Title: Area law of non-critical ground states in 1D long-range interacting systems

Speakers: Tomotaka Kuwahara

Series: Perimeter Institute Quantum Discussions

Date: March 11, 2020 - 4:00 PM

URL: http://pirsa.org/20030012

Abstract: The area law for entanglement provides one of the most important connections between information theory and quantum many-body physics. It is not only related to the universality of quantum phases, but also to efficient numerical simulations in the ground state (i.e., the lowest energy state). Various numerical observations have led to a strong belief that the area law is true for every non-critical phase in short-range interacting systems [1]. The so-called area-law conjecture states that the entanglement entropy is proportional to the surface region of subsystem if the ground state is non-critical (or gapped).

However, the area law for long-range interacting systems is still elusive as the long-range interaction results in correlation patterns similar to the ones in critical phases. Here, we show that for generic non-critical one-dimensional ground states, the area law robustly holds without any corrections even under long-range interactions [2]. Our result guarantees an efficient description of ground states by the matrix-product state in experimentally relevant long-range systems, which justifies the density-matrix renormalization algorithm. In the present talk, I will give an overview of the results, and show ideas of the proof if the time allows.

[1] J. Eisert, M. Cramer, and M. B. Plenio, ``Colloquium: Area laws for the entanglement entropy," Rev. Mod. Phys. 82, 277â€“306 (2010).

[2] T. Kuwahara and K. Saito, ``Area law of non-critical ground states in 1d long-range interacting systems," arXiv preprint arXiv:1908.11547 (2019),

Area law of non-critical ground states in 1D long-range interacting systems

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Perimeter institute 11th March

With Keiji Saito (Keio university)

T Kuwahara and K Saito, arXiv:1908.11547

Contents

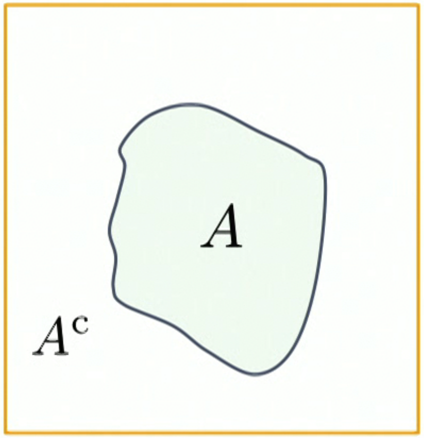
- Introduction on the area law conjecture
- Results on the long-range area law
- The proof outline
- Summary

Entanglement area law

- Ground state $|0\rangle$
- Entanglement entropy

$$S(\rho_A) = -\mathrm{tr}(\rho_A \log \rho_A)$$

ρ_A: reduced density matrix



- Area law: $S(\rho_A) \leq c|\partial A| \quad (c : \text{constant})$

⭐ Area law conjecture:

for short-range interacting Hamiltonians with O(1) spectral gap, the ground state satisfies the area law.

Why is area law important?

- Tensor network representation of g.s. on D-dimensional lattice

⬇ Necessary condition !

- The ground state satisfies the area law

- Violation of the area law

＝Breakdown of all the algorithm using the tensor network ansatz

➡ Area law or not is critically important problem

Why is area law important?

- Tensor network representation of g.s. on D-dimensional lattice

⇓ ⇑✖ Area law is not sufficient condition

Y. Ge, J. Eisert, New J. Phys. **18**, 083026 (2016)

- The ground state satisfies the area law

- Violation of the area law

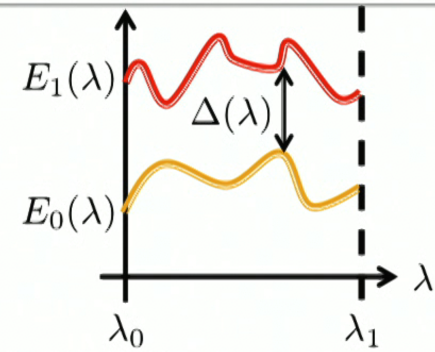
= Breakdown of all the algorithm using the tensor network ansatz

➡ Area law or not is critically important problem

Several approaches for area law conj.
1. quasi-adiabatic continuation

- Parameterized Hamiltonian $H(\lambda)$

$$\frac{d}{d\lambda}|0(\lambda)\rangle = -iD(\lambda)|0(\lambda)\rangle \quad |0(\lambda)\rangle\text{: g.s. of } H(\lambda)$$



- If H(λ) is gapped, D(λ) is quasi-local.

M. B. Hastings and Xiao-Gang Wen, Phys. Rev. B **72**, 045141 (2005)

- Target ground state: $|0(\lambda_1)\rangle$, Trivial ground state: $|0(\lambda_0)\rangle$

Assume that $H(\lambda_0)$ and $H(\lambda_1)$ are connected via a gapped path

- $|0(\lambda_1)\rangle = \mathcal{T}e^{-i\int_{\lambda_0}^{\lambda_1} D(\lambda)d\lambda}|0(\lambda_0)\rangle \quad D(\lambda)\text{: quasi-local operator}$

Small Incremental Entangling (SIE) theorem

- Area law of $|0(\lambda_1)\rangle$

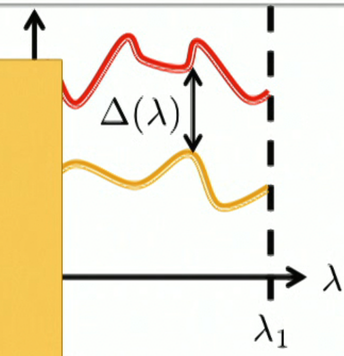
Karel Van Acoleyen, et al., Phys. Rev. Lett. **111**, 170501 (2013)

Several approaches for area law conj.

1. quasi-adiabatic continuation

- Parameterized Hamiltonian $H(\lambda)$

$\frac{d}{d\lambda}|0(\lambda$



Usually, the existence of the gapped path is extremely difficult to prove...

T. Cubitt, D. Perez-Garcia, M. M. Wolf Nature, 528, 207–211(2015)
J. Bausch, T. Cubitt, A. Lucia, D. Perez-Garcia, arXiv: 1810.01858

- If H(λ) i

5)

- Target

$))\rangle$

Assume that $H(\lambda_0)$ and $H(\lambda_1)$ are connected via a gapped path

- $|0(\lambda_1)\rangle = \mathcal{T}e^{-i\int_{\lambda_0}^{\lambda_1} D(\lambda)d\lambda}|0(\lambda_0)\rangle$ $D(\lambda)$: quasi-local operator

Small Incremental Entangling (SIE) theorem

Karel Van Acoleyen, et al., Phys. Rev. Lett. **111**, 170501 (2013)

- Area law of $|0(\lambda_1)\rangle$

Several approaches for area law conj. 2. utilizing clustering property

9

- Clustering theorem for gapped ground states:

$$\langle 0|A_X B_Y|0\rangle - \langle 0|A_X|0\rangle\langle 0|B_Y|0\rangle \leq \|A_X\| \cdot \|B_Y\| e^{-\mathrm{dist}_{X,Y}/\xi} \qquad \xi = \mathcal{O}(1/\Delta)$$

M. B. Hastings, Phys. Rev. Lett. **93**, 140402 (2004)

- Finite correlation length

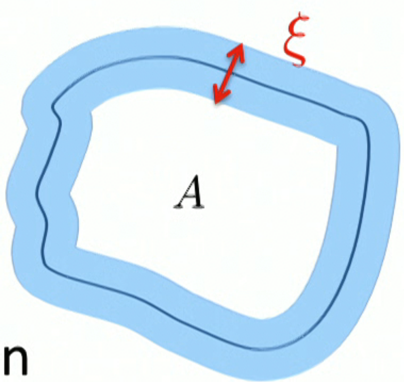
→ ? A is only entangled with A^c at the boundary: area law

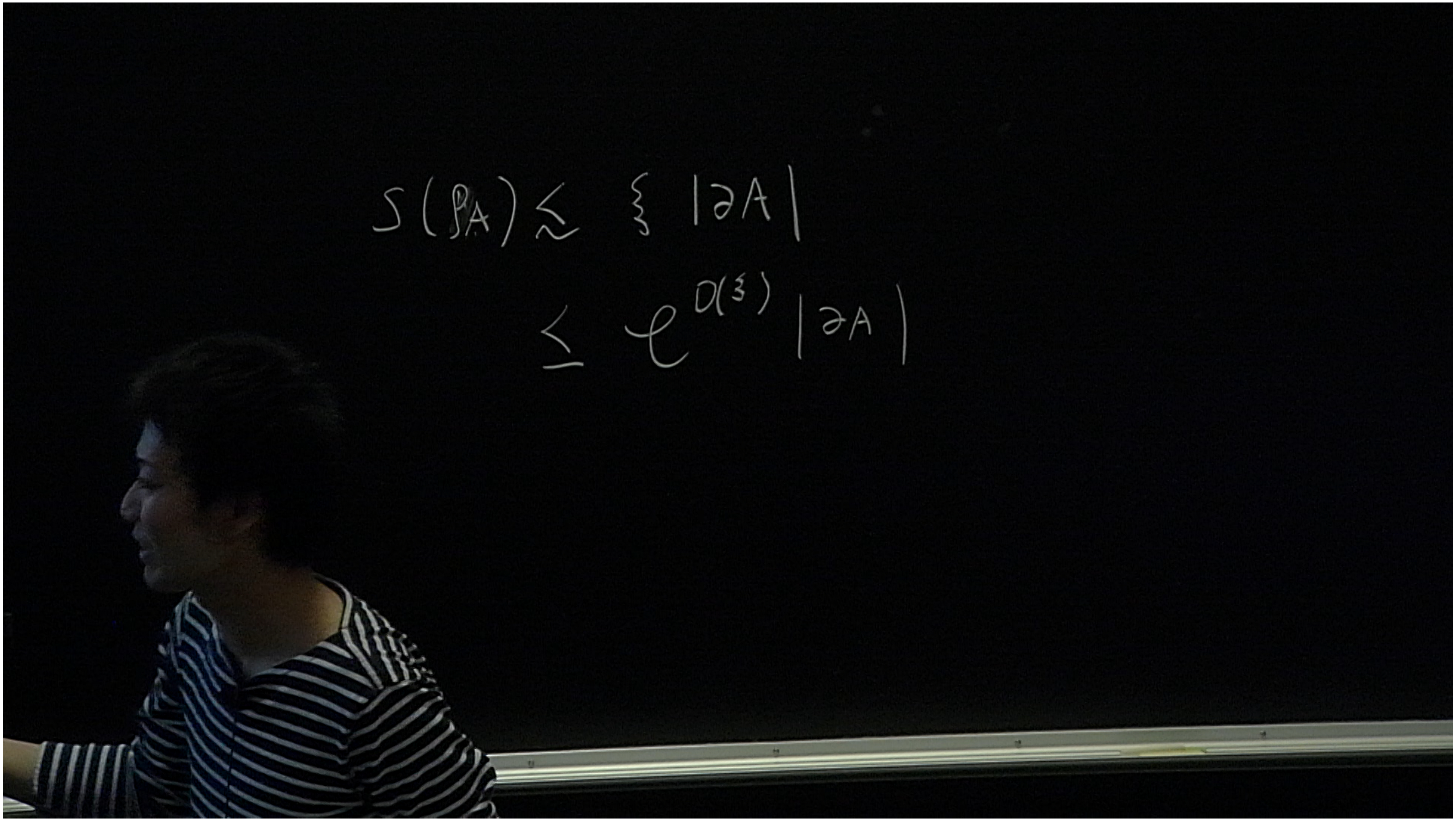
↓

- The intuition is true (at least) in 1D

F. Brandao and M. Horodecki, Nature Physics **9** (2013), 721

- Generalization to high D: completely open




$S(\rho_A) \lesssim \xi |\partial A|$
$\leq e^{O(\xi)} |\partial A|$

Several approaches for area law conj. 3. AGSP based approach

- AGSP K : Approximate ground state projection
- Relationship between the AGSP and E.E.

AGSP: $K \approx |0\rangle\langle 0|$ → Entanglement Entropy: $S(|0\rangle)$

approximation error: ϵ
Schmidt rank: D

- 1D area law is completely proved by AGSP approach

M. B. Hastings, J. Stat. Mech. (2007) P08024
I. Arad, A. Kitaev, Z. Landau, and U. Vazirani, arXiv:1301.1162

- Beyond 1D, there are several works.

Tree graph with fractal dimension (D<2): N. Abrahamsen, arXiv:1907.04862

2D subvolume law for frustration free Hamiltonians:
A. Anshu, I. Arad, and D. Gosset, arXiv:1905.11337

Set up

- 1D long-range Hamiltonian (system size : n , local dimension: d)

$$H = \sum_{i<j} h_{i,j} + \sum_{i=1}^{n} h_i \quad \text{with} \quad \|h_{i,j}\| \leq \frac{g}{r_{i,j}^{\alpha}}$$

$\alpha > 0, \quad g = \mathcal{O}(1)$

$\| \cdots \|$: operator norm

➡ Generalized to k-body interaction

➡ α is experimentally controllable

e.g.) $H = \sum_{i<j} \frac{J_{i,j}}{r_{i,j}^{\alpha}} \sigma_i^x \sigma_j^x + B \sum_i \sigma_i^z$

α is tuned in $0 < \alpha < 3$

P. Jurcevic, et al., Nature **511**, 202 (2014).
J. Zhang, et al., Nature **551**, 601 (2017).

- Gapped unique ground states: $|0\rangle$, gap: Δ

Area law and main questions

- 1D Area law : $S(\rho_L) \leq \text{const.}, \quad \rho_L = \text{tr}_R(|0\rangle\langle 0|)$

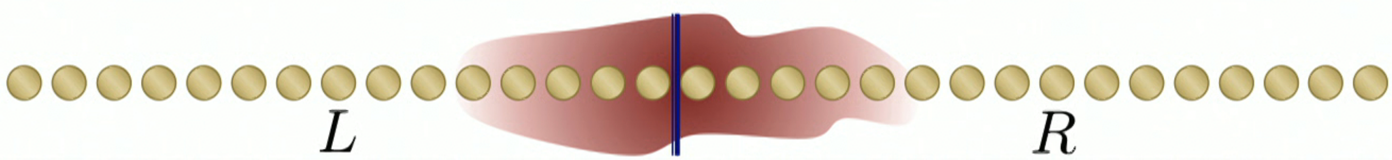
Interaction decay: $r^{-\alpha}$

For $\alpha \geq 3$, the universality class is the same as short-range limit ($\alpha = \infty$)

M. E. Fisher, et al., Phys. Rev. Lett. 29, 917–920 (1972).

Question.

- **Can we prove area law for $\alpha < 3$?**
- **What is the critical α_c to ensure the area law?**



Several known results

- 1D Area law : $S(\rho_L) \leq \text{const.}, \quad \rho_L = \text{tr}_R(|0\rangle\langle 0|)$

Interaction decay: $r^{-\alpha}$

- Explicit counterexample for $\alpha \leq 1$

(Free fermion with long-range hopping)

J. Eisert and T. J. Osborne, Phys. Rev. Lett. 97, 150404 (2006).

- Sub-logarithmic violation was numerically indicated for $\alpha \leq 3$

(Transverse Ising model with long-range interactions)

T. Koffel, et al., Phys. Rev. Lett. 109, 267203 (2012).

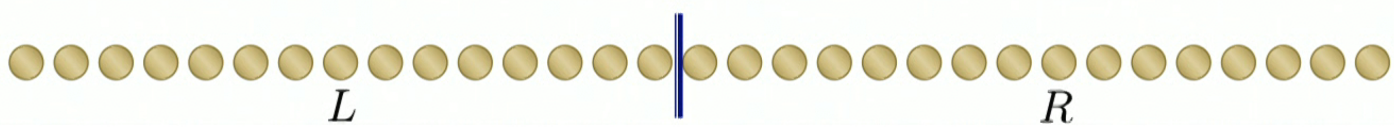
- Area law at finite temperatures: $I(L:R) \leq \text{const} \times \|H_{\partial L}\|$

M. M. Wolf, et al., Phys. Rev. Lett. 100, 070502 (2008).

$I(A:B) := S(\rho_A) + S(\rho_B) - S(\rho_{AB})$

$H_{\partial L}$: boundary interactions

→ Gibbs states satisfies area law for $\alpha > 2$



Several known results

- 1D Area law : $S(\rho_L) \leq \text{const.}, \quad \rho_L = \text{tr}_R(|0\rangle\langle 0|)$

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J. Eisert and T. J. Osborne, Phys. Rev. Lett. 97, 150404 (2006).

- Sub $\alpha \leq 3$

(Transv

T. Koffel

The condition $\alpha > 2$ seems to be necessary for the long-range area law of ground state…

- Area ... tures : $I(L:R) \leq \text{const} \times \|H_{\partial L}\|$

M. M. Wolf, et al., Phys. Rev. Lett. 100, 070502 (2008).

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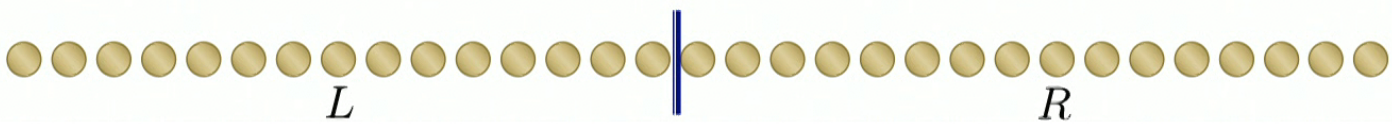


Approaches to long-range Area law

- Area law : $S(\rho_L) \leq \text{const.}, \quad \rho_L = \text{tr}_R(|0\rangle\langle 0|)$

1. quasi-adiabatic continuation

Existence of adiabatic path to trivial g.s. implies area law for $\alpha > 4$

Z-X. Gong et al., Phys. Rev. Lett. 119, 050501 (2017).

2. utilizing clustering property: hopeless

Gapped g.s. : power-law decay of the correlation

M. B. Hastings and T. Koma, Commun. Math. Phys. **265**, 781–804 (2006).

⟷ There exists a counterexample

M. B. Hastings, Quantum Info. Comput. **16**, 1228–1252 (2016).

3. AGSP based approach: Our approach!

Our area law results

- Entanglement entropy $S(\rho_L)$ is upper-bounded by

$$S(\rho_L) \leq c \log^2(d) \left(\frac{\log(d)}{\Delta}\right)^{1+2/\bar{\alpha}+\eta}$$

$(\log(d)/\Delta)^{\eta} = \text{poly-log}(\log(d)/\Delta)$ c: constant , d: local dimension, Δ: gap

➡ The bound diverges for $\bar{\alpha} \to 0$ (or $\alpha \to 2$)

- Is there an explicit example to violate the area law for $\alpha \leq 2$?
- Can we prove the sub-volume law for $\alpha \leq 2$?

➡ In short-range limit ($\bar{\alpha} \to \infty$),

$$S(\rho_L) \lesssim \frac{\log^3(d)}{\Delta}$$

The bound by Arad-Kitaev-Landau-Vazirani is recovered!

I. Arad, et al., arXiv:1301.1162

MPS representation of g.s.

- There exists a matrix product state $|\psi_D\rangle$ such that

$$\| |\psi_D\rangle - |0\rangle \|_1 \leq 1/\mathrm{poly}(n),$$

where $D = \exp[c' \log^{5/2}(n)]$ c': constant

➡ Quasi-polynomial bond dimension is enough to approximate g.s.

- Can we improve it to polynomial form?

➡ It is still non-trivial how to construct the MPS

- Can we extend the technique by Landau-Vazirani-Vidick?

Z. Landau, et a., Nature Physics 11, 566 (2015).

Contents

Proof sketch : AGSP-based approach

- Approximate-ground-state-projection (AGSP) : K

(Rough Def.) $K|0\rangle \simeq |0\rangle \quad \text{and} \quad \|K(1-|0\rangle\langle 0|)\| \simeq 0$

- Three parameters $\{\delta_K, \epsilon_K, D_K\}$

$\exists|\tilde{0}\rangle \quad \text{s.t.} \quad K|\tilde{0}\rangle = |\tilde{0}\rangle$

SR(K) : Schmidt rank along a given cut

$$\||0\rangle - |\tilde{0}\rangle\| \le \delta_K, \quad \|K(1-|\tilde{0}\rangle\langle\tilde{0}|)\| \le \epsilon_K, \quad \text{and} \quad \mathrm{SR}(K) \le D_K$$

➡ Good AGSP ensures overlap between g.s. and low-rank quantum state

Proof sketch : AGSP-based approach

- Approximate-ground-state-projection (AGSP) : K

(Rough Def.) $K|0\rangle \simeq |0\rangle$ and $\|K(1-|0\rangle\langle 0|)\| \simeq 0$

- Three parameters $\{\delta_K, \epsilon_K, D_K\}$

$\exists|\tilde{0}\rangle$ s.t $K|\tilde{0}$ … given cut

$\||0\rangle - |\tilde{0}$ … $\leq D_K$

➡ Go…
an…

(Bootstapping lemma)

If $\epsilon_K^2 D_K \leq \dfrac{1}{2}$, there exists a quantum state $|\psi\rangle$

s.t. $\mathrm{SR}(|\psi\rangle) \leq D_K$

and $\||\psi\rangle - |0\rangle\| \leq \epsilon_K \sqrt{2D_K} + \delta_K$

Interaction truncation

- Truncate the long-range interactions, but **only around the boundary**

➡ Decompose total system into q+2 blocks (q: even)

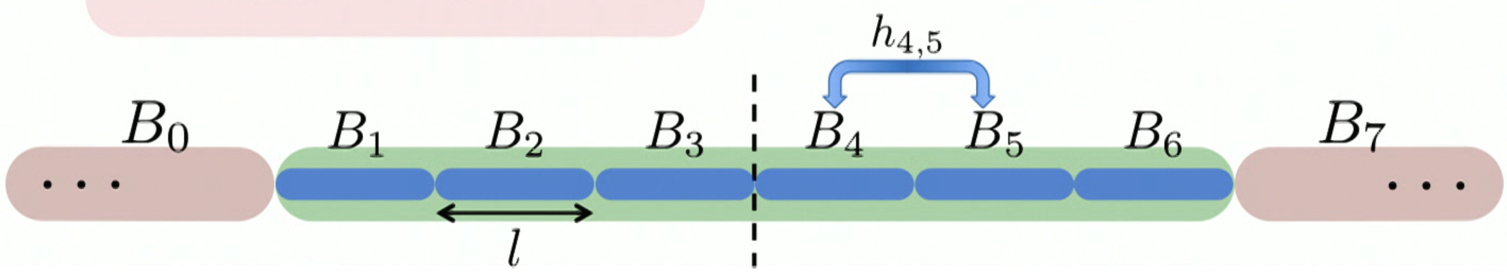
Edge blocks: B_0, B_{q+1} , Bulk blocks (size l) : $B_1, B_2, \ldots, B_q$

➡ Truncate all the interactions between non-adjacent blocks

$$H_t = \sum_{s=0}^{q+1} h_s + \sum_{s=0}^{q} h_{s,s+1}$$

h_s: internal interaction in block B_s

$h_{s,s+1}$: interaction between blocks B_s and B_{s+1}



Interaction truncation

- Truncate the long-range interactions, but **only around the boundar**

Roughly speaking, we have

$\Delta_{\mathrm{t}} \gtrsim \Delta - q l^{-\bar{\alpha}}$

$\| |0\rangle - |0_{\mathrm{t}}\rangle \| \lesssim q l^{-\bar{\alpha}} / \Delta$

➡ $q l^{-\bar{\alpha}} \ll 1$ is necessary

➡ Decom

Edge bl

➡ Truncate all th... ...ctions between non-adjacent blocks

$$H_t = \sum_{s=0}^{q+1} h_s + \sum_{s=0}^{q} h_{s,s+1}$$

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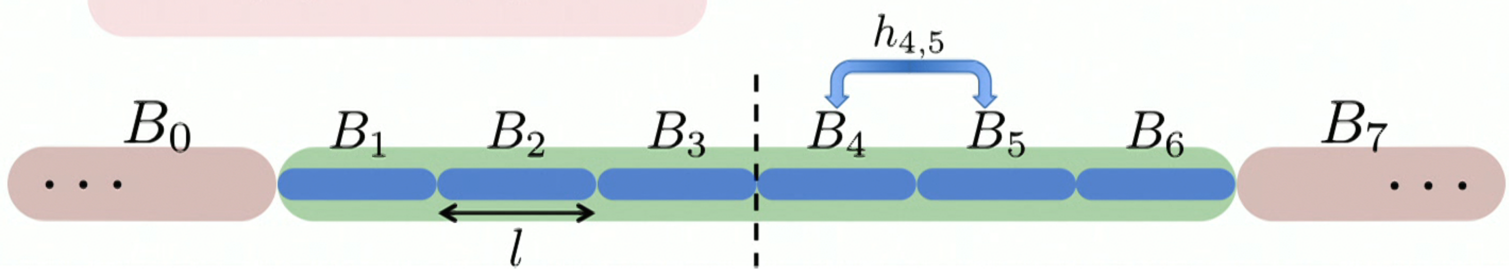
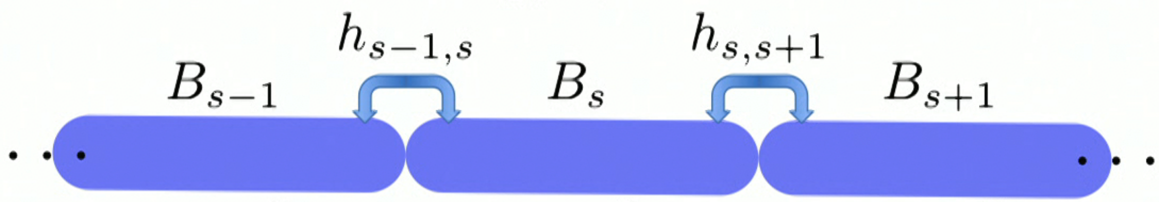
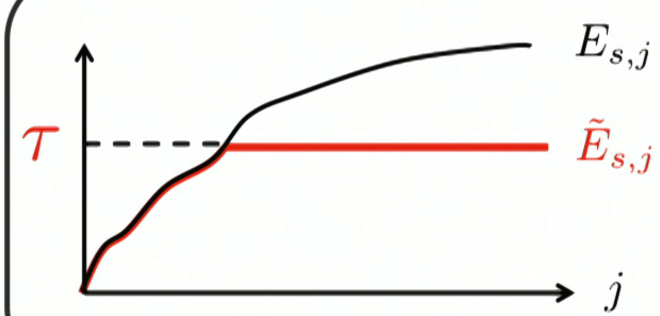


Multi-energy truncation

- Truncate the energy spectrum for each block : $H_t \to \tilde{H}_t$



$h_{s-1} \to \tilde{h}_{s-1}$ $h_s \to \tilde{h}_s$ $h_{s+1} \to \tilde{h}_{s+1}$ $\|\tilde{H}_t\| = \mathcal{O}(q\tau)$



Spectrums of h_s and $\tilde{h}_s$

Extend the analysis by Arad-Kuwahara-Landau

I. Arad, **T. Kuwahara**, Z. Landau, J. Stat. Mech, 033301 (2016)

For $\tau \gtrsim \log(q)$,

$\tilde{\Delta}_t \geq \Delta_t/2$ and $\||\tilde{0}_t\rangle - |0_t\rangle\| \lesssim qe^{-\mathcal{O}(\tau)}$

$\|\tilde{H}_t\| = \mathcal{O}(q\tau) = \mathcal{O}(q\log q)$

Multi-energy truncation

- Truncate the ene[...]ctrum for each block : $H_t \to \tilde{H}_t$

By using $\tilde{H}_t$, we construct Chebyshev-based AGSP, which gives

$$\epsilon_K \sim e^{-m/\sqrt{q \log q}} \quad D_K \sim e^{m \log(l)/q + ql}$$

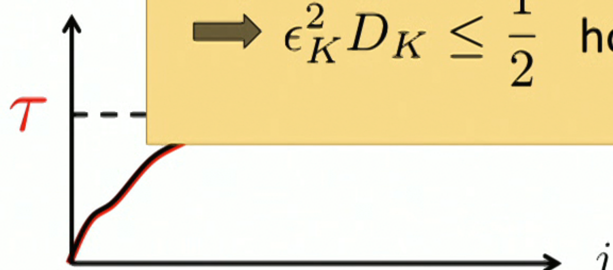
with $\quad ql^{-\bar{\alpha}} \lesssim 1$

➡ $\epsilon_K^2 D_K \le \frac{1}{2}$ holds for some $\{q, m, l\}$

h_{s-} … $\mathcal{O}(q\tau)$

Landau

…1 (2016)



Spectrums of h_s and $\tilde{h}_s$

For $\tau \gtrsim \log(q)$,

$\tilde{\Delta}_t \ge \Delta_t/2$ and $\| |\tilde{0}_t\rangle - |0_t\rangle \| \lesssim q e^{-\mathcal{O}(\tau)}$

$\|\tilde{H}_t\| = \mathcal{O}(q\tau) = \mathcal{O}(q \log q)$

Derivation of area law

- Prove existence of state $|\phi\rangle$ s.t. $\||\phi\rangle - |0\rangle\| \leq 1/2$ with

$$\log[\mathrm{SR}(|\phi\rangle)] \lesssim \log^2(d) \left(\frac{\log(d)}{\Delta}\right)^{1+2/\bar{\alpha}+\eta}$$

- Construct a set of AGSP $\{K_p\}_{p=1}^{\infty}$ with $\{\delta_p, \epsilon_p, D_p\}_{p=1}^{\infty}$

$$\bar{\gamma}_p := \frac{\epsilon_p}{1/2 - \delta_p} + \delta_p \quad \text{with} \quad \bar{\gamma}_0 = 1 \qquad \left(\lim_{p\to\infty} \epsilon_p = 0, \quad \lim_{p\to\infty} \delta_p = 0\right)$$

→ $$S(|0\rangle) \leq \log[\mathrm{SR}(|\phi\rangle)] - \sum_{p=0}^{\infty} \bar{\gamma}_p^2 \log \frac{\bar{\gamma}_p^2}{3D_{p+1}}$$ → Area law

→ $$\left\| \frac{K_p e^{-i\theta_p}|\phi\rangle}{\|K_p|\phi\rangle\|} - |0\rangle \right\| \leq \bar{\gamma}_p$$ → MPS representation

Summary

- 1D Long-range interaction: $1/r^{\alpha}$
- Area law: $S(\rho_L) \leq c \log^2(d) \left(\frac{\log(d)}{\Delta}\right)^{1+2/\bar{\alpha}+\eta} \quad (\bar{\alpha} > \alpha - 2)$
- MPS approximation : $\| |\psi_D\rangle - |0\rangle \|_1 \leq 1/\mathrm{poly}(n),$

$$D = \exp[c' \log^{5/2}(n)]$$

- Open questions
  - Is there an explicit example to violate the area law for $\alpha \leq 2$?
  - Can we prove the sub-volume law for $\alpha \leq 2$?
  - Can we improve the bond dimension of MPS to polynomial form?
  - Is there an efficient algorithm to construct the MPS?