrm{A} \times 2V_\mathrm{A}$ matrix with $i, j \in A$.

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where $[F]_A$ is the $2V_{\text{A}} \times 2V_{\text{A}}$ matrix with $i, j \in A$.

One can prove that the eigenvalues λ_j of $[F]_A$ are real with $|\lambda_j| \leq 1$ so

$$0 \leq \text{Tr}[F]_{\text{A}}^{2(m+1)} \leq \text{Tr}[F]_{\text{A}}^{2m} \leq 2V_{\text{A}}$$

Hence, the expansion

$$S_{\text{vN}} = V_{\text{A}} \ln 2 - \sum_{n=1}^{\infty} \frac{\text{Tr}[F]_{\text{A}}^{2n}}{4n(2n-1)}$$

is convergent. Higher-order terms lower the average entanglement entropy.

S_{vN} in eigenstates of quadratic fermionic Hamiltonians

We are interested in the average over all eigenstates

$$\langle S_{\mathsf{vN}} \rangle = V_{\mathrm{A}} \ln 2 - \sum_{n=1}^{\infty} \frac{\langle \mathrm{Tr}[F]_{\mathrm{A}}^{2n} \rangle}{4n(2n-1)}$$

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$$V_{\text{A}} \ln 2 - \frac{\langle \text{Tr}[F]_{\text{A}}^{2} \rangle}{2} \ln 2 \leq \langle S_{\text{vN}} \rangle \leq V_{\text{A}} \ln 2 - \frac{\langle \text{Tr}[F]_{\text{A}}^{2} \rangle}{4}$$

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The "first order" (results for $n = 1$) bounds are universal in the thermodynamic limit

$$V_{\text{A}} \left(\ln 2 - \ln 2 \frac{V_{\text{A}}}{V} \right) \leq \langle S_{\text{vN}} \rangle \leq V_{\text{A}} \left(\ln 2 - \frac{1}{2} \frac{V_{\text{A}}}{V} \right)$$

Typical eigenstates have maximal entanglement entropy for $V_{\text{A}}/V \to 0$.

S_{vN} in eigenstates of quadratic fermionic Hamiltonians

We are interested in the average over all eigenstates

$$\langle S_{\mathrm{vN}}\rangle = V_{\mathrm{A}} \ln 2 - \sum_{n=1}^{\infty} \frac{\langle \mathrm{Tr}[F]_{\mathrm{A}}^{2n}\rangle}{4n(2n-1)}$$

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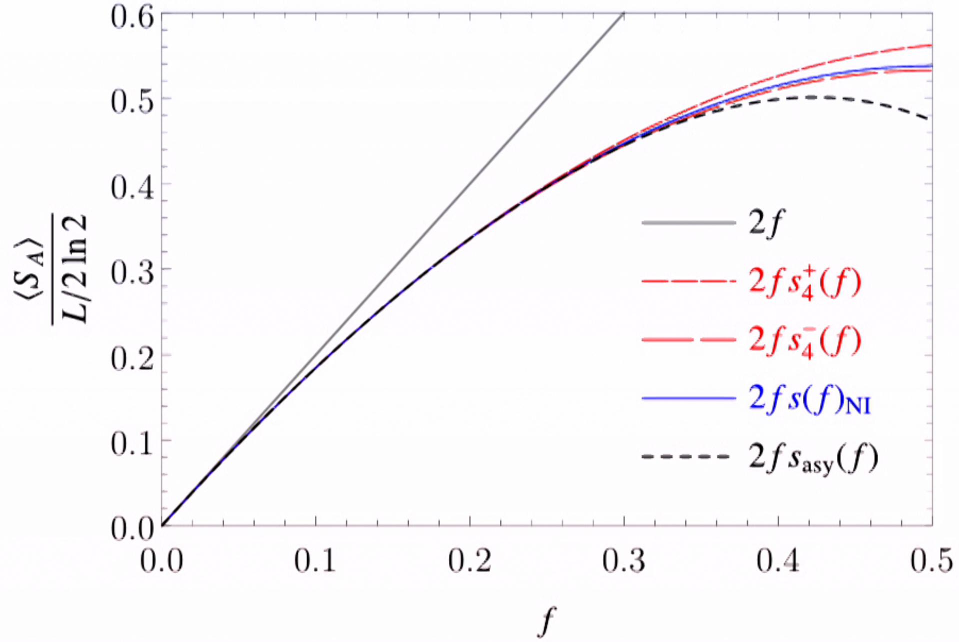
The "second order" bounds (results for up to $n=2$) for free fermions in 1D are

$$L_{\mathrm{A}} \left[\ln 2 - \frac{1}{2}\frac{L_{\mathrm{A}}}{L} - (2\ln 2 - 1)\left(\frac{4}{3}\frac{L_{\mathrm{A}}^2}{L^2} - \frac{L_{\mathrm{A}}^3}{L^3}\right)\right] \leq \langle S_{\mathrm{vN}}\rangle \leq L_{\mathrm{A}} \left[\ln 2 - \frac{1}{2}\frac{L_{\mathrm{A}}}{L} - \frac{2}{9}\frac{L_{\mathrm{A}}^2}{L^2} + \frac{1}{6}\frac{L_{\mathrm{A}}^3}{L^3}\right]$$

Typical eigenstates have maximal entanglement entropy for $V_{\mathrm{A}}/V \to 0$.

S_{vN} in eigenstates of quadratic fermionic Hamiltonians

Free (NI) fermions in 1D ($\hat{H} = -\sum_{\langle i,j \rangle}^{L} (\hat{f}_i^\dagger \hat{f}_j + \mathrm{H.c.})$, $L = 36$ sites)



L. Hackl, L. Vidmar, MR, and E. Bianchi, Phys. Rev. B **99**, 075123 (2019).

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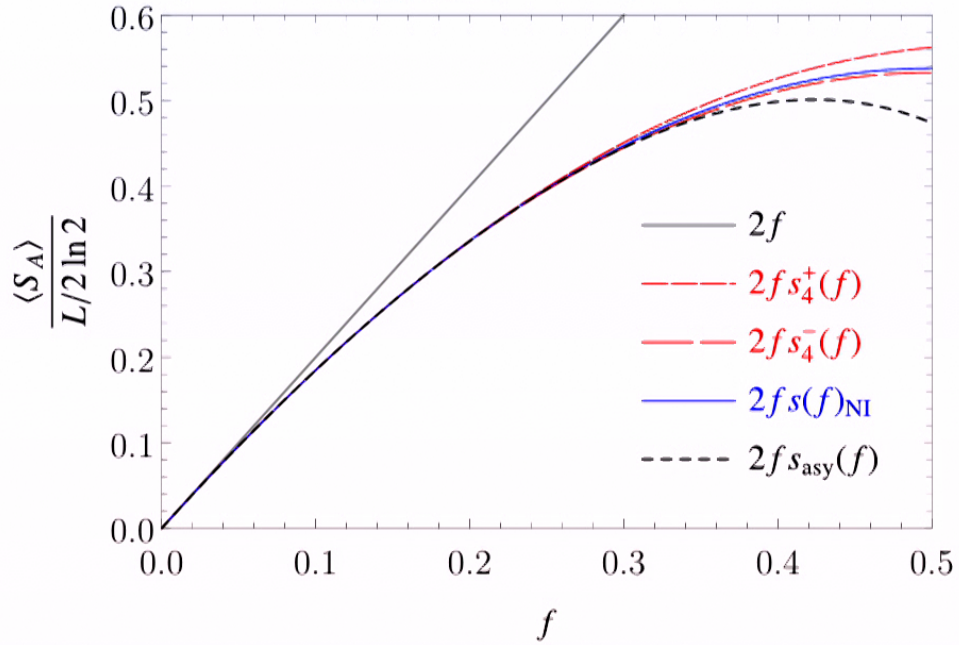

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Normalized width of the distribution of S_{vN} for eigenstates

$$\Delta = \frac{\sqrt{\langle S^2 \rangle - \langle S \rangle^2}}{L_A \ln 2}$$

vanishes as $1/\sqrt{L}$ or faster.

Entanglement in the 1D TFIM

Hamiltonian: $\hat{H} = J \sum_{\langle \mathbf{i},\mathbf{j} \rangle} \hat{\sigma}_{\mathbf{i}}^z \hat{\sigma}_{\mathbf{j}}^z + g \sum_{\mathbf{i}} \hat{\sigma}_{\mathbf{i}}^x$

PHYSICAL REVIEW LETTERS **121**, 220602 (2018)

Volume Law and Quantum Criticality in the Entanglement Entropy of Excited Eigenstates of the Quantum Ising Model

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[3] *Max Planck Institute of Quantum Optics, Hans-Kopfermann-Straße 1, D-85748 Garching bei München, Germany*
[4] *Department of Physics, The Pennsylvania State University, University Park, Pennsylvania 16802, USA*
[5] *Institute for Gravitation and the Cosmos, The Pennsylvania State University, University Park, Pennsylvania 16802, USA*

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Much has been learned about universal properties of entanglement entropies in ground states of quantum many-body lattice systems. Here we unveil universal properties of the average bipartite entanglement entropy of eigenstates of the paradigmatic quantum Ising model in one dimension. The leading term exhibits a volume-law scaling that we argue is universal for translationally invariant quadratic models. The subleading term is constant at the critical field for the quantum phase transition and vanishes otherwise (in the thermodynamic limit); i.e., the critical field can be identified from subleading corrections to the average (over all eigenstates) entanglement entropy.

S_{vN} in eigenstates of quadratic fermionic Hamiltonians

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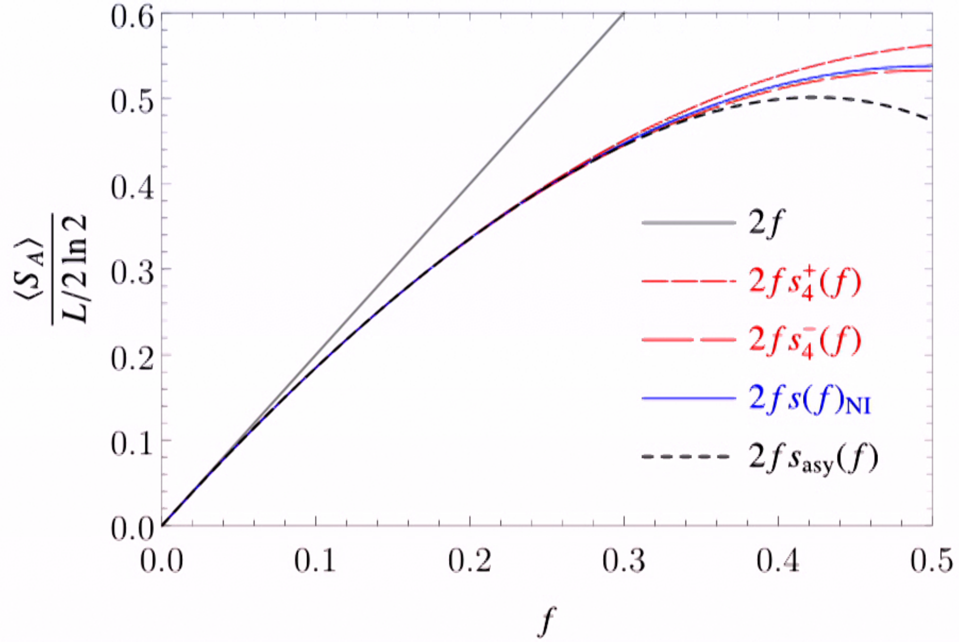

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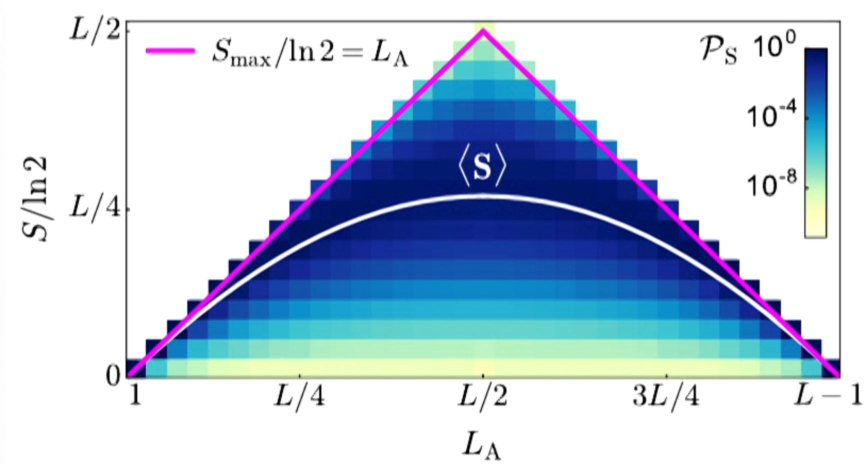

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It is diagonalizable via a Bogoliubov transformation: $\hat{f}_i = \sum_{l=1}^{V} (\alpha_{il} \hat{c}_l + \beta_{il} \hat{c}_l^\dagger)$.
It is straightforward find all many-body eigenstates $|m\rangle$.

Outline

$S_{\rm vN}$ in eigenstates of quantum chaotic Hamiltonians

Entanglement entropy in eigenstates of quantum chaotic Hamiltonians

Znidaric'07, Santos *et al.*'12, Deutsch *et al.*'13, Beugeling *et al.*'15, Yang *et al.*'15, Garrison & Grover'15, Vivo *et al.*'16, Dymarsky *et al.*'16, Fujita *et al.*'17, MR & Vidmar'17, Huang'17...

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What is the effect of adding a conservation law?

Focus on the conservation of the number of particles

S_{vN} in eigenstates of quantum chaotic Hamiltonians

Hard-core bosons in one dimension ($t = t' = 1$ and $V = V' = 1.1$)

$$\hat{H} = \sum_{i=1}^{L} \left\{ -t \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + V \hat{n}_i \hat{n}_{i+1} - t' \left(\hat{b}_i^\dagger \hat{b}_{i+2} + \text{H.c.} \right) + V' \hat{n}_i \hat{n}_{i+2} \right\}$$

Random canonical states (z_j: normally distributed real random number)

$$|\psi_N\rangle = \frac{1}{\sqrt{\mathcal{D}_N}} \sum_{j=1}^{\mathcal{D}_N} z_j |j\rangle,$$

$|j\rangle$ are base kets for N particles in the site basis

S_{vN} in eigenstates of quantum chaotic Hamiltonians

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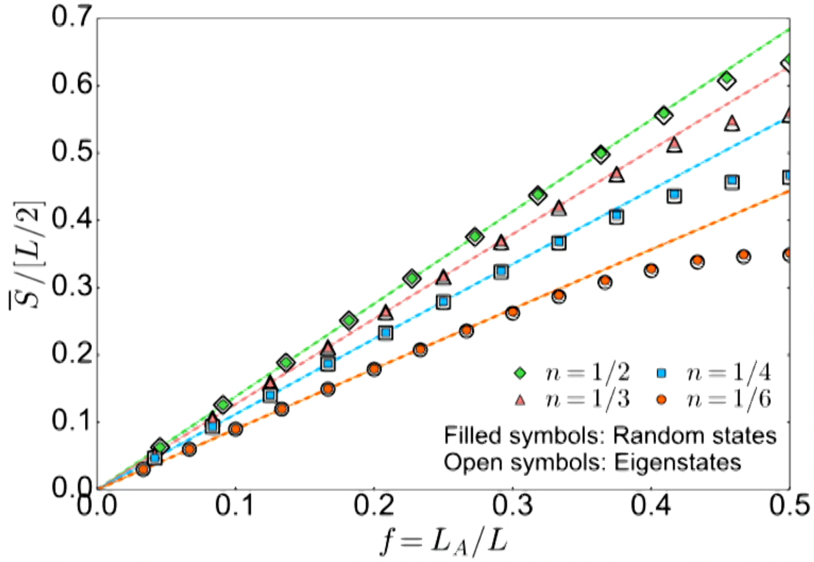
$\bar{S}_{\text{vN}}$ vs subsystem fraction $f = L_A/L$ [$L = 22$ (n=1/2), 24 (n=1/3,1/4) and 30 (n=1/6)]



L. Vidmar and MR, PRL **119**, 220603 (2017).

S_{vN} in eigenstates of quantum chaotic Hamiltonians

To calculate $\bar{S} = -\overline{\text{Tr}\{\hat{\rho}_A \ln(\hat{\rho}_A)\}}$, we define

$$\hat{M} = (\hat{\bar{\rho}}_A)^{-1}(\hat{\rho}_A - \hat{\bar{\rho}}_A)\,,$$

so that

$$\bar{S} = -\overline{\text{Tr}\left\{\hat{\bar{\rho}}_A(\hat{I} + \hat{M})\ln\left[\hat{\bar{\rho}}_A(\hat{I} + \hat{M})\right]\right\}} = S_{\text{MF}} + \bar{S}_0 + \bar{S}_{\text{fluct}}$$

$$S_{\text{MF}} = -\text{Tr}\left\{\hat{\bar{\rho}}_A \ln \hat{\bar{\rho}}_A\right\}\,,\; S_0 = -\text{Tr}\left\{\hat{\bar{\rho}}_A \hat{M} \ln \hat{\bar{\rho}}_A\right\}\,,\; S_{\text{fluct}} = -\text{Tr}\left\{\hat{\bar{\rho}}_A(\hat{I} + \hat{M})\ln(\hat{I} + \hat{M})\right\}$$

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For large systems, using Stirling's approximation and replacing $\sum_{N_A} \to \int \mathrm{d}n$:

$$S^{*}_{\text{MF}} = -L_A\left[n\ln n + (1-n)\ln(1-n)\right] + \frac{f+\ln(1-f)}{2}$$

Leading order: J. R. Garrison and T. Grover, Phys. Rev. X **8**, 021026 (2018).

One can prove that $\bar{S}_{\text{fluct}} \leq S^{\text{bound}}_{\text{fluct}}$, where for large systems and $f = 1/2$

$$S^{\text{bound}*}_{\text{fluct}} = -\sqrt{L_A}\,\ln\left(\frac{1-n}{n}\right)\sqrt{\frac{n(1-n)}{\pi}} + \frac{1}{\sqrt{L_A}}\,\frac{(1-2n)}{3\sqrt{\pi n(1-n)}}$$

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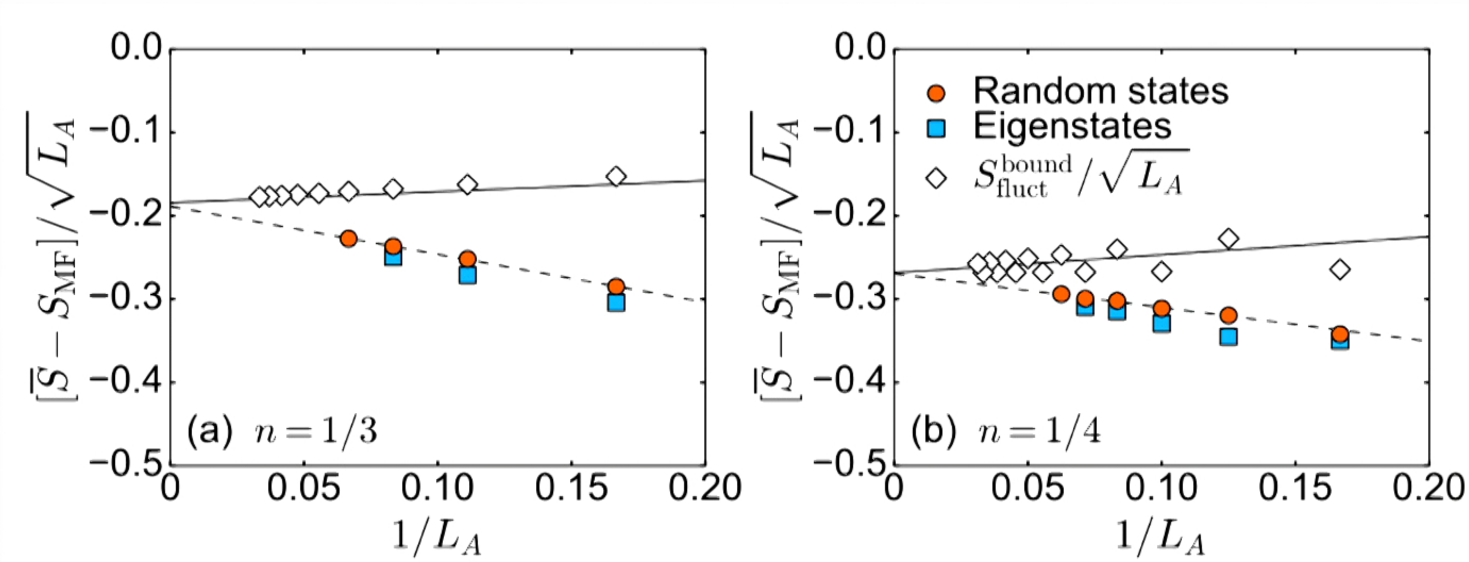
For the canonical ensemble at infinite temperature

$$S_A = \ln \mathcal{D}_N/2 \approx -L_A\left[n\ln n + (1-n)\ln(1-n)\right] - (1/4)\ln(L_A) + C_n$$

S_{vN} in eigenstates of quantum chaotic Hamiltonians

Fluctuation contribution to the average entanglement entropy for $f = 1/2$



L. Vidmar and MR, PRL **119**, 220603 (2017).

Outline

1. Introduction
   - Experiments with ultracold gases in one dimension
   - Classical and quantum: integrability vs chaos

2. Entanglement entropy
   - Entanglement entropies and experiments
   - Integrability vs nonintegrability
   - Quadratic fermionic Hamiltonians
   - Quantum chaotic Hamiltonians
   - Quantum chaos and eigenstate thermalization

3. Summary

Matrix elements of Hermitian operators within RMT

Let $\hat{O} = \sum_i O_i |i\rangle\langle i|$, where $\hat{O}|i\rangle = O_i|i\rangle$,

$$O_{\alpha\beta} \equiv \langle\alpha|\hat{O}|\beta\rangle = \sum_i O_i \langle\alpha|i\rangle\langle i|\beta\rangle = \sum_i O_i (\psi_i^\alpha)^* \psi_i^\beta$$

$|\alpha\rangle$ and $|\beta\rangle$ are eigenvectors of a random matrix. Averaging over $|\alpha\rangle$ and $|\beta\rangle$ (random orthogonal unit vectors in arbitrary bases): $\overline{(\psi_i^\alpha)^*(\psi_i^\beta)} = \frac{1}{\mathcal{D}}\delta_{\alpha\beta}$.

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This means that (to leading order):

$$\overline{O_{\alpha\alpha}} = \frac{1}{\mathcal{D}}\sum_i O_i \equiv \bar{O}, \quad \text{while} \quad \overline{O_{\alpha\beta}} = 0 \quad \text{for} \quad \alpha \neq \beta.$$

Matrix elements of Hermitian operators within RMT

Let $\hat{O} = \sum_i O_i |i\rangle\langle i|$, where $\hat{O}|i\rangle = O_i|i\rangle$,

$$O_{\alpha\beta} \equiv \langle\alpha|\hat{O}|\beta\rangle = \sum_i O_i \langle\alpha|i\rangle\langle i|\beta\rangle = \sum_i O_i (\psi_i^\alpha)^* \psi_i^\beta$$

$|\alpha\rangle$ and $|\beta\rangle$ are eigenvectors of a random matrix. Averaging over $|\alpha\rangle$ and $|\beta\rangle$ (random orthogonal unit vectors in arbitrary bases): $\overline{(\psi_i^\alpha)^*(\psi_i^\beta)} = \frac{1}{\mathcal{D}}\delta_{\alpha\beta}$.

This means that (to leading order):

$$\overline{O_{\alpha\alpha}} = \frac{1}{\mathcal{D}}\sum_i O_i \equiv \bar{O}, \quad \text{while} \quad \overline{O_{\alpha\beta}} = 0 \quad \text{for} \quad \alpha \neq \beta.$$

One can also show that ($\eta = 2$ for GOE and $\eta = 1$ for GUE):

$$\overline{O_{\alpha\alpha}^2} - \overline{O_{\alpha\alpha}}^2 = \eta\overline{|O_{\alpha\beta}|^2} = \frac{\eta}{\mathcal{D}^2}\sum_i O_i^2 \equiv \frac{\eta}{\mathcal{D}}\overline{O^2}.$$

Matrix elements of Hermitian operators within RMT

Let $\hat{O} = \sum_i O_i |i\rangle\langle i|$, where $\hat{O}|i\rangle = O_i|i\rangle$,

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Combining these results one can write

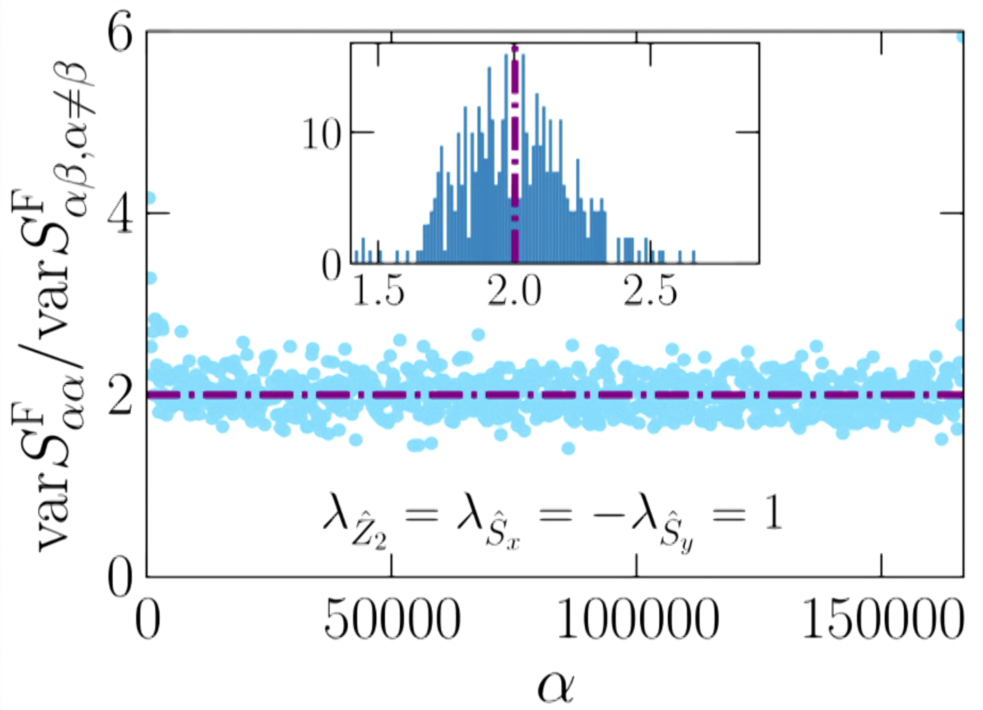
$$O_{\alpha\beta} \approx \bar{O}\delta_{\alpha\beta} + \sqrt{\frac{\overline{O^2}}{\mathcal{D}}} R_{\alpha\beta},$$

where $R_{\alpha\beta}$ is a random variable (real for GOE and complex for GUE).

Ratio of variances in the 2D F-TFIM

Hamiltonian: $\hat{H} = -J \sum_{\langle \mathbf{i},\mathbf{j} \rangle} \hat{\sigma}^z_{\mathbf{i}} \hat{\sigma}^z_{\mathbf{j}} + g \sum_{\mathbf{i}} \hat{\sigma}^x_{\mathbf{i}}$

Ratio of variances for the ferromagnetic structure factor



R. Mondaini and MR, PRE **96**, 012157 (2017).

Eigenstate thermalization hypothesis

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M. Srednicki, J. Phys. A **32**, 1163 (1999), L. D'Alessio *et al.*, Adv. Phys. **65**, 239 (2016).

$$O_{\alpha\beta} = O(E)\delta_{\alpha\beta} + e^{-S(E)/2} f_O(E,\omega) R_{\alpha\beta}$$

where $E \equiv (E_\alpha + E_\beta)/2$, $\omega \equiv E_\alpha - E_\beta$, $S(E)$ is the thermodynamic entropy at energy E, and $R_{\alpha\beta}$ is a random number with zero mean and unit variance.

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Transverse (+ longitudinal) field Ising model in two dimensions

$$\hat{H} = J \sum_{\langle \mathbf{i},\mathbf{j}\rangle} \hat{\sigma}^z_{\mathbf{i}} \hat{\sigma}^z_{\mathbf{j}} + g \sum_{\mathbf{i}} \hat{\sigma}^x_{\mathbf{i}} + \varepsilon \sum_{\mathbf{i}} \hat{\sigma}^z_{\mathbf{i}}$$

Integrable for $J = 0$ or for $g = 0$
(not mappable onto noninteracting fermions)

$\varepsilon \neq 0$ breaks the Z_2 symmetry of the TFIM

Mondaini, Fratus, Srednicki, and MR, PRE **93**, 032104 (2016).

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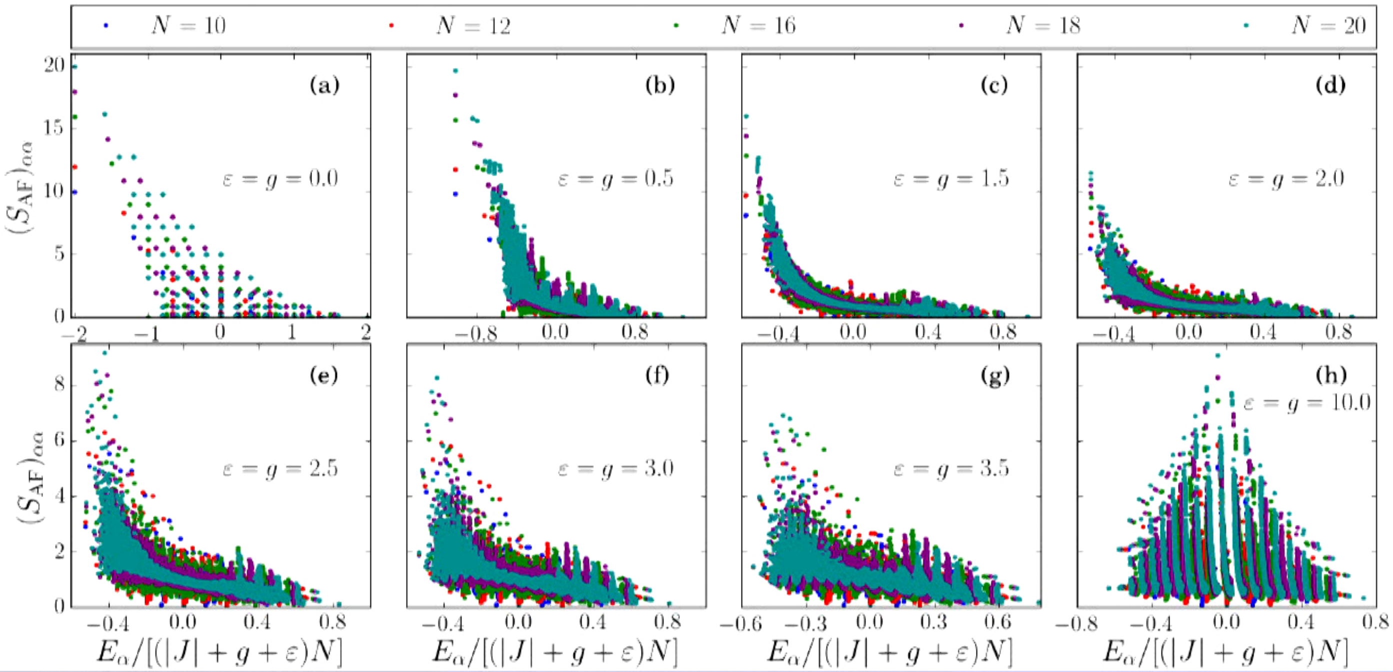
Diagonal matrix elements (2D AF-TFIM)



Summary

- The entanglement properties of typical eigenstates of integrable models are different from those of typical eigenstates of nonintegrable ones
  - Need a nonvanishing subsytem fraction to detect them!
- The leading and first subleading [when $O(1)$ or larger] terms in the von Neumann entanglement entropy of typical eigenstates of quantum chaotic Hamiltonians are universal and given by random matrix theory.

Summary

- The entanglement properties of typical eigenstates of integrable models are different from those of typical eigenstates of nonintegrable ones
  - Need a nonvanishing subsytem fraction to detect them!
- The leading and first subleading [when $O(1)$ or larger] terms in the von Neumann entanglement entropy of typical eigenstates of quantum chaotic Hamiltonians are universal and given by random matrix theory.

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