Title: The theory of entanglement-assisted quantum metrology

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Abstract: The quantum Fisher information (QFI) measures the amount of information that a quantum state carries about an unknown parameter. Given a quantum channel, its entanglement-assisted QFI is the maximum QFI of the output state assuming an input state over the system and an arbitrarily large ancilla. Consider N identical copies of a quantum channel, the channel QFI grows either linearly or quadratically with N asymptotically. Here we obtain a simple criterion that determines whether the scaling is linear or quadratic. In both cases, we found a quantum error correction protocol achieving the channel QFI asymptotically and an SDP to solve the optimal code.

Quantum metrology for quantum channels

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2020/02/12

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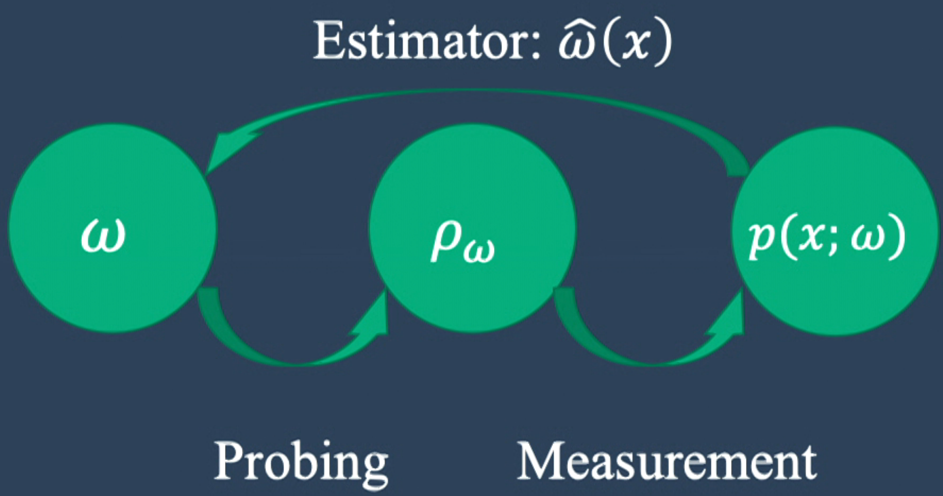
Quantum Cramér-Rao bound



Estimation precision:

$$\Delta\omega = (\mathbb{E}[(\widehat{\omega}(x) - \omega)^2])^{\frac{1}{2}}$$

For unbiased estimators $\mathbb{E}[\widehat{\omega}(x)] = \omega$, we have the quantum Cramér-Rao bound:

$$\Delta\omega \geq \frac{1}{\sqrt{N_{\mathrm{expr}} \cdot F(\rho_\omega)}}$$

N_{expr}: Number of experiments

$F(\rho_\omega)$: Quantum Fisher information

Helstrom 1976, Holevo 1982,
Braunstein & Caves. *PRL* 72(22), 3439. (1994)

Quantum Fisher information

$$F(\rho_\omega) = \mathrm{Tr}(\rho_\omega L^2),$$

where the Hermitian operator L is the symmetrized logarithmic derivatives of ρ_ω defined by

$$\frac{\partial \rho_\omega}{\partial \omega} = \frac{1}{2}(L\rho_\omega + \rho_\omega L).$$

- Connection to the Bures distance: $\frac{1}{4}F(\rho_\omega)(d\omega)^2 = d_B^2(\rho_\omega, \rho_{\omega+d\omega})$.
- Additivity: $F\left(\rho_\omega^{(1)} \otimes \rho_\omega^{(2)}\right) = F\left(\rho_\omega^{(1)}\right) + F\left(\rho_\omega^{(2)}\right)$.
- Monotonicity: $F\big(\mathcal{E}(\rho_\omega)\big) \le F(\rho_\omega)$.
- Convexity: $F(p_1\rho_\omega + p_2\sigma_\omega) \le p_1 F(\rho_\omega) + p_2 F(\sigma_\omega)$.

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Quantum Fisher information

Example 1 (pure state):

$\rho_\omega = |\psi_\omega\rangle\langle\psi_\omega|$, $|\psi_\omega\rangle = e^{-i\omega H t}|\psi_0\rangle$,

$$F(\rho_\omega) = 4t^2\langle\Delta^2 H\rangle = \Theta(t^2),$$

where $\langle\Delta^2 H\rangle = (\langle\psi_0|H^2|\psi_0\rangle - \langle\psi_0|H|\psi_0\rangle^2)$.

- For a single qubit state, when $H = Z/2$, the optimal initial state is

$$|\psi_0\rangle = \frac{(|0\rangle + |1\rangle)}{\sqrt{2}}.$$

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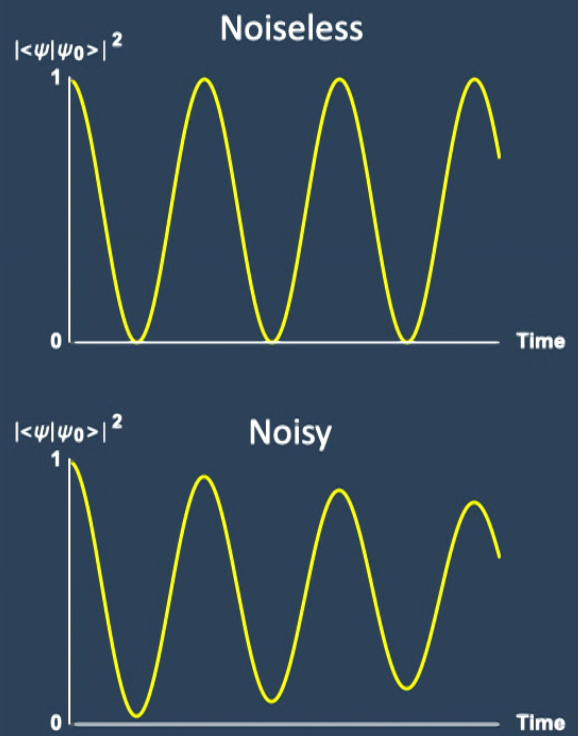
Quantum Fisher information



Example 2 (single-qubit dephasing channel):

$$\frac{d\rho}{dt} = -i\left[\frac{\omega Z}{2}, \rho\right] + \frac{\gamma}{2}(Z\rho Z - \rho)$$

Input state: $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

Noiseless case ($\gamma = 0$): $F(\rho_\omega(t)) = t^2 = \Theta(t^2)$.

Noisy case ($\gamma > 0$): $F(\rho_\omega(t)) = t^2 e^{-2\gamma t}$.

Normalized QFI: $\mathfrak{F} = \max_{t>0} F(\rho_\omega(t))/t = 1/2e\gamma$.

If we measure and renew the qubit every constant time, we get QFI $= \Theta(t)$.

Quantum Fisher information

Example 3 (N-qubit dephasing channel):

Input state:

$$|\psi_0\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)^{\otimes N}$$

Noiseless case ($\gamma = 0$): $F(\rho_\omega(t)) = Nt^2 = \Theta(N)$.

Noisy case ($\gamma > 0$): $F(\rho_\omega(t)) = Nt^2 e^{-2\gamma t}$.

Normalized QFI: $\mathfrak{F} = N/2e\gamma = \Theta(N)$.

Quantum Fisher information

Example 3 (N-qubit dephasing channel):

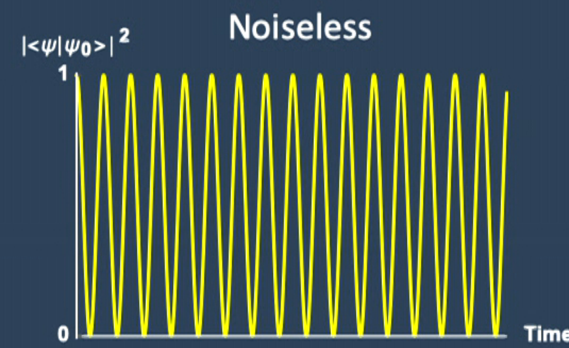
Input state (GHZ state):

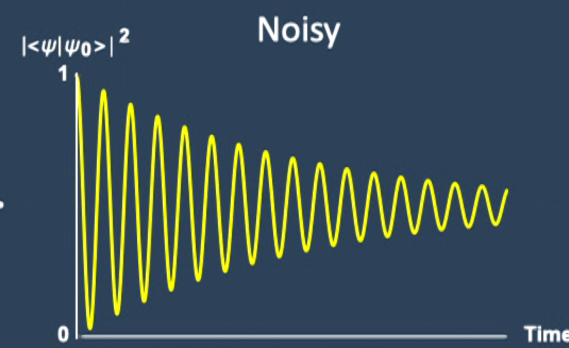
$$|\psi_0\rangle = \frac{|0\rangle^{\otimes N} + |1\rangle^{\otimes N}}{\sqrt{2}}$$

Noiseless case ($\gamma = 0$): $F(\rho_\omega(t)) = N^2 t^2 = \Theta(N^2)$.

Noisy case ($\gamma > 0$): $F(\rho_\omega(t)) = N^2 t^2 e^{-2N\gamma t}$.

Normalized QFI: $\mathfrak{F} = N/2e\gamma = \Theta(N)$.





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Quantum Fisher information

In quantum metrology, we care about the scaling of QFI with respect to the number of channels used N or the probing time t. There are two types of estimation precision limits:

- The Heisenberg limit (HL): QFI $= \Theta(N^2)$ or $\Theta(t^2)$

— the ultimate estimation precision limit allowed by quantum mechanics.

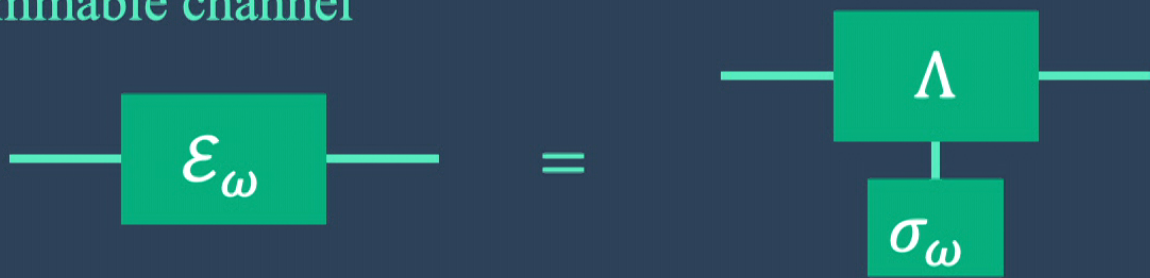
- The standard quantum limit (SQL): QFI $= \Theta(N)$ or $\Theta(t)$

— achievable using "classical" strategy (no need to maintain the coherence in & between probes for a long time).

QFI upper bounds for quantum channels

To clarify the role of noise in quantum metrology, many different upper bounds of the QFI were derived.

- Programmable channel

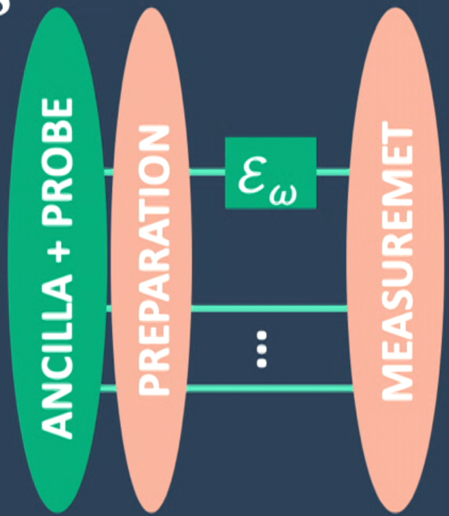


$$F\left(\mathcal{E}_\omega(\rho)^{\otimes N}\right) = F\left(\Lambda^{\otimes N}\left(\rho \otimes \sigma_\omega^{\otimes N}\right)\right) \leq F\left(\rho \otimes \sigma_\omega^{\otimes N}\right) = NF(\sigma_\omega)$$

Example (depolarizing): $\mathcal{E}_\omega(\rho) = p\frac{I}{2} + (1-p)e^{-i\omega Zt/2}\rho e^{i\omega Zt/2}$ $(p > 0)$.

Ji, *et al.* *IEEE:TIT 54*(11), 5172-5185. (2008)
Pirandola & Lupo. *PRL* 118(10), 100502. (2017)

QFI upper bounds for quantum channels



- Channel-extension method

Purification-based definition of the QFI:

$$F(\rho_\omega) = 4 \min_{|\psi_\omega\rangle} \langle \dot{\psi}_\omega | \dot{\psi}_\omega \rangle, \quad \rho_\omega = \mathrm{Tr}_E(|\psi_\omega\rangle\langle\psi_\omega|),$$

where $|\dot{\psi}_\omega\rangle = \partial_\omega |\psi_\omega\rangle$. Let $\mathcal{E}_\omega(\rho) = \sum_i K_i \rho K_i^\dagger$,

$$F(\rho_\omega) = 4 \min_{\{K_i'\}} \langle \psi_0 | \sum_i \dot{K}_i'^\dagger \dot{K}_i' \, |\psi_0\rangle, \quad \rho_\omega = (\mathcal{E}_\omega \otimes I)(|\psi_0\rangle\langle\psi_0|),$$

minimized over all possible Kraus operators K_i' representing $\mathcal{E}_\omega(\rho)$.

Fujiwara & Imai. *JPA: Math & Theor* 41,255304. (2008)

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QFI upper bounds for quantum channels

Define the (entanglement-assisted) channel QFI of $\mathcal{E}_\omega$,

$$\mathcal{F}_1(\mathcal{E}_\omega) := \max_{|\psi_0\rangle} F\big((\mathcal{E}_\omega \otimes I)(|\psi_0\rangle\langle\psi_0|)\big) = 4 \min_{\{K_i'\}} \|\alpha\|,$$

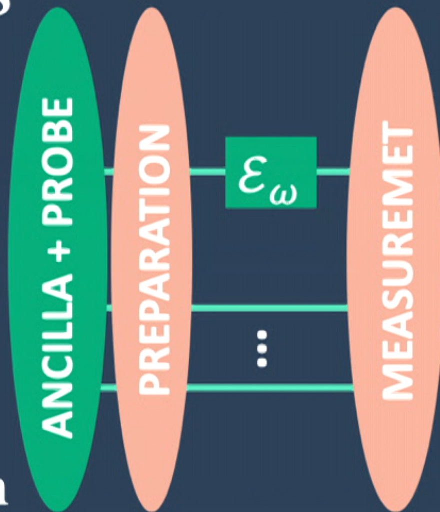
where $\alpha = \sum_i \dot{K}_i'^\dagger \dot{K}_i'$.

Let $\boldsymbol{K} = (K_1, K_2, \cdots, K_r)^T$ and $\boldsymbol{K}' = u\boldsymbol{K}$ where $u^\dagger u = I$. Then

$$\dot{\boldsymbol{K}}'^\dagger \dot{\boldsymbol{K}}' = \big(\dot{\boldsymbol{K}} - ih\boldsymbol{K}\big)^\dagger \big(\dot{\boldsymbol{K}} - ih\boldsymbol{K}\big), \quad h = iu^\dagger \dot{u},$$

where h is an $r \times r$ Hermitian matrix. Then

$$\mathcal{F}_1(\mathcal{E}_\omega) = 4 \min_h \|\alpha\|, \quad \alpha = \big(\dot{\boldsymbol{K}} - ih\boldsymbol{K}\big)^\dagger \big(\dot{\boldsymbol{K}} - ih\boldsymbol{K}\big).$$



Fujiwara & Imai. *JPA: Math & Theor* 41,255304. (2008)

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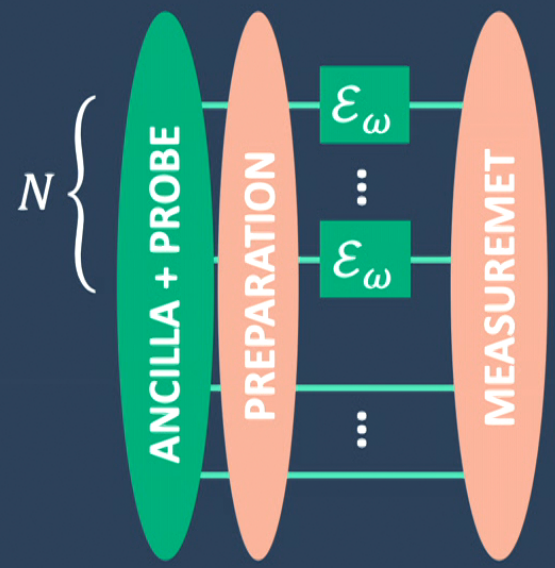
QFI upper bounds for quantum channels

Consider N identical copies of a quantum channel $\mathcal{E}_\omega$,

$$\mathcal{F}_N(\mathcal{E}_\omega) = \mathcal{F}_1\left(\mathcal{E}_\omega^{\otimes N}\right) = 4 \min_{\{K'_{N,i}\}} \|\alpha_N\|.$$



If we only consider $\{K'_{N,i}\}$ which are tensor products,

we can find an upper bound of $\mathcal{F}_N(\mathcal{E}_\omega)$,

$$\mathcal{F}_N(\mathcal{E}_\omega) \le \min_h \; 4N\|\alpha\| + 4N(N-1)\|\beta\|^2,$$

where $\alpha = \dot{\boldsymbol{K}}'^\dagger \dot{\boldsymbol{K}}' = \left(\dot{\boldsymbol{K}} - ih\boldsymbol{K}\right)^\dagger \left(\dot{\boldsymbol{K}} - ih\boldsymbol{K}\right)$, $\beta = i\boldsymbol{K}'^\dagger \dot{\boldsymbol{K}}' = i\boldsymbol{K}^\dagger\left(\dot{\boldsymbol{K}} - ih\boldsymbol{K}\right)$.

Fujiwara & Imai. *JPA: Math & Theor* 41,255304. (2008)
Demkowicz-Dobrzanski, *et al*. *Ncomms* 3, 1063 (2012).

QFI upper bounds for quantum channels

$$\mathcal{F}_N(\mathcal{E}_\omega) \le \min_h \; 4N\|\alpha\| + 4N(N-1)\|\beta\|^2.$$

When $\exists h$, such that $\beta = 0$, we must have

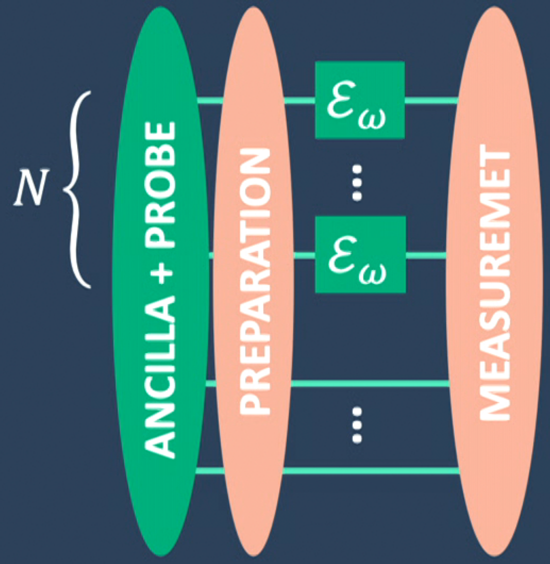
$$\mathcal{F}_N(\mathcal{E}_\omega) \le \min_{h:\beta=0} \; N\|\alpha\|.$$

Hamiltonian: $H = i\boldsymbol{K}^\dagger \dot{\boldsymbol{K}}$,

Kraus span: $\mathcal{S} = \mathrm{span}_{\mathrm{Herm}}\{K_i^\dagger K_j, \forall i, j\}$.

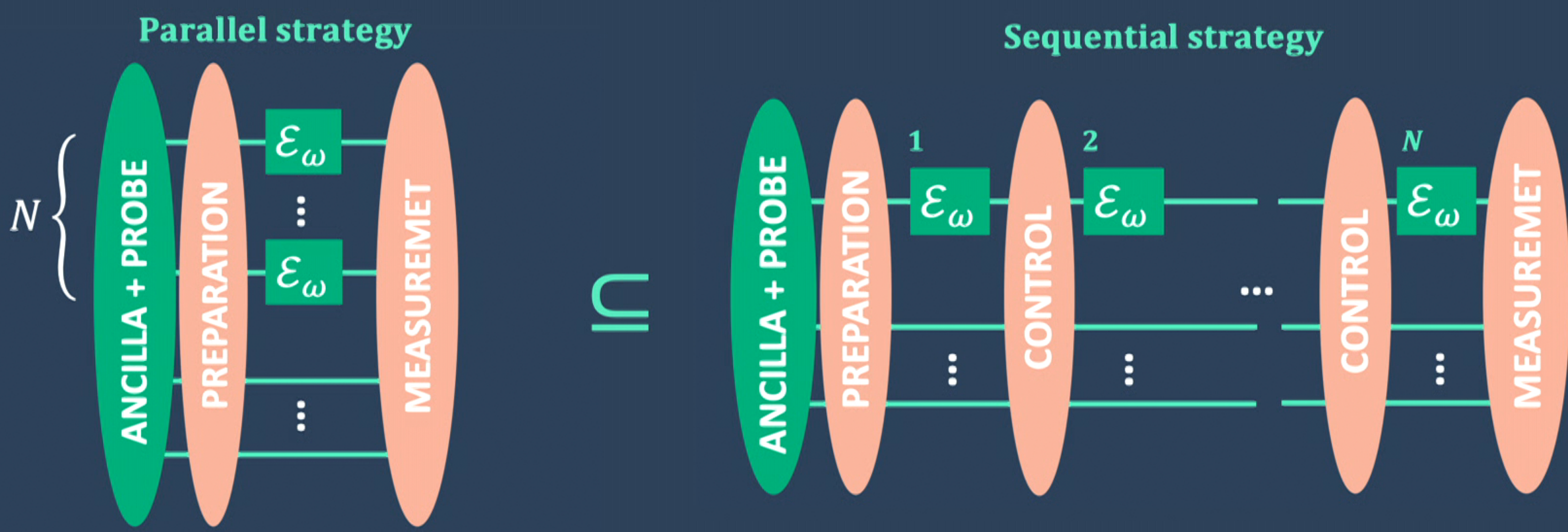
$\exists h$ such that $\beta = 0$ if and only if $H \in \mathcal{S}$.

Theorem: $\mathcal{F}_N(\mathcal{E}_\omega) = O(N)$ if $H \in \mathcal{S}$.



Demkowicz-Dobrzanski, *et al. Ncomms* 3, 1063 (2012).
Demkowicz-Dobrzanski & Maccone. *PRL* 113, 250801 (2014).

QFI upper bounds for quantum channels



$$F \le 4N\|\alpha\| + 4N(N-1)\|\beta\|^2 \qquad F \le 4N\|\alpha\| + 4N(N-1)\|\beta\|\left(\|\beta\| + 2\sqrt{\|\alpha\|}\right)$$

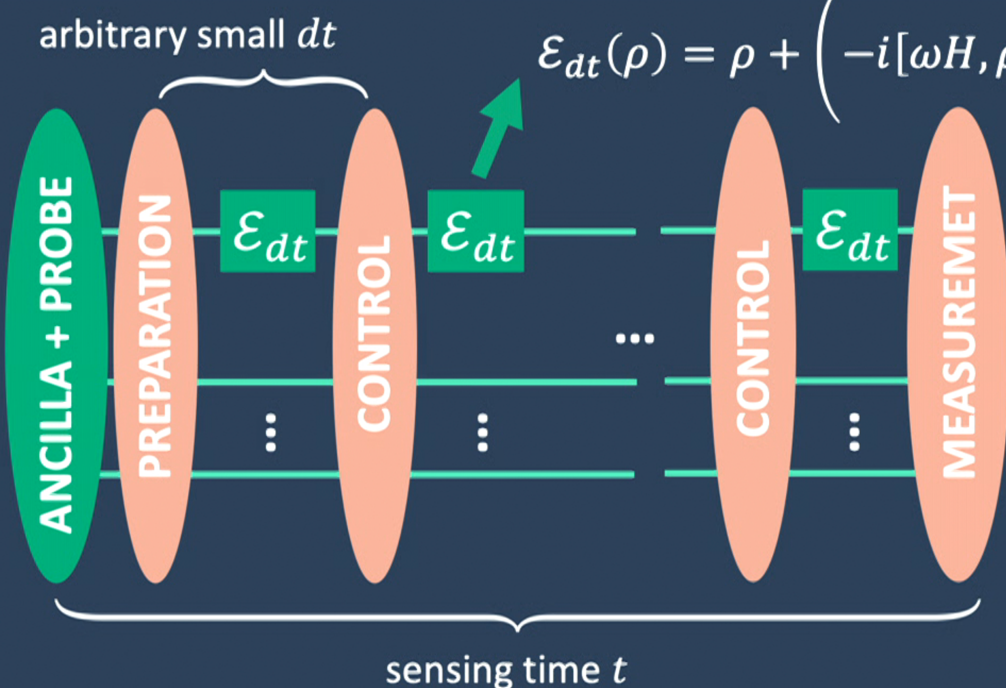
Examples ($H \in \mathcal{S}$): Dephasing, erasure, amplitude-damping channel, etc.

Demkowicz-Dobrzanski, *et al. Ncomms* 3, 1063 (2012).
Demkowicz-Dobrzanski & Maccone. *PRL* 113, 250801 (2014).

Estimating Hamiltonian under Markovian noise

$$\mathcal{E}_{dt}(\rho) = \rho + \left(-i[\omega H, \rho] + \sum_{i=1}^{r} L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\}\right) dt + O(dt^2)$$

$$\rho \in \mathcal{H}_P,\ \dim \mathcal{H}_P = d.$$



We consider the case where quantum controls are performed arbitrarily fast.

Sekatski *et al. Quantum* 1, 27 (2017).

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Estimating Hamiltonian under Markovian noise

$$F(t) \le 4\left(\frac{t}{dt}\right)\|\alpha_{dt}\| + \left(\frac{t}{dt}\right)^2 \|\beta_{dt}\|(\|\beta_{dt}\| + 2\sqrt{\|\alpha_{dt}\|})$$

$$\alpha_{dt} = \alpha^{(0)} + \alpha^{(1)}\sqrt{dt} + \alpha^{(2)}dt + O(dt^{3/2})$$

$$\beta_{dt} = \beta^{(0)} + \beta^{(1)}\sqrt{dt} + \beta^{(2)}dt + \beta^{(3)}dt^{3/2} + O(dt^2)$$

It is easy to find Kraus operators s.t. $\alpha^{(0,1)} = \beta^{(0,1,3)} = 0$.

$$\alpha^{(2)} = (\mathbf{h}I + \hbar\mathbf{L})^\dagger(\mathbf{h}I + \hbar\mathbf{L}), \quad \beta^{(2)} = H + \mathfrak{h}I + \mathbf{h}^\dagger\mathbf{L} + \mathbf{L}^\dagger\mathbf{h} + \mathbf{L}^\dagger\hbar\mathbf{L},$$

where $\mathfrak{h} \in \mathbb{C}$, $\mathbf{h} \in \mathbb{C}^r$, $\hbar \in \mathbb{C}^{r\times r}$ is Hermitian.

Sekatski *et al. Quantum* 1, 27 (2017).
Demkowicz-Dobrzanski *et al. PRX* 7, 041009 (2017).
SZ *et al. Ncomms 9, 78* (2018).

Estimating Hamiltonian under Markovian noise

Lindblad span: $\mathcal{S} = \text{span}_{\text{Herm}}\{I, L_i, L_i^\dagger, L_i^\dagger L_j, \forall i, j\}$

$\exists h$ such that $\beta^{(2)} = 0$ if and only if $H \in \mathcal{S}$.

Theorem: $F(t) = O(t)$ if $H \in \mathcal{S}$.

Examples ($H \in \mathcal{S}$):

Dephasing channel: $H = Z$, $L_1 = Z$, $\mathcal{S} = \text{span}\{I, Z\}$.

Depolarizing channel: $H = Z$, $L_{1,2,3} = X, Y, Z$, $\mathcal{S} =$ all Hermitian matrices.

Photon loss channel (truncated at $d - 1$ photons):

$$H = a^\dagger a,\ L_1 = a,\ \mathcal{S} = \text{span}_{\text{Herm}}\{I, a^\dagger a, a, a^\dagger\}.$$

Demkowicz-Dobrzanski *et al. PRX* 7, 041009 (2017).
SZ *et al. Ncomms 9, 78* (2018).

Estimating Hamiltonian under Markovian noise

What happens if $H \notin \mathcal{S}$?

For example, $H = Z$, $L_1 = X$, $\mathcal{S} = \text{span}\{I, X\}$;

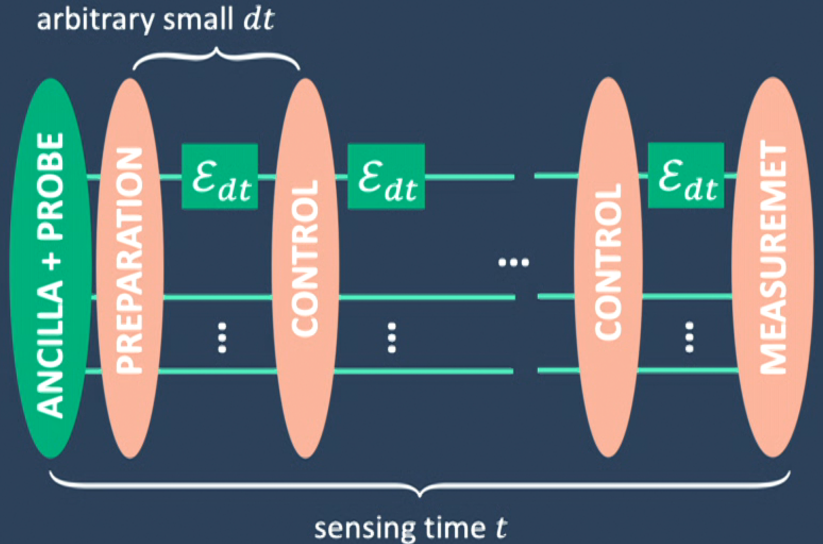
or $H = \left(a^\dagger a\right)^2$, $L_1 = a$, $\mathcal{S} = \text{span}_{\text{Herm}}\{I, a^\dagger a, a, a^\dagger\}$.

Theorem (HNLS): The HL $F(t) = \Theta(t^2)$ is achievable if and only if $H \notin \mathcal{S}$. When $H \notin \mathcal{S}$, there exists a quantum error correction protocol achieving the HL.

Demkowicz-Dobrzanski *et al. PRX* 7, 041009 (2017).
SZ *et al. Ncomms 9, 78* (2018).

Estimating Hamiltonian under Markovian noise

Theorem (HNLS): The Heisenberg limit $F(t) = \Theta(t^2)$ is achievable if and only if $H \notin \mathcal{S}$. When $H \notin \mathcal{S}$, there exists a quantum error correction protocol achieving the HL.



- Noiseless ancilla
- Fast & accurate quantum control

Demkowicz-Dobrzanski *et al. PRX* 7, 041009 (2017).
SZ *et al. Ncomms 9, 78* (2018).

Quantum error correction

Consider quantum channel $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$. The Knill-Laflamme condition says

$\exists$A recovery channel $\mathcal{R}$, s.t. $\rho = \mathcal{R} \circ \mathcal{E}(\rho)$ for all $\rho = P\rho P$,

$$\text{if and only if} \qquad PE_j^\dagger E_k P \propto P, \forall j, k.$$

Similarly, $\exists$A recovery channel $\mathcal{R}$, s.t. $\frac{d\rho}{dt} = -i[\omega PHP, \rho]$ for all $\rho = P\rho P$,

$$\text{if and only if} \quad PL_k P \propto P, \; PL_j^\dagger L_k P \propto P.$$

To recover the HL, we need

$$PSP \propto P, \quad PHP \neq \text{constant} \cdot P.$$

Knill & Laflamme. *PRA* 55(2):900. (1997)

Estimating Hamiltonian under Markovian noise

When $H \in \mathcal{S}$, we have

$$F(t) \leq 4 \min_{h,\mathbf{h},\mathfrak{h}} \left\|\alpha^{(2)}\right\|_{\beta^{(2)}=0} t \quad \Rightarrow \mathfrak{F} = \max_{t>0} \frac{F(t)}{t} \leq 4 \min_{h,\mathbf{h},\mathfrak{h}} \left\|\alpha^{(2)}\right\|_{\beta^{(2)}=0}.$$

where $\alpha^{(2)} = (\mathbf{h}I + \mathfrak{h}\mathbf{L})^\dagger(\mathbf{h}I + \mathfrak{h}\mathbf{L})$, $\beta^{(2)} = H + hI + \mathbf{h}^\dagger\mathbf{L} + \mathbf{L}^\dagger\mathbf{h} + \mathbf{L}^\dagger\mathfrak{h}\mathbf{L}$.

Theorem: When $H \in \mathcal{S}$, for any small $\eta > 0$, there exists an approximate quantum error correction protocol such that $\mathfrak{F} > 4 \min_{h,\mathbf{h},\mathfrak{h}} \|\alpha\|_{\beta=0} - \eta$.

Why approximate QEC?

If the noise ($\mathcal{S}$) is fully corrected, the Hamiltonian (H) is also fully corrected.

SZ & Jiang. *arXiv*:1910.08472 (2019)

Estimating Hamiltonian under Markovian noise

- Quantum error correction solves the problem of obtaining the optimal QFI when we estimate Hamiltonian under Markovian noise.
- The QEC codes could be solved via an SDP.

- We can only detect the parameter in the Hamiltonian;
- We need to perform fast and frequent quantum controls;
- It is not clear what the physical meaning of $\mathfrak{F} = 4 \min_{h,\mathbf{h},\mathfrak{h}} \|\alpha^{(2)}\|_{\beta^{(2)}=0}$ is.

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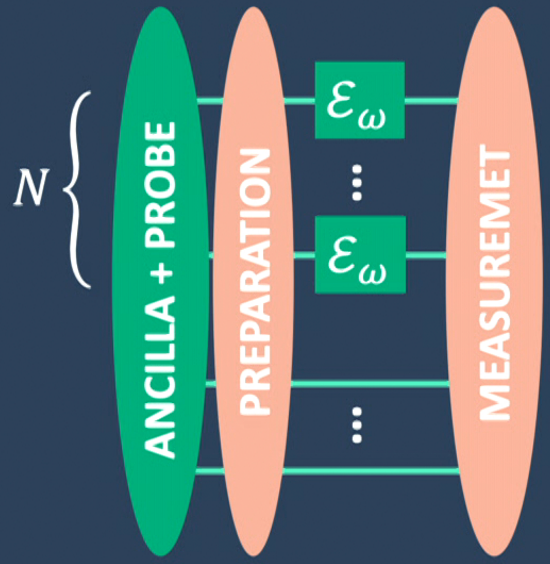
Asymptotic QFI of quantum channels

$$\mathcal{F}_N(\mathcal{E}_\omega) \le \min_h \; 4N\|\alpha\| + 4N(N-1)\|\beta\|^2.$$



Hamiltonian: $H = i\boldsymbol{K}^\dagger \dot{\boldsymbol{K}}$,

Kraus span: $\mathcal{S} = \mathrm{span}_{\mathrm{Herm}}\{K_i^\dagger K_j, \forall i, j\}$.

Theorem 1 (HNKS):

$$\mathcal{F}_N(\mathcal{E}_\omega) = \Theta(N^2) \text{ if and only if } H \notin \mathcal{S}.$$

SZ & Jiang. *In preparation.*

Asymptotic QFI of quantum channels

Theorem 2: When $H \notin \mathcal{S}$, $\mathcal{F}_{\mathrm{HL}}(\mathcal{E}_\omega) := \lim_{N\to\infty} \frac{\mathcal{F}_N(\mathcal{E}_\omega)}{N^2} = 4 \min_h \|\beta\|^2$,

There exists an input state $|\psi_N\rangle$ solvable via an SDP such that

$$\lim_{N\to\infty} F\left((\mathcal{E}_\omega^{\otimes N} \otimes I)(|\psi_N\rangle) \right) / N^2 = \mathcal{F}_{\mathrm{HL}}(\mathcal{E}_\omega).$$

Theorem 3: When $H \in \mathcal{S}$, $\mathcal{F}_{\mathrm{SQL}}(\mathcal{E}_\omega) = \lim_{N\to\infty} \frac{\mathcal{F}_N(\mathcal{E}_\omega)}{N} = 4 \min_{h:\beta=0} \|\alpha\|$,

$\forall \eta > 0$, there exists an input state $|\psi_{N,\eta}\rangle$ solvable via an SDP such that

$$\lim_{N\to\infty} F\left((\mathcal{E}_\omega^{\otimes N} \otimes I)(|\psi_{N,\eta}\rangle) \right) / N = \mathcal{F}_{\mathrm{SQL}}(\mathcal{E}_\omega) - \eta.$$

SZ & Jiang. *In preparation.*

Asymptotic QFI of quantum channels

- From the QFI upper bound $\mathcal{F}_N(\mathcal{E}_\omega) \le \min_h \; 4N\|\alpha\| + 4N(N-1)\|\beta\|^2$, we can obtain upper bounds $\mathcal{F}_{\mathrm{HL}}^{(u)}(\mathcal{E}_\omega) = 4\min_h \|\beta\|^2$ and $\mathcal{F}_{\mathrm{SQL}}^{(u)}(\mathcal{E}_\omega) = 4 \min_{h:\beta=0} \|\alpha\|$ for $\mathcal{F}_{\mathrm{HL}}(\mathcal{E}_\omega)$ and $\mathcal{F}_{\mathrm{SQL}}(\mathcal{E}_\omega)$. Theorem 2 and Theorem 3 shows that they are equal.
- Recall that the channel QFI $\mathcal{F}_1(\mathcal{E}_\omega) = 4\min_h \|\alpha\|$ and $\mathcal{F}_{\mathrm{SQL}}(\mathcal{E}_\omega) = 4 \min_{h:\beta=0} \|\alpha\|$. It shows that in general the channel QFI non-additive.
- For the sequential strategy, Theorem 1 and Theorem 3 still holds, because the SQL upper bound is the same as the parallel strategy. However, the HL upper bound is different. There is still possibility of improvement when $H \notin \mathcal{S}$.

SZ & Jiang. *In preparation.*

Sketch of the proof – Dephasing channels

(1) Theorem 2 and 3 are true for single-qubit dephasing channels:

$$\mathcal{D}_\omega = \mathcal{D}_p \circ \mathcal{U}_\phi,$$

where $\mathcal{U}_\phi(\cdot) = e^{-\frac{i\phi Z}{2}}(\cdot)e^{\frac{i\phi Z}{2}}$ and $\mathcal{D}_p(\cdot) = (1-p)(\cdot) + pZ(\cdot)Z$. Both ϕ and p are functions of ω. Let $\xi := (1-2p)e^{-i\phi}$,

$$\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \xrightarrow{\mathcal{D}_\omega} \begin{pmatrix} \rho_{00} & \xi\rho_{01} \\ \xi^*\rho_{10} & \rho_{11} \end{pmatrix}.$$

SZ & Jiang. *In preparation.*

Sketch of the proof – Dephasing channels

For dephasing channels, HNKS holds if and only if $p = 0$, i.e. there is no noise.

$$\text{When } p = 0, \quad \mathcal{F}_{\mathrm{HL}}(\mathcal{D}_\omega) = |\dot{\xi}|^2,$$

achievable using GHZ states.

$$\text{When } p > 0, \quad \mathcal{F}_{\mathrm{SQL}}(\mathcal{D}_\omega) = \frac{|\dot{\xi}|^2}{1-|\xi|^2},$$

achievable using spin-squeezed states.

SZ & Jiang. *In preparation.*

Sketch of the proof – QEC protocol

(2) Quantum error correction protocol. We find an encoding channel $\mathcal{E}_{\text{enc}}$ and a decoding channel $\mathcal{R}$ such that $\mathcal{R} \circ \mathcal{E}_\omega \circ \mathcal{E}_{\text{enc}} = \mathcal{D}_{L,\omega}$.

Encoding: $|0_L\rangle = \sum_{ij}^d A_{0,ij} |i\rangle_P \, |j,0\rangle_A$, $|1_L\rangle = \sum_{ij}^d A_{1,ij} |i\rangle_P \, |j,1\rangle_A$.

$\dim \mathcal{H}_P = d$. $\dim \mathcal{H}_A = 2d$. $A_0, A_1 \in \mathbb{C}^{d\times d}$.

Decoding: $\mathcal{R}(\cdot) = \sum_m T_m(\cdot)T_m^\dagger$,

where $T_m = |0_L\rangle\langle R_m, 0| + |1_L\rangle\langle Q_m, 1|$. $R = (|R_1\rangle \cdots |R_M\rangle)$, $Q = (|Q_1\rangle \cdots |Q_M\rangle)$. $RR^\dagger = QQ^\dagger = I$.

SZ & Jiang. *In preparation.*

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Sketch of the proof – QEC protocol

Let $|A\rangle\rangle = \sum_{ij} A_{ij} |i\rangle |j\rangle$.

$$E_0 = (|K_1 A_0\rangle\rangle \cdots |K_r A_0\rangle\rangle),$$

$$E_1 = (|K_1 A_1\rangle\rangle \cdots |K_r A_1\rangle\rangle),$$

$$T = QR^\dagger,\ \|T\| \le 1.$$

SZ & Jiang. *In preparation.*

Sketch of the proof – QEC protocol

(2) Quantum error correction protocol. We find an encoding channel $\mathcal{E}_{\text{enc}}$ and a decoding channel $\mathcal{R}$ such that $\mathcal{R} \circ \mathcal{E}_\omega \circ \mathcal{E}_{\text{enc}} = \mathcal{D}_{L,\omega}$.

Encoding: $|0_L\rangle = \sum_{ij}^{d} A_{0,ij} |i\rangle_P \, |j, 0\rangle_A$, $|1_L\rangle = \sum_{ij}^{d} A_{1,ij} |i\rangle_P \, |j, 1\rangle_A$.

$\dim \mathcal{H}_P = d$. $\dim \mathcal{H}_A = 2d$. $A_0, A_1 \in \mathbb{C}^{d\times d}$.

Decoding: $\mathcal{R}(\cdot) = \sum_m T_m(\cdot)T_m^\dagger$,

where $T_m = |0_L\rangle\langle R_m, 0| + |1_L\rangle\langle Q_m, 1|$. $R = (|R_1\rangle \cdots |R_M\rangle)$, $Q = (|Q_1\rangle \cdots |Q_M\rangle)$. $RR^\dagger = QQ^\dagger = I$.

SZ & Jiang. *In preparation.*

Sketch of the proof – QEC protocol

Let $|A\rangle\rangle = \sum_{ij} A_{ij} |i\rangle|j\rangle$.

$$E_0 = (|K_1 A_0\rangle\rangle \cdots |K_r A_0\rangle\rangle),$$

$$E_1 = (|K_1 A_1\rangle\rangle \cdots |K_r A_1\rangle\rangle),$$

$$T = QR^\dagger,\ \|T\| \leq 1.$$

$$\Rightarrow \quad \xi = \mathrm{Tr}\left(T E_0 E_1^\dagger\right).$$

SZ & Jiang. *In preparation.*

Sketch of the proof – HL

(3) Achieving the HL upper bound.

$$\exists T, \text{ s.t. } \xi = 1 \quad \text{if and only if} \quad PSP \propto P, \forall S \in \mathcal{S}$$

$$\tilde{C} = A_0 A_0^\dagger - A_1 A_1^\dagger,$$

$$PSP \propto P, \forall S \in \mathcal{S}, \;\Leftrightarrow\; \mathrm{Tr}\left(\tilde{C} S\right) = 0, \forall S \in \mathcal{S}.$$

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$$\dot{\xi} = -i\mathrm{Tr}(\tilde{C} H), \quad \mathcal{F}_{\mathrm{HL}}(\mathcal{D}_\omega) = |\dot{\xi}|^2.$$

The optimal code could be solved via the following SDP:

$$\text{maximize } \mathrm{Tr}(\tilde{C} H) \text{ subject to } \|\tilde{C}\|_1 \le 2 \text{ and } \mathrm{Tr}(\tilde{C} S), S \in \mathcal{S}.$$

The solution is $2\|H - \mathcal{S}\| = 2\min_h \|\beta\|$.

SZ & Jiang. *In preparation.*

Sketch of the proof – SQL

(4) Achieving the SQL upper bound.

We introduce the perturbation code:

$$A_0 = C + \epsilon D,\ A_1 = C - \epsilon D,\ T = e^{i\epsilon G}.$$

$$|\dot{\xi}|^2 = O(\epsilon^2)\,,\ \ 1 - |\xi|^2 = O(\epsilon^2),\ \ \mathcal{F}_{\mathrm{SQL}}(\mathcal{D}_{L,\omega}) = \frac{|\dot{\xi}|^2}{1-|\xi|^2}.$$

SZ & Jiang. *In preparation.*

Sketch of the proof – SQL

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First optimize $\mathcal{F}_{\mathrm{SQL}}(\mathcal{D}_{L,\omega})$ over G,

$$\max_G \mathcal{F}_{\mathrm{SQL}}(\mathcal{D}_{L,\omega}) = f(C, \tilde{C}),$$

$$\tilde{C} = CD^\dagger + DC^\dagger \propto A_0 A_0^\dagger - A_1 A_1^\dagger.$$

SZ & Jiang. *In preparation.*

Sketch of the proof – SQL

$$\max_{C,\tilde{C}} f(C,\tilde{C}) \Leftrightarrow \max_{C} \min_{h:\beta=0} 4\mathrm{Tr}\left(C^{\dagger}\alpha C\right) \Leftrightarrow 4 \min_{h:\beta=0} \|\alpha\|$$

SZ & Jiang. *In preparation.*

Sketch of the proof – SQL

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- $\tilde{C}$ is introduced as a the Lagrange multiplier associated with $\beta = 0$.

$$\min_{h:\beta=0} 4\mathrm{Tr}\left(C^{\dagger}\alpha C\right) = \max_{\tilde{C}} \min_{h} \ 4\mathrm{Tr}\left(C^{\dagger}\alpha C\right) + \mathrm{Tr}(\tilde{C}\beta).$$

SZ & Jiang. *In preparation.*

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- Using the minimax theorem,

$$\max_{C} \min_{h:\beta=0} 4\mathrm{Tr}\left(C^{\dagger}\alpha C\right) = \min_{h:\beta=0} \max_{C} 4\mathrm{Tr}\left(C^{\dagger}\alpha C\right).$$

SZ & Jiang. *In preparation.*

Conclusions and outlook

- We showed the achievability of the asymptotic QFI upper bounds for general quantum channels. The input state is solvable via SDPs.
- The multi-parameter HNLS criterion & code optimization[1]; removing the noiseless ancilla requirement for diagonal noise[2].
- Future directions: difference between the parallel strategy and the sequential strategy when HNKS holds; generalization to infinite dimensional systems; reduction of the size of the ancillary system.

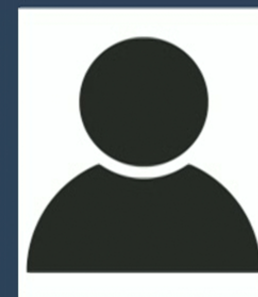
[1] Gorecki & SZ *et al. arXiv*:1901.00896 (2019).
[2] Layden & SZ *et al. PRL* 122, 040502 (2019).

Acknowledgements

Liang Jiang

Mengzhen Zhang

David Layden



Wojciech Górecki

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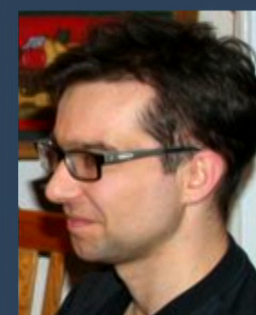

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