Title: Quantum Field Theory for Cosmology - Lecture 18

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (Kempf)

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QFT for Cosmology, Achim Kempf, Lecture 16

Note Title

Recall:

- Using different choices of mode functions, $v_k(\eta)$, $\tilde{v}_k(\eta)$, we can write $\hat{x}_k(\eta)$ in different ways:

$$\hat{x}_k(\eta) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k(\eta)\, a_{-k}^+\right) \qquad (A)$$

$$= \frac{1}{\sqrt{2}}\left(\tilde{v}_k^*(\eta)\, \tilde{a}_k + \tilde{v}_k(\eta)\, \tilde{a}_{-k}^+\right)$$

- Since for each k the space of possible mode functions is complex 2-dimensional, there exist complex α_k, β_k so that:

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$$= \frac{1}{\sqrt{2}}\left(\tilde{v}_k^*(\eta)\, \tilde{a}_k + \tilde{v}_k(\eta)\, \tilde{a}_{-k}^+\right)$$

- Since for each k the space of possible mode functions is 2-dimensional (complex), there exist complex α_k, β_k so that:

$$\tilde{v}_k(\eta) = \alpha_k\, v_k(\eta) + \beta_k\, v_k^*(\eta) \qquad (B)$$

(Recall: Because $\tilde{v}_k(\eta)$ must obey the Wronskian condition, α_k and β_k must obey $|\alpha_k|^2 - |\beta_k|^2 = 1$)

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- From (A) and (B) we obtain (exercise):

$$\boxed{a_k = \alpha_k^* \tilde{a}_k + \beta_k \tilde{a}_{-k}^+}$$

- Thus, $a_k|0\rangle = 0$ becomes $(\alpha_k^* \tilde{a}_k + \beta_k \tilde{a}_{-k}^+)|0\rangle = 0$, which yields:

$$|0\rangle = \left[\prod_k \frac{1}{|\alpha_k|^{1/2}} e^{-\frac{\beta_k}{2\alpha_k^*} \tilde{a}_k^+ \tilde{a}_{-k}^+}\right] |\tilde{0}\rangle \qquad (T)$$

needed for normalization

$\Rightarrow$ We can now express all basis vectors $|0\rangle$, $a_k^+|0\rangle$, $a_{k'}^+ a_{k''}^+|0\rangle \ldots$ in terms of the basis vectors $|\tilde{0}\rangle$, $\tilde{a}_r^+|0\rangle$, $\tilde{a}_v^+ \tilde{a}_{v'}^+|0\rangle \ldots$

Example scenario:

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Example scenario:

✱ Assume $v_k(\eta), \tilde{v}_k(\eta)$ chosen so that $|0\rangle, |\tilde{0}\rangle$ are vacuum at η_1, η_2.

Thus, $a_k|0\rangle = 0$ becomes $(\alpha_k^* \tilde{a}_k + \beta_k \tilde{a}_{-k}^\dagger)|0\rangle = 0$, which yields:

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Example scenario:

* Assume $v_k(\eta), \tilde{v}_k(\eta)$ chosen so that $|0\rangle, |\tilde{0}\rangle$ are vacuum at η_1, η_2.
* Assume system is in vacuum state at η_1, i.e. $|\Omega\rangle = |0\rangle$.
* Then system's state $|\Omega\rangle$ at η_2 is an excited state, i.e., a state with particles!

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The extent of particle creation?

- Eqn. (T) shows that there is a finite probability amplitude for finding arbitrarily many particles at time t_2. Does that mean ∞ many get created (at ∞ energy expense and thus halting the expansion?)

- Let us calculate the expected number of created particles:

  * Definition (QM):

$$\hat{N} := a^+ a \text{ is called a "Number operator"}$$

  * Why? It is a self-adjoint observable with eigenbasis:

$$\hat{N}(a^+)^n|0\rangle = n\,(a^+)^n|0\rangle$$

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* Why? It is a self-adjoint observable with eigenbasis:

$$\hat{N}(a^+)^n|0\rangle = n\,(a^+)^n|0\rangle$$

* Exercise: verify.

* Definition (QFT): $\hat{N}_k := a_k^+ a_k$

Interpretation of $\hat{N}_k$ in QFT

* Assume that at some time, η, the state $|0\rangle$ is the vacuum.

* Thus, at η, for example the state $(a_k^+)^n|0\rangle$ is a state with n particles of momentum k.

* Now assume that at η the system is in an arbitrary state $|\Omega\rangle$.

* Then, at η, the expected number of particles of momentum k is:

$$\bar{N}_k = \langle\Omega|\hat{N}_k|\Omega\rangle$$

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Calculation in the above scenario for $\tilde{N}_k := \tilde{a}_k^+ \tilde{a}_k$ at time η_2

$$\bar{\tilde{N}}_k = \langle \Omega | \hat{\tilde{N}}_k | \Omega \rangle$$

$$= \langle 0 | \tilde{a}_k^+ \tilde{a}_k | 0 \rangle$$

Now use that $a_k = \alpha_k^* \tilde{a}_k + \beta_k \tilde{a}_{-k}^+$, i.e.

also, that $\tilde{a}_k = \tilde{\alpha}_k^* a_k + \tilde{\beta}_k a_{-k}^+$

Exercise: Calculate $\tilde{\alpha}_k, \tilde{\beta}_k$ in terms of α_k, β_k.

$$= \langle 0 | (\tilde{\alpha}_k a_k^+ + \tilde{\beta}_k^* a_{-k})(\tilde{\alpha}_k^* a_k + \tilde{\beta}_k a_{-k}^+) | 0 \rangle$$

$$= \langle 0 | \tilde{\beta}_k^* \tilde{\beta}_k a_{-k} a_{-k}^+ + \cancel{* a^+ a} + \cancel{* a a^+} + \cancel{* a a} | 0 \rangle$$

$$= \tilde{\beta}_k^* \tilde{\beta}_k \langle 0 | \cancel{a_{-k}^+ a_{-k}} + 1 | 0 \rangle$$

(using infrared regularization we have $[\tilde{a}_k, \tilde{a}_{k'}^+] = \delta_{k,k'}$)

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$$= \tilde{\beta}_k^* \tilde{\beta}_k$$

The expected total number of particles at time η_2 is then:

$$\bar{N} = \sum_k \langle \Omega | \hat{N}_k | \Omega \rangle = \sum_k \tilde{\beta}_k^* \tilde{\beta}_k$$

Note:

* We assumed here an infrared, i.e., a box regularization. (Else the number of created particles can only be 0 or ∞) ← Exercise: Why?
* Else, $\bar{N}$ may come out infinite, but that can be ok.
* This happens even for photon creation through moving charges.
* But we always must have of course finite "energy":

$$\langle \Omega | \hat{H}(\eta) | \Omega \rangle < \infty$$

Identification of the vacuum state

How can we identify, at any arbitrary fixed time, η, that Hilbert space vector, say $|\text{vacuum at } \eta\rangle$, which describes the vacuum, i.e., the no particle state, at that time, η?

Q: Is $|\text{vacuum at } \eta\rangle$ one of the (infinitely many) states

$$|0\rangle, |\tilde{0}\rangle, |\tilde{\tilde{0}}\rangle, \dots$$

that come with choices of mode functions

$$v_k, \tilde{v}_k, \tilde{\tilde{v}}_k, \dots$$

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through $a_k|0\rangle = 0$, $\tilde{a}_k|\tilde{0}\rangle = 0$, $\tilde{\tilde{a}}|\tilde{\tilde{0}}\rangle = 0, \dots$?

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A: A ... ille

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A: As we will see:

Yes, if or when |vacuum at η> exists at all,

then there exist suitable mode functions, v_k,

(namely exactly one, up to a phase, for each k)

so that with

$$\hat{x}_k = \frac{1}{\sqrt{2}}\left(v_k^* a_k + v_k a_{-k}^+\right)$$

the state $|0\rangle$ defined through $a_k|0\rangle = 0$

is the vacuum state at the time η:

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But how to specify $|\text{vacuum at } \eta\rangle$?

We notice: To specify $|\text{vacuum at } \eta\rangle$ by specifying a suitable vector $|0\rangle$

is equivalent to

specifying a suitable mode function v_k (i.e. a suitable solution to the K.G. and Wronskian equations)

is equivalent to

specifying at time η that $v_k(\eta) = r_k$, $v_k'(\eta) = s_k$ for a suitable choice of $r_k, s_k \in \mathbb{C}$.

(because with the KG equation being 2^{nd} order in time:

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1st attempt:

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Ansatz:

Let us try to define the vacuum state at a time η as that Hilbert space vector (up to a phase) which at time η minimizes the Hamiltonian, $\hat{H}^{(x)}(\eta)$.

To this end, we will choose $r_k, s_k \in \mathbb{C}$ suitably, so that $v_k(\eta) = r_k$, $v_k'(\eta) = s_k$ define that mode function v_k so that its $|0\rangle$ is the lowest energy state.

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Calculation of the lowest energy state at some arbitrary fixed time, η_1.

$$\langle 0|\,\hat{H}^{(x)}(\eta_1)\,|0\rangle = \langle 0|\,\frac{1}{2}\int_{box} \hat{\chi}'^2(\eta_1,x) + \sum_{i=1}^{3} \hat{\chi}_{,i}^2(\eta_1,x)$$

$$+ \left(m^2 a^2(\eta_1) - \frac{a''(\eta_1)}{a(\eta_1)}\right)\hat{\chi}^2(\eta_1,x)\, d^3x\,|0\rangle$$

Exercise:

Use Fourier and use

$$\hat{\chi}_k(\eta_1) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta_1)\,a_k + v_k(\eta_1)\,a_{-k}^+\right)$$

to evaluate this energy expectation value.

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Result:

$$\langle 0|\,\hat{H}^{(x)}(\eta_1)\,|0\rangle = \langle 0|\,\frac{1}{4}\sum_k \left(v_k'^2(\eta_1)+\omega_k^2(\eta_1)\,v_k^2(\eta_1)\right)a_k^+a_{-k}^+$$

$$+\frac{1}{4}\sum_k \left(v_k'^{*2}(\eta_1)+\omega_k^2(\eta_1)\,v_k^{*2}(\eta_1)\right)a_k\,a_{-k}$$

$$+\frac{1}{2}\sum_k \left(|v_k'(\eta_1)|^2+\omega_k^2(\eta_1)\,|v_k(\eta_1)|^2\right)\left(a_k^+a_k+\frac{1}{2}\right)|0\rangle$$

$$=\frac{1}{4}\sum_k |v_k'(\eta_1)|^2+\omega_k^2(\eta_1)\,|v_k(\eta_1)|^2$$

Here: the time-dependent frequency reads: $\omega_k^2(\eta) := k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}$

Note: We assume $\omega_k^2(\eta) > 0$ because, else, the potential is inverted

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Recall:

* We defined $r_k := v_k(\eta_1)$, $s_k := v_k'(\eta_1)$

* We need to determine $r_k, s_k \in \mathbb{C}$

* This will determine a full mode function v_k with its a_k

* This determines a corresponding $|0\rangle$ obeying $a_k|0\rangle = 0$

* Our ansatz is then that:

$$|\text{vacuum at } \eta_1\rangle = |0\rangle$$

Concretely:

* From above, the energy at η_1 is:

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* Using the definitions $r_k = V_k(\eta_0)$, $s_k = V_k'(\eta_0)$:

$$\langle 0|\, \hat{H}^{(x)}(\eta_0)\,|0\rangle = \frac{1}{4}\sum_k s_k s_k^* + \omega_k^2(\eta_0)\, r_k r_k^* \qquad (E)$$

* We want to minimize this expression, subject to the Wronskian condition

$$V_k'(\eta_0)\, V_k^*(\eta_0) - V_k(\eta_0)\, V_k^*(\eta_0)' = 2i$$

i.e., subject to the constraint:

$$s_k r_k^* - r_k s_k^* = 2i \qquad (C)$$

* Use Lagrange multiplier λ and extremize

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* Use Lagrange multiplier λ and extremize

* We have to solve:

$$\frac{\partial S'}{\partial s_k^*} = 0 \quad \text{i.e.,} \quad s_k - \lambda r_k = 0$$

$$\frac{\partial S'}{\partial r_k^*} = 0 \quad \text{i.e.,} \quad \omega_k^2 r_k + \lambda s_k = 0$$

along with the constraint (C): $s_k r_k^* - r_k s_k^* = 2i$

* Exercise:

Show that the solution is:

$$\frac{\partial S'}{\partial r_k^*} = 0 \quad \text{i.e.,} \quad \omega_k^2 r_k + \lambda s_k = 0$$

along with the constraint (C): $s_k r_k^* - r_k s_k^* = 2i$

* Exercise:

Show that the solution is:

$$r_k = \frac{1}{\sqrt{\omega_k}} e^{i\theta} \qquad s_k = i\sqrt{\omega_k}\, e^{i\theta}$$

where $\theta \in [0, 2\pi)$ is arbitrary. We'll choose $\theta = 0$.

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where $\theta \in [0, 2\pi)$ is arbitrary. We'll choose $\theta = 0$.

$\Rightarrow$ These conditions at time η_i:

$$\boxed{v_k(\eta_i) = \frac{1}{\sqrt{\omega_k(\eta_i)}} \quad ; \quad v_k'(\eta_i) = i\sqrt{\omega_k(\eta_i)}}$$

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$\Rightarrow$ These conditions at time η_1,

$$\boxed{v_k(\eta_1) = \frac{1}{\sqrt{\omega_k(\eta_1)}} \quad , \quad v_k'(\eta_1) = i\sqrt{\omega_k(\eta_1)}}$$

define a mode function v_k for all η so that

$$\hat{x}_k(\eta_1) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta_1)\, a_k + v_k(\eta_1)\, a_{-k}^+\right)$$

and the corresponding state $|0\rangle$ obeying $a_k|0\rangle = 0$

is the lowest energy state of the Hamiltonian $\hat{H}^{(k)}(\eta_1)$,

Special case: Minkowski space

- Minkowski space is the special case $a(\eta) = 1$ for all η.

Then, $\omega_k^2(\eta) = \vec{k}^2 + m^2$ is a constant. Also: $\eta = t$.

- We conclude that $|0\rangle$ is the state of lowest energy at a time η_0 if we choose the mode functions which obey these conditions:

$$v_k(\eta_0) = \frac{1}{\sqrt{\omega_k}}, \quad v_k'(\eta_0) = i\sqrt{\omega_k}$$

- Solving the K.G. eqn, we find that these mode fctns are:

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□ Solving the K.G. eqn, we find that these mode fctns are:

$$v_k(\eta) = \frac{1}{\sqrt{\omega_k}} e^{i(\eta-\eta_1)\omega_k} = \frac{1}{\sqrt{\omega_k}} e^{i(t-t_1)\omega_k}$$

Exercise:

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Exercise:

* Verify that the state $|0\rangle$ that we have found for Minkowski space agrees with the state that we identified as the Minkowski space vacuum at the beginning of the course.

* Show that, if we, similarly, determine the lowest energy state at another time, η_2, then we obtain the same mode function v_k (up to an irrelevant phase).

* This means that the same vector $|0\rangle$ minimizes

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At an arbitrary time η_1 the vacuum (no particle) state is that state which is the lowest energy state $|0\rangle$ at time η_1:

$$|\text{vacuum at } \eta_1\rangle = |0\rangle$$

- Implied prediction:

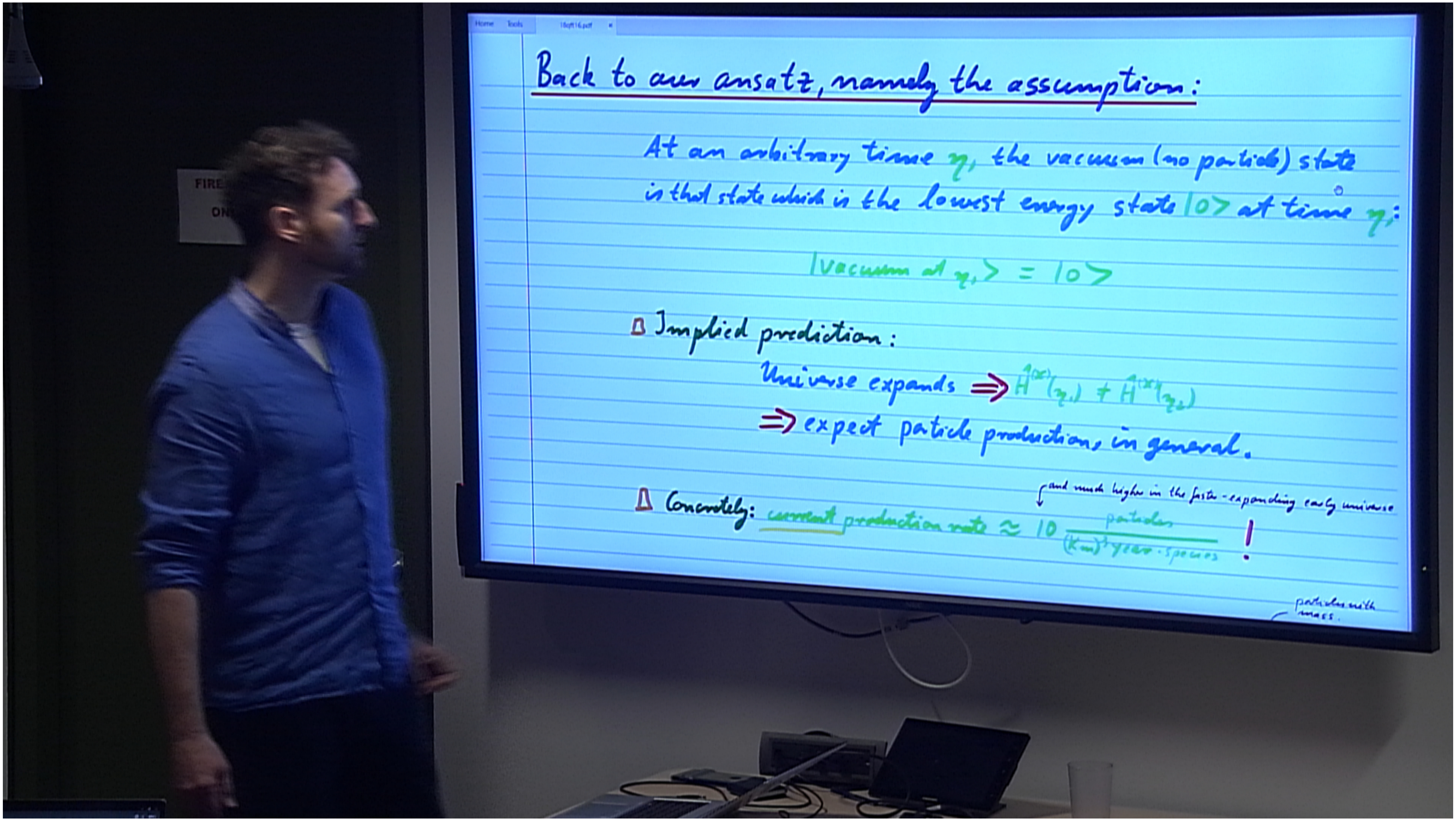
  Universe expands $\Rightarrow \hat{H}^{(x)}(\eta_1) \neq \hat{H}^{(x)}(\eta_2)$

  $\Rightarrow$ expect particle production, in general.

and much higher in the faster-expanding early universe

- Concretely: current production rate $\approx 10 \dfrac{\text{particles}}{(\text{Km})^3\,\text{year}\cdot\text{species}}$!

particles with mass.



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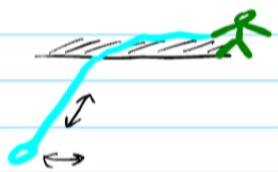
(particles with mass.)

Reconsider:

- Recall that any quantum system does not get excited (or only very little), if we change its parameters (e.g. the $\omega_k(\eta)$) "slowly".
- For the oscillator, "slow", is slow compared to the natural frequency of the oscillator.



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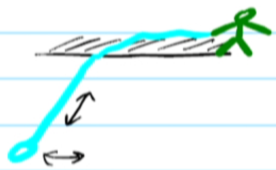

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→ How to improve our ansatz for vacuum identification?

Preliminary consideration

- Consider models where the universe is initially Minkowski and then undergoes an expansion whose parameter change (of $\omega_k(\eta)$) is slow, i.e., adiabatic.

  Note: the overall change may still be large!

$\Rightarrow$ We expect essentially no particle creation.

$\Rightarrow$ The vacuum state (i.e. no particle state) should always be essentially the same Hilbert space vector.

$\Rightarrow$ Since there is only one vacuum state, $|0\rangle$, for all time,

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How can we find this mode function v_k ?

- **Easy:** We know $v_k(\eta)$ at very early times, when the universe was still Minkowski:

$$v_k(\eta) = \frac{1}{\sqrt{\omega_k}} e^{i\omega_k(\eta-\eta_0)}$$

arbitrary reference time

Then: the K.G. eqn. yields $v_k(\eta)$ at all time!

- **Proposotion:**

$$v_k(\eta) = \frac{1}{\sqrt{\omega_k(\eta)}} e^{i\int_{\eta_0}^{\eta} \omega_k(\eta')\,d\eta'} \qquad (S)$$

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▫ Definition:

We say that a mode k evolves adiabatically slow, if:

Intuition:
$\frac{\omega'}{\omega^2}$ and $\frac{\omega''}{\omega^3}$ are rate of change of frequency compared to the frequency, and also rate of acceleration of frequency compared to the frequency.

$$\frac{\omega_k'(\eta)}{\omega_k^2(\eta)} \ll 1 \quad \text{and} \quad \frac{\omega_k''(\eta)}{\omega_k^3(\eta)} \ll 1 \qquad (AC)$$

Note: The denominators are chosen so that the quotients are unitless, because only pure numbers can reasonably be said to be small or large.

▫ Exercise: Prove the proposition.

Hint: Show that (S) obeys the K.G. eqn

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* Try to identify the v_k whose $|0\rangle$ is the adiabatically defined vacuum without referring to what v_k would look like in an earlier Minkowski period of the universe.

* Namely, try to identify v_k by a characteristic property that it has at all time.

* Indeed, we notice: (Exercise: check this)

Our v_k of (S) above satisfies at all times:

$$v_k(\eta) = e^{i\theta} \frac{1}{\sqrt{\omega_k(\eta)}}, \quad v_k'(\eta) = \left(i\omega_k(\eta) - \frac{1}{2}\frac{\omega_k'(\eta)}{\omega_k(\eta)}\right)\frac{e^{i\theta}}{\sqrt{\omega_k(\eta)}} \qquad (AV)$$

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"The general adiabatic vacuum identification"

Definition:

* Consider an arbitrary time η_1.
* Assume that the evolution of ω_k is adiabatically slow for mode k, at time η_1.
* We then identify that state as the vacuum $|0\rangle$ (i.e. as the no particle state) at η_1 whose mode function v_k is specified by the conditions (AV) at η_1:

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(AV)

* We call this $|0\rangle$ the "adiabatic vacuum" at η_1.

Remarks:

- Recall that the criteria for chosing v_k so that its $|0\rangle$ is the lowest energy vacuum at time η_1 are:

$$v_k(\eta_1) = \frac{1}{\sqrt{\omega_k(\eta_1)}}\, e^{i\theta}, \quad v_k'(\eta_1) = i\sqrt{\omega_k(\eta_1)}\, e^{i\theta} \qquad \text{(EV)}$$

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- Note that AV and EV generally differ!

  $\Rightarrow$ The adiabatically-defined vacuum is generally not the lowest energy state!

- Note that the adiabatic vacuum criterion should only be applied when the evolution of the mode under consideration is actually adiabatic.

- No vacuum criterion for generic spacetimes is known.