Title: Quantum Field Theory for Cosmology - Lecture 17

Speakers: Achim Kempf

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ote Title

QFT for Cosmology, Achim Kempf, Lecture 15

Solving the quantized K.G. eqn. on FRW spacetimes

Recall:

1.) We obtain the solution $\hat{\phi}(x,t)$ through the ansatz

$$\hat{\phi}(x,t) = \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+ \qquad (*)$$

* if we use operators a_k obeying $[a_k, a_{k'}^+] = \delta^3(k-k')$ and

* if we find classical solutions $\{u_k(x,t)\}$ of the K.G.

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* if we use operators a_k obeying $[a_k, a_{k'}^+] = \delta^3(k-k')$ and

* if we find classical solutions $\{u_k(x,t)\}$ of the K.G. eqn., called mode functions, which obey:

$$\sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) = i\, \delta^3(\vec{x} - \vec{x}') \qquad (\varsigma)$$

2.) Then, we can use the $\{a_k\}$ to build a convenient basis in the Hilbert space:

- Namely: $|0\rangle$ is the vector obeying $a_k|0\rangle = 0$
- The other basis vectors are:

$$a_k^+|0\rangle, \ldots \frac{1}{\sqrt{n!}}(a_k^+)^n|0\rangle, \ldots a_k^+ a_{k'}^+|0\rangle, \ldots$$

$$\frac{1}{\sqrt{n_1!}} \cdots \frac{1}{\sqrt{n_m!}} (a_{k_1}^+)^{n_1} \cdots (a_{k_m}^+)^{n_m}|0\rangle, \ldots \text{ etc.}$$

3.) Choosing a different set of classical solutions $\{\tilde{u}_k(x,t)\}$ which obey (C) yields the same $\hat{\phi}(x,t)$, namely

$$\hat{\phi}(x,t) = \sum_k \tilde{u}_k(x,t)\,\tilde{a}_k + \tilde{u}_k^*(x,t)\,\tilde{a}_k^+$$

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3.) Choosing a different set of classical solutions $\{\tilde u_k(x,t)\}$ which obey (C) yields the same $\hat\phi(x,t)$, namely

$$\hat\phi(x,t) = \sum_k \tilde u_k(x,t)\,\tilde a_k + \tilde u_k^*(x,t)\,\tilde a_k^+$$

but the basis of vectors $|\tilde 0\rangle, \tilde a_k^+|\tilde 0\rangle, \tilde a_k^+ \tilde a_{k'}^+|\tilde 0\rangle \dots$ is a different basis. Recall: Stone von Neumann theorem.

Application to FRW spacetime

- For convenience (namely, to avoid a "friction"-type term) we aim to solve not for $\hat{\phi}(x,t)$ directly, but instead for:

$$\hat{\chi}(\eta,x) := a(\eta)\,\hat{\phi}(\eta,x)$$

- In terms of $\hat{\chi}(\eta,x)$ the quantum K.G. eqn. reads:

$$\hat{\chi}''(\eta,x) - \Delta\hat{\chi}(\eta,x) + \left(m^2a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right)\hat{\chi}(\eta,x) = 0$$

- **Note:** This is a partial differential equation

- Observation: The derivatives $\frac{\partial}{\partial x^i}$ become multiplication operators ik_i under spatial Fourier transform.

- Plan: Before trying to solve it, use Fourier to transform the K.G. eqn. from a partial DE into a more managable set of ordinary DEs.

- Define: $\hat{x}_k(\eta) := \int \frac{1}{(2\pi)^{3/2}} \hat{x}(\eta, x) e^{-ikx} d^3x$

i.e.: $\hat{x}(\eta, x) = \int \frac{1}{(2\pi)^{3/2}} \hat{x}_k(\eta) e^{ikx} d^3k$

- Analogously:

- **Plan:** Before trying to solve it, use Fourier to transform the K.G. eqn. from a partial DE into a more manageable set of ordinary DEs.

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  i.e.: $$\hat{x}(\eta,x) = \int \frac{1}{(2\pi)^{3/2}} \, \hat{x}_k(\eta) \, e^{ikx} \, d^3k$$

- **Analogously:**

$$\hat{\pi}^{(x)}_k(\eta) := \int \frac{1}{(2\pi)^{3/2}} \, \hat{\pi}^{(x)}(\eta,x) \, e^{-ikx} \, d^3x$$

- Thus, in terms of $\hat{x}_k(\eta)$, the K.G. eqn. reads:

$$\boxed{\hat{x}_k''(\eta) + \left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right)\hat{x}_k(\eta) = 0} \qquad \text{(EoM)}$$

$\Rightarrow$ for each comoving Fourier mode k the K.G. eqn. is the eqn. of a harmonic oscillator with time-dependent frequency

$$\hat{x}_k''(\eta) + \omega_k^2(\eta)\,\hat{x}(\eta) = 0$$

with: $\quad \omega_k(\eta) := \sqrt{k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}}$

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In extreme cases:

The frequency $\omega_k(\eta)$ may become imaginary, namely if $a''(\eta)$ is large enough, i.e., if the expansion is rapid enough. Note that the discriminant also depends on k, i.e., some modes may have imaginary frequencies while others don't.

$\square$ Exercise:

* Show that $\hat{x}_k^{+}(\eta) = \hat{x}_{-k}(\eta)$, $\hat{\pi}_k^{(x)+}(\eta) = \hat{\pi}_{-k}^{(x)}(\eta)$ (HC)

* Show that

$$[\hat{x}_k(\eta), \hat{\pi}_{k'}(\eta)] = i\,\delta^3(k+k') \qquad \text{(CCR)}$$

$$\boxed{x_k''(\eta) + \left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right) x_k(\eta) = 0}$$

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* Show that $\hat{\chi}_k^+(\eta) = \hat{\chi}_{-k}(\eta)$, $\hat{\pi}_k^{(\chi)+}(\eta) = \hat{\pi}_{-k}^{(\chi)}(\eta)$ (HC)

* Show that

$$[\hat{\chi}_k(\eta), \hat{\pi}_{k'}(\eta)] = i\,\delta^3(k+k') \qquad \text{(CCR)}$$

i.e. $[\hat{\chi}_k(\eta), \hat{\pi}_{k'}^+(\eta)] = i\,\delta^3(k-k')$

- In order to solve EoM, HC, CCR for $x_k(\eta)$, we we make this ansatz:

$$\hat{x}_k(\eta) := \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_{-k}(\eta)\, a_{-k}^+\right) \qquad \text{(A)}$$

(convenient later)

- Exercise: Express the mode functions $u_k(\eta, x)$ of $(*)$ in terms of the functions $v_k(\eta)$.

- Proposition: The ansatz (A)

1.) solves the hermiticity condition (HC) by construction.

2.) solves the (EoM), if the $v_k(\eta)$ each solve (EoM)

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(*Note*: The equation depends only on $|k|$, not on the direction of k. Thus if $v_k(\eta)$ is a solution for one k then it is solution for all k' with $|k'| = |k|$. $\Rightarrow$ We can and will choose $v_k(\eta) = v_{|k|}(\eta)$)

$$\boxed{v_k''(\eta) + \left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right) v_k(\eta) = 0 \qquad (M)}$$

3.) the commutation relations (CCR) if the v_k are chosen such that they also obey:

$$\boxed{v_k'(\eta)\, v_k^*(\eta) - v_k(\eta)\, v_k^*(\eta)' = 2i \qquad (W)}$$

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☐ Exercise:

a) Prove the proposition.

b.) Assume that $v_k(\eta)$ is any solution of (EoM). Show that if (W) holds at one time then it holds at all time.

(Note: The LHS of (W) is the "Wronskian" of the ODE)

Conclusion:

In order to obtain the solution $\hat{\phi}(\eta, x)$, we do:

A) Find for each mode $k \in \mathbb{R}^3$ a solution $v_k(\eta)$ to **(M)**, i.e., a solution to the classical harmonic oscillator with time-dependent frequency.

B) Make sure $v_k(\eta)$ obeys **(W)**, if need be by multiplying with a constant. (Recall exercise b))

C) Build a basis in the Hilbert space:

$a|0\rangle = 0, \quad a^+|0\rangle, \quad a^+ a^+ |0\rangle, \quad$ etc.

the v_k are chosen such that they also obey:

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Exercise:

a) Prove the proposition.

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B) Make sure $v_k(\eta)$ obeys (W), if need be by multiplying with a constant. (Recall exercise b))

C) Build a basis in the Hilbert space:

$a_k |0\rangle = 0$, $a_k^+ |0\rangle$, $a_k^+ a_{k'}^+ |0\rangle$, etc ...

Choice of mode solutions $\{v_k(\eta)\}$

- For each choice, say $\{v_k(\eta)\}_{k\in\mathbb{R}^3}$ or $\{\tilde{v}_k(\eta)\}_{k\in\mathbb{R}^3}$ we obtain the same $\hat{\phi}(x,t)$ but the bases

  $|0\rangle,\ a_k^+|0\rangle,\ a_k^+a_{k'}^+|0\rangle, \dots$

  and

  $|\tilde{0}\rangle,\ \tilde{a}_k^+|\tilde{0}\rangle,\ \tilde{a}_k^+\tilde{a}_{k'}^+|\tilde{0}\rangle, \dots$

  will of course be different.

- We will often find it convenient to use the basis $|0\rangle,\ a_k^+|0\rangle,\ a_k^+a_{k'}^+|0\rangle, \dots$ that comes with one set of mode functions $\{v_k(\eta)\}$ at one time (say initially) and then the basis obtain the same $\hat{\phi}(x,t)$ but the bases

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Why?

- In the Heisenberg picture, the system's state vector is always the same Hilbert space vector.

- But the observables evolve in time!

$\Rightarrow$ The meanings of all Hilbert space vectors change over time

$\rightsquigarrow$ We may, e.g., choose a set $\{v_\alpha\}$ whose vector $|0\rangle$ happens to be the vacuum state

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- In the Heisenberg picture, the system's state vector is always the same Hilbert space vector.

- But the observables evolve in time!

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We may, e.g., choose a set $\{v_k\}$ whose vector $|0\rangle$ happens to be the vacuum state at one time and we may choose another set $\{\tilde{v}_k\}$ whose vector $|\tilde{0}\rangle$ happens to be the vacuum state at another time.

How many possible choices of

$$\{v_k(\eta)\}, \{\tilde{v}_k(\eta)\}, \{\hat{\tilde{v}}_k(\eta)\}, \ldots$$

▫ How many possible choices of

$$\{v_k(\eta)\}_{k\in\mathbb{R}^3}, \{\tilde{v}_k(\eta)\}_{k\in\mathbb{R}^3}, \{\hat{\tilde{v}}_k(\eta)\}_{k\in\mathbb{R}^3}, \dots$$

do exist?

▫ Let us consider each mode, k, separately:

▫ The solution space of (M), for fixed k,

$$v_k''(\eta) + \left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right) v_k(\eta) = 0$$

has of course 2 complex dimensions.

Note: Every solution obeying (M) must be complex-valued. Why?

▫ If v_k is a complex-valued solution, then

$k \in \mathbb{R}$ $\quad$ $k \in \mathbb{R}^3$ $\quad$ $k \in \mathbb{R}^5$

do exist?

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- The solution space of (M), for fixed k,

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Note: Every solution obeying (M) must be complex-valued. Why?

- If v_k is a complex-valued solution, then

$$v_k \text{ and } v_k^*$$

form a basis in the solution space.

$\Longrightarrow$ Every solution, $\tilde{v}_k$, is a linear combination of v_k, v_k^*, i.e., there must exist $\alpha, \beta \in \mathbb{C}$, so that:

$$\tilde{v}_k(\eta) = \alpha_k\, v_k(\eta) + \beta_k\, v_k^*(\eta)$$

- The actual dimensionality is 3 !

The solution space thus has 4 real dimensions, but one real dimension is lost because the solutions $v_k, \tilde{v}_k$, etc must also obey (W), i.e.:

$$v_k'(\eta)\, v_k^*(\eta) - v_k(\eta)\, v_k^*(\eta)' = 2i \qquad \text{(W)}$$

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- **The actual dimensionality is 3 !**

The solution space thus has 4 real dimensions, but one real dimension is lost because the solutions $v_k, \tilde{v}_k$, etc must also obey (W), i.e.:

$$v_k'(\eta)\, v_k^*(\eta) - v_k(\eta)\, v_k^*(\eta)' = 2i \qquad \text{(W)}$$

(i.e. $\mathrm{Im}\,(v' v^*) = 1$, which is only one real equation)

Proposition:

Assume v_k obeys (W). Then, $\tilde{v}_k$ defined through

$$\tilde{v}_k(\eta) = \alpha_k\, v_k(\eta) + \beta_k\, v_k^*(\eta) \qquad (B)$$

also obeys (W), iff the coefficients α_k, β_k obey:

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

Proof: Exercise

⇒ we easily obtain: $\hat{x}_k(\eta) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k(\eta)\, a_{-k}^+\right)$

$$= \frac{1}{\sqrt{2}}\left(\tilde{v}_k^*(\eta)\, \tilde{a}_k + \tilde{v}_k(\eta)\, \tilde{a}_{-k}^+\right) \quad (P)$$

$$\tilde{v}_k(\eta) = \alpha_k v_k(\eta) + \beta_k v_k^*(\eta) \qquad (B)$$

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⇒ we easily obtain:

$$\left.\begin{aligned} \hat{x}_k(\eta) &= \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k(\eta)\, a_{-k}^+\right) \\ &= \frac{1}{\sqrt{2}}\left(\tilde{v}_k^*(\eta)\, \tilde{a}_k + \tilde{v}_k(\eta)\, \tilde{a}_{-k}^+\right) \end{aligned}\right\} (P)$$

$$= \ldots$$

□ Terminology: Such a transformation from one choice $\{v_k\}$, a_k and corresponding basis

$$|0\rangle,\ a_k^+|0\rangle,\ a_k^+ a_{k'}^+|0\rangle,\ \dots$$

to some $\{\tilde{v}_k\}$, $|\tilde{0}\rangle$, $\tilde{a}_k$ and their basis

$$|\tilde{0}\rangle,\ \tilde{a}_k^+|0\rangle,\ \tilde{a}_k^+ \tilde{a}_{k'}^+|\tilde{0}\rangle,\ \dots$$

is called a "Bogolubov transformation".

Strategy: We have two tasks now:

* Make Bogolubov Hilbert basis transforms explicit.

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Strategy: We have two tasks now:

* Make Bogolubov Hilbert basis transforms explicit.

(E.g. so that $|0\rangle$ is, at least at one time, the vacuum.) * Find out when which choice of $\{v_K\}$ is convenient.

Bogolubov transformations of Hilbert bases

- How can we express the basis vectors

Bogolubov transformations of Hilbert bases

- How can we express the basis vectors

$$|\tilde{0}\rangle,\ \tilde{a}_k^+|\tilde{0}\rangle,\ \frac{1}{\sqrt{2!}}\tilde{a}_k^{+2}|\tilde{0}\rangle,\ \dots,\ \tilde{a}_k^+\tilde{a}_{k'}^+|\tilde{0}\rangle,\ \dots$$

as linear combinations of the basis

$$|0\rangle,\ a_k^+|0\rangle,\ \frac{1}{\sqrt{2!}}a_k^{+2}|0\rangle,\ \dots,\ a_k^+a_{k'}^+|0\rangle,\ \dots\ ?$$

- Proposition: Equations (B) & (P) yield: $\boxed{a_k = \alpha_k^* \tilde{a}_k + \beta_k \tilde{a}_{-k}^+}$

Proof: Exercise.

- Now we observe that $a_k|0\rangle = 0$ becomes:

- How can we express the basis vectors

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- Now we observe that $a_k|0\rangle = 0$ becomes:

$$\left(\alpha_k^* \tilde{a}_k + \beta_k \tilde{a}_{-k}^+\right)|0\rangle = 0$$

- Try to solve for $|0\rangle$ using ansatz: $|0\rangle := \left(\prod_k f_k(\tilde{a}_k^+, \tilde{a}_{-k}^+)\right)|\tilde{0}\rangle$
- Proposition:

$$|0\rangle = \left[\prod_k \frac{1}{|\alpha_k|^{1/2}} e^{-\frac{\beta_k}{2\alpha_k^*}\tilde{a}_k^+\tilde{a}_{-k}^+}\right]|\tilde{0}\rangle \quad (T)$$

needed for normalization

- Proof: Exercise.

  Hint: Use $a e^{\lambda a^+} = e^{\lambda a^+} a + \lambda e^{\lambda a^+} \dots$

Interpretation of (T):

- Assume, e.g., that $|0\rangle$ and $|\tilde{0}\rangle$ are those Hilbert

Now can we express the states

$$|\tilde{0}\rangle,\ \tilde{a}_k^{+}|\tilde{0}\rangle,\ \frac{1}{\sqrt{2!}}\tilde{a}_k^{+2}|\tilde{0}\rangle, \dots,\ \tilde{a}_k^{+}\tilde{a}_{k'}^{+}|\tilde{0}\rangle, \dots$$

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- <u>Proof:</u> Exercise.

Hint: Use $a e^{\lambda a^+} = e^{\lambda a^+} a + \lambda e^{\lambda a^+} \ldots$

Interpretation of (T):

$$|\tilde{0}\rangle,\ \tilde{a}_k^{\dagger}|\tilde{0}\rangle,\ \frac{1}{\sqrt{2!}}\tilde{a}_k^{\dagger 2}|\tilde{0}\rangle,\ \ldots,\ \tilde{a}_k^{\dagger}\tilde{a}_{k'}^{\dagger}|\tilde{0}\rangle,\ \ldots$$

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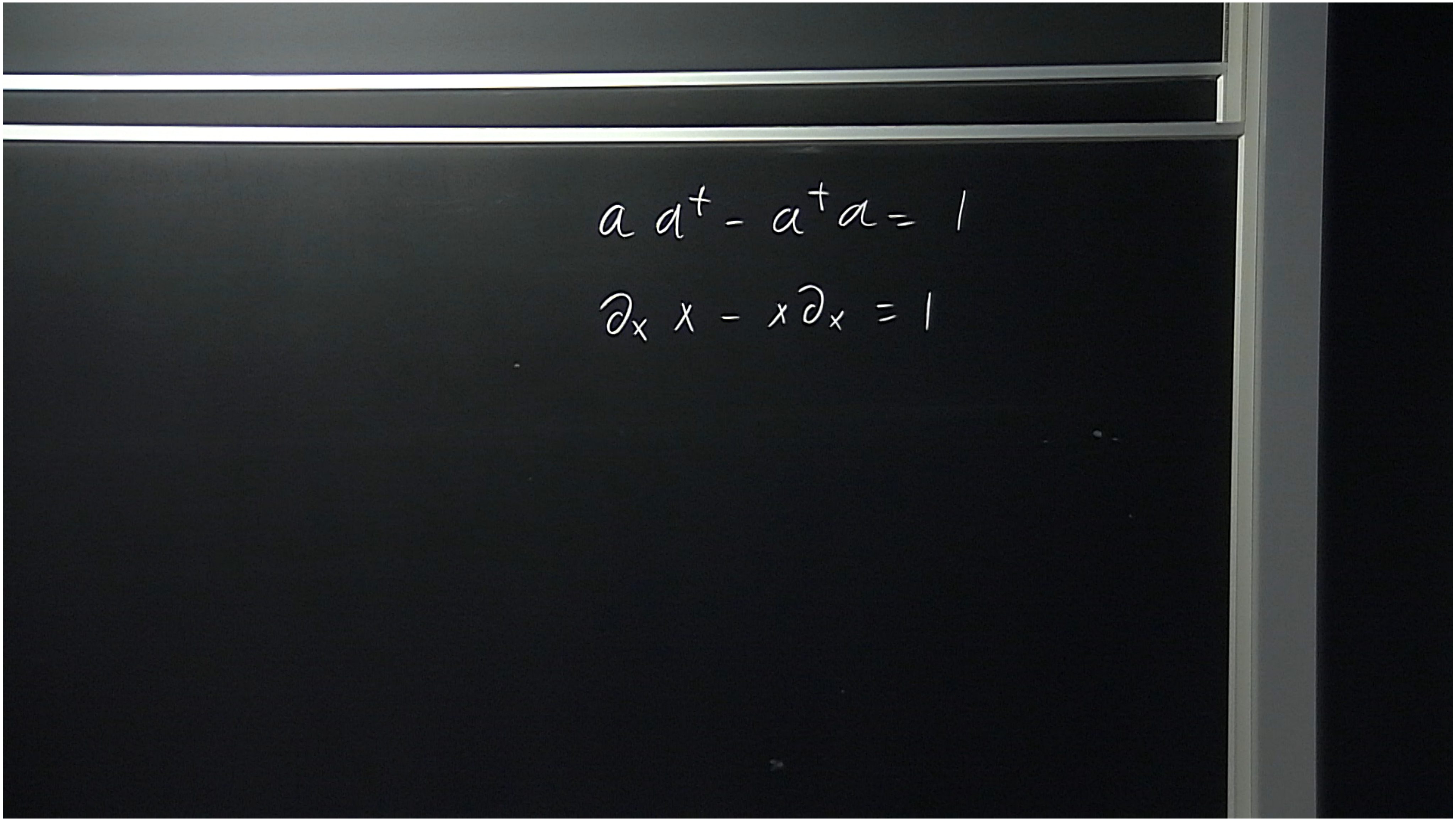
$a a^{+} - a^{+} a = 1$
$\partial_x x - x \partial_x = 1$

$$|0\rangle = \left[\prod_k \frac{1}{|\alpha_k|^{1/2}} e^{-\frac{\beta_k}{2\alpha_k^*}\tilde{a}_k^+\tilde{a}_{-k}^+}\right]|\tilde{0}\rangle \quad (T)$$

needed for normalization

- Proof: Exercise.
  Hint: Use $a e^{\lambda a^+} = e^{\lambda a^+} a + \lambda e^{\lambda a^+} \ldots$

Interpretation of (T):

- Assume, e.g., that $|0\rangle$ and $|\tilde{0}\rangle$ are those Hilbert space vectors which happen to be the vacuum state vectors at the times η_1 and η_2 respectively.