Title: PSI 2019/2020 - QFT III - Lecture 9

Speakers:

Collection: PSI 2019/2020 - QFT III

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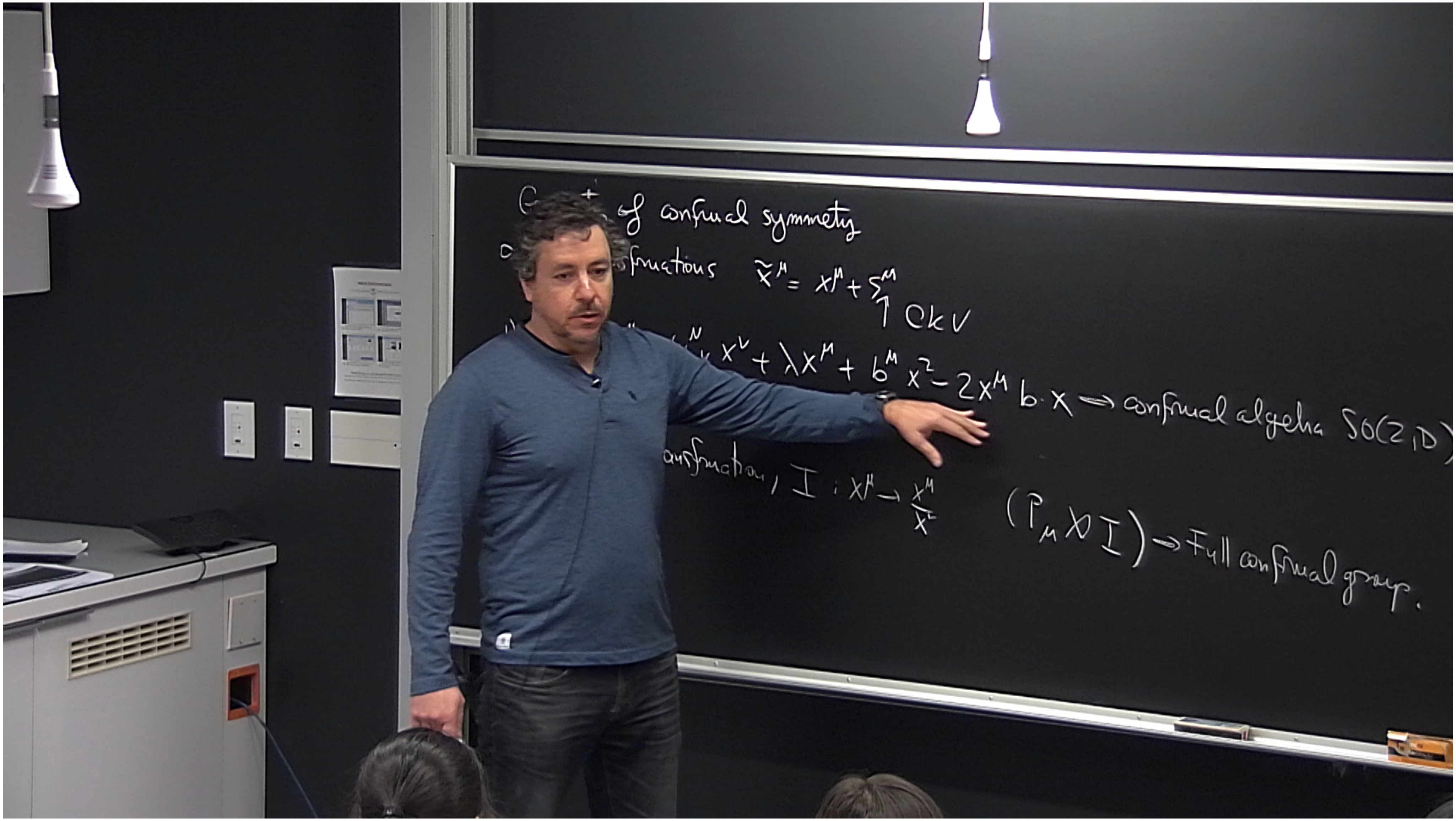

of conformal symmetry
transformations $\tilde{x}^\mu = x^\mu + \xi^\mu$
CKV
$x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu b \cdot x$ ⟹ conformal algebra SO(2,D)
transformation, $I : x^\mu \to \frac{x^\mu}{x^2}$
$(P_\mu \otimes I)$ ⟹ Full conformal group.

Geometry of conformal symmetry

∞'al transformations $\tilde{x}^\mu = x^\mu + \xi^\mu$

$\uparrow$ CKV

1) $\xi^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu b\cdot x \Rightarrow$ conformal algebra $SO(2,D)$

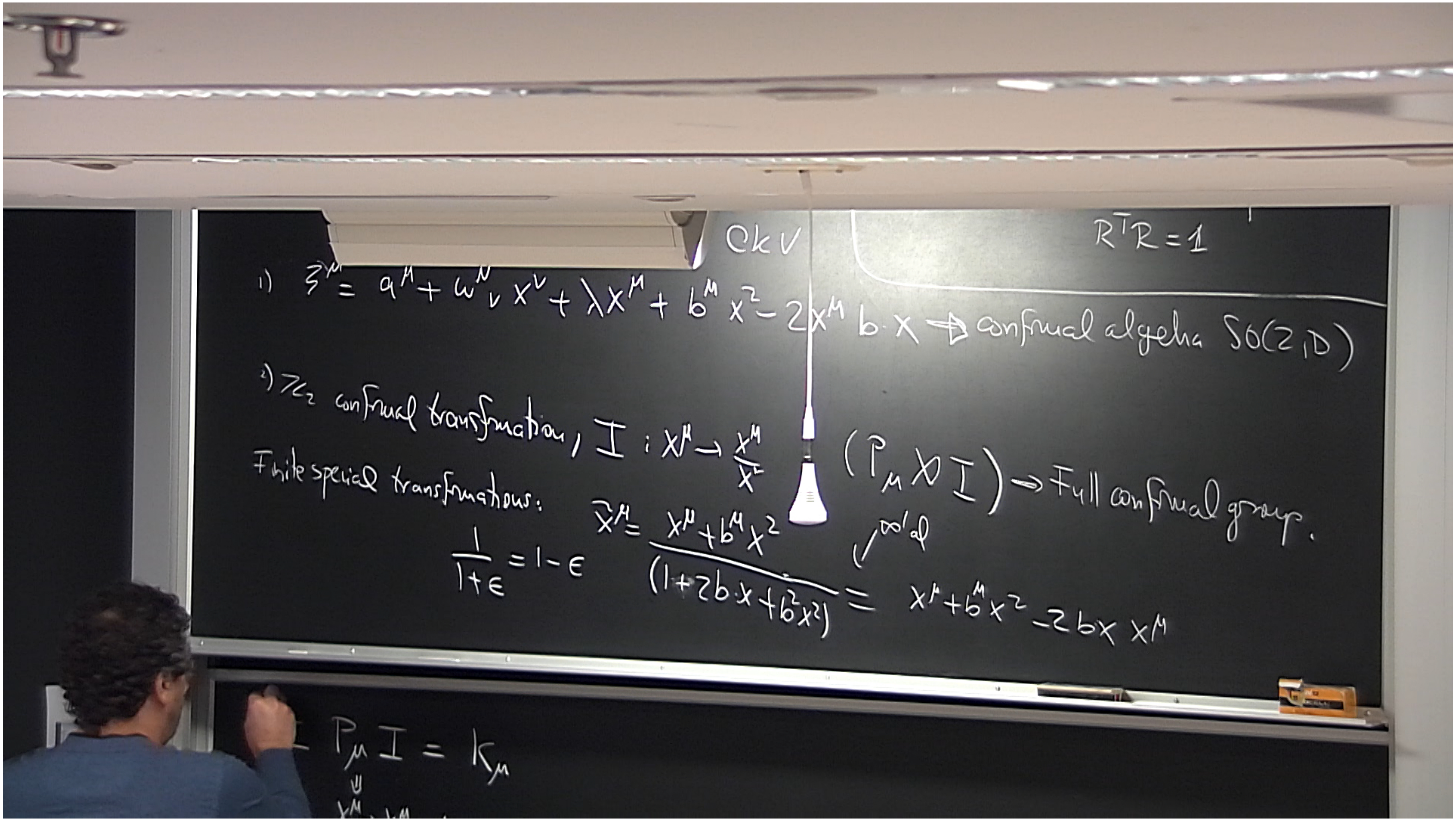
2) $\mathbb{Z}_2$ conformal transformation, $I : x^\mu \to \frac{x^\mu}{x^2}$

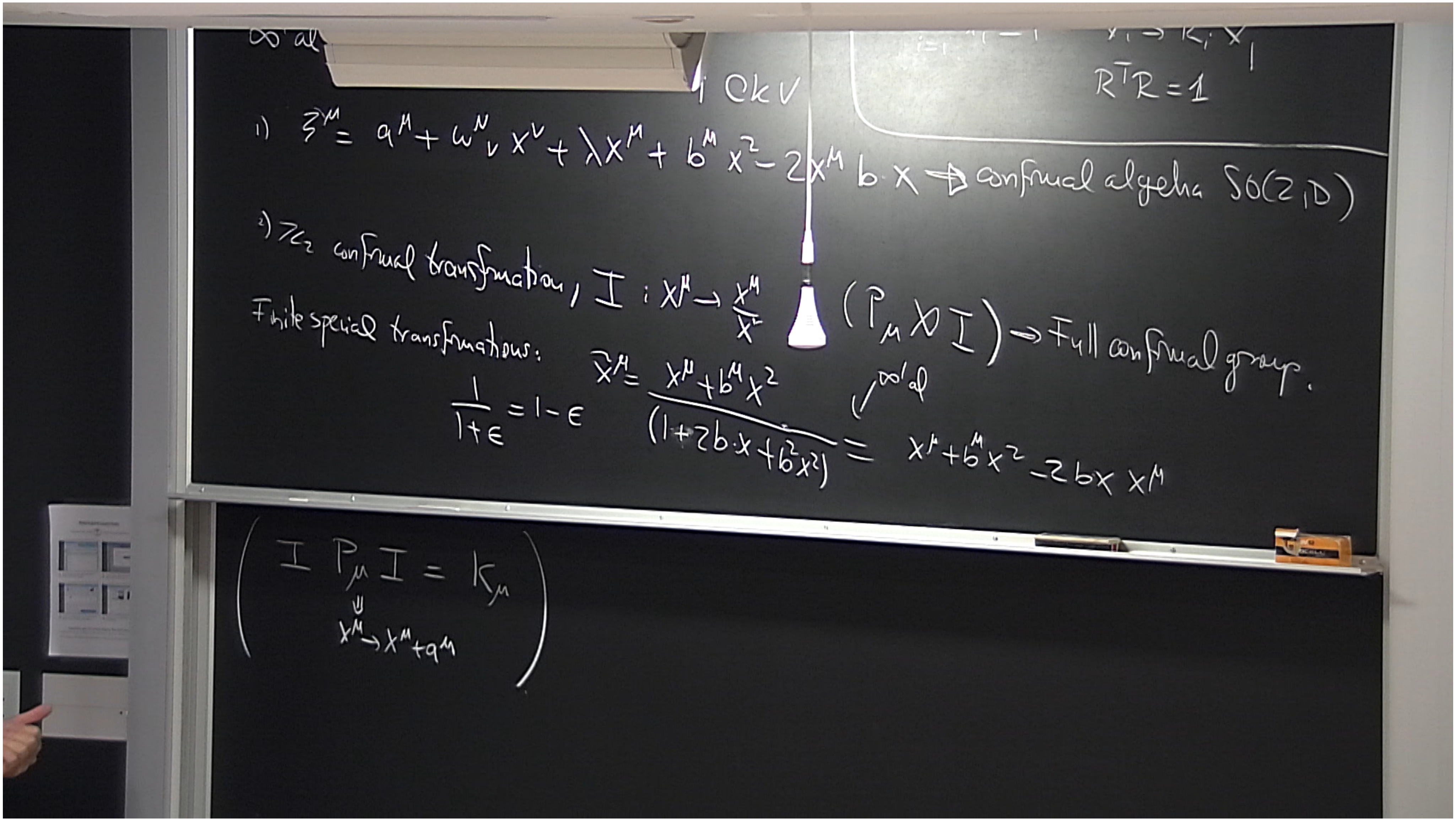
$(P_\mu \rtimes I) \Rightarrow$ Full conformal group.

S^2 $SO(3)$ symmetry

$\sum_{i=1}^{3} x_i^2 = 1$ $x_i \to R_i{}^j x_j$

$R^T R = \mathbb{1}$


CKV
$R^T R = 1$
1) $\xi^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2 x^\mu b \cdot x$ ⇒ conformal algebra $SO(2,D)$
2) $\mathbb{Z}_2$ conformal transformation, $I : x^\mu \to \frac{x^\mu}{x^2}$
$(P_\mu \rtimes I) \Rightarrow$ Full conformal group.
Finite special transformations:
$\tilde{x}^\mu = \frac{x^\mu + b^\mu x^2}{(1 + 2 b \cdot x + b^2 x^2)} = x^\mu + b^\mu x^2 - 2 b x \, x^\mu$
$\frac{1}{1+\epsilon} = 1 - \epsilon$
$I P_\mu I = K_\mu$


CkV
$R^T R = 1$
1) $\xi^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu b\cdot x$ ⇒ conformal algebra $SO(2,D)$
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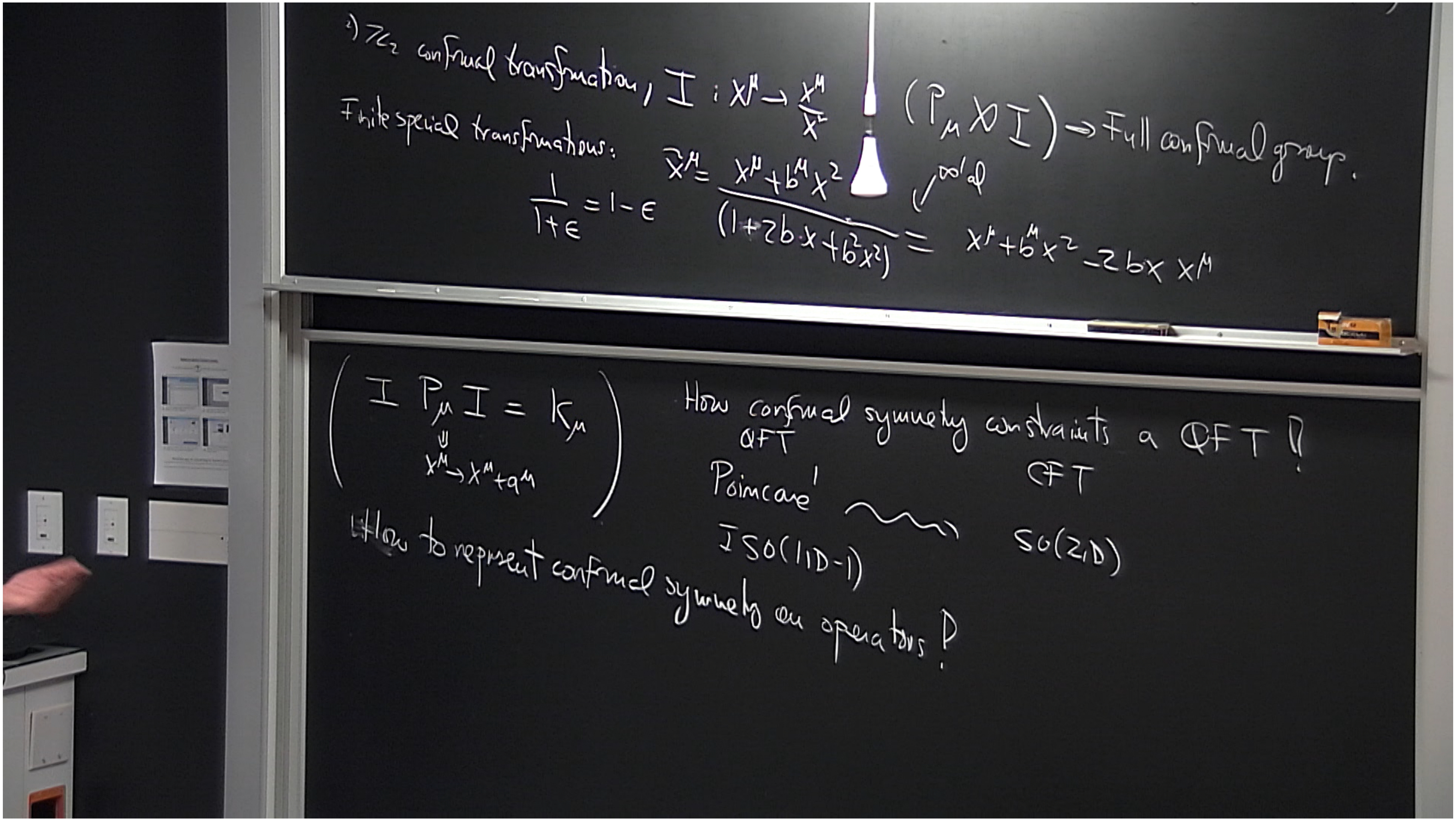
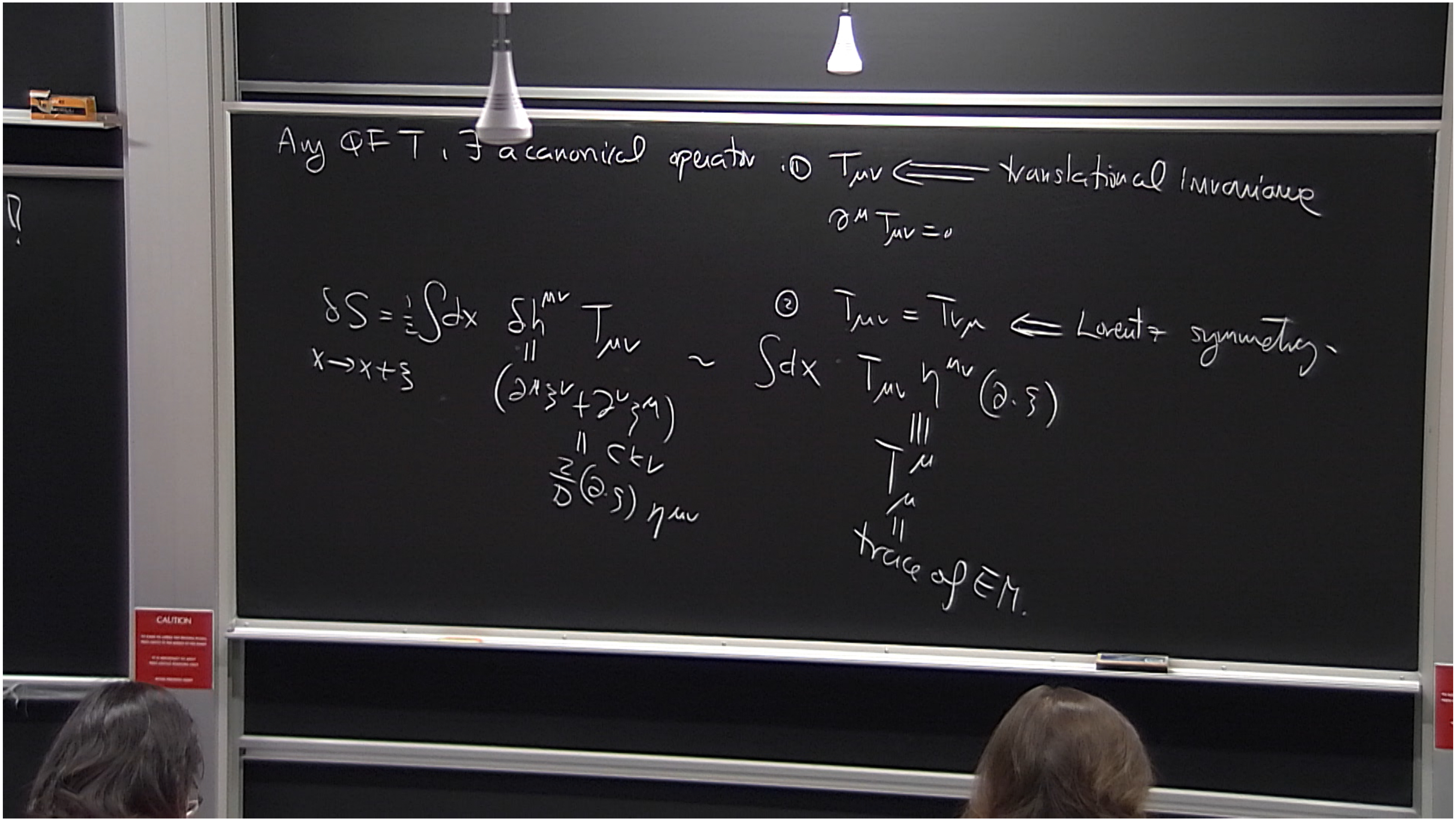
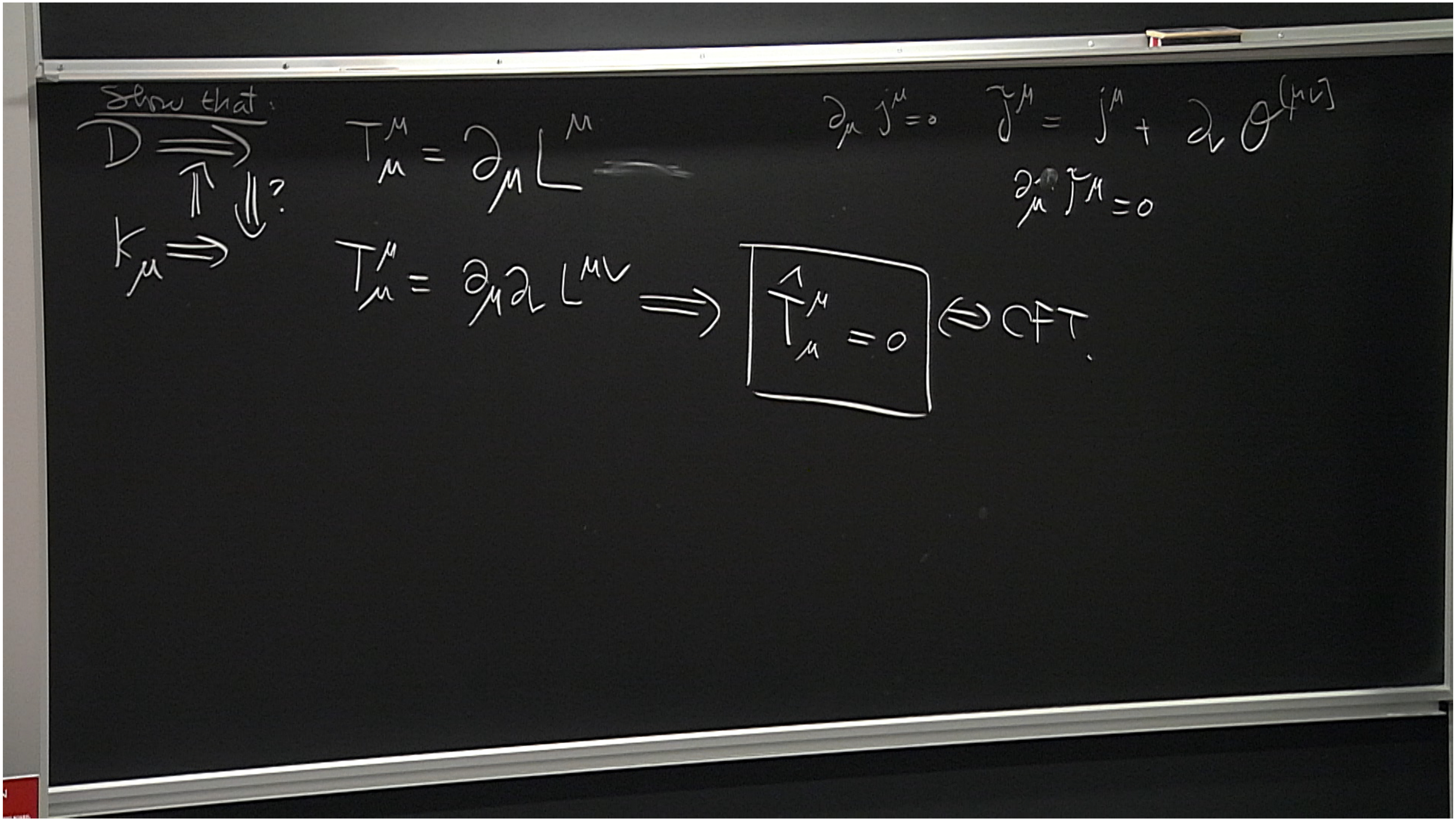
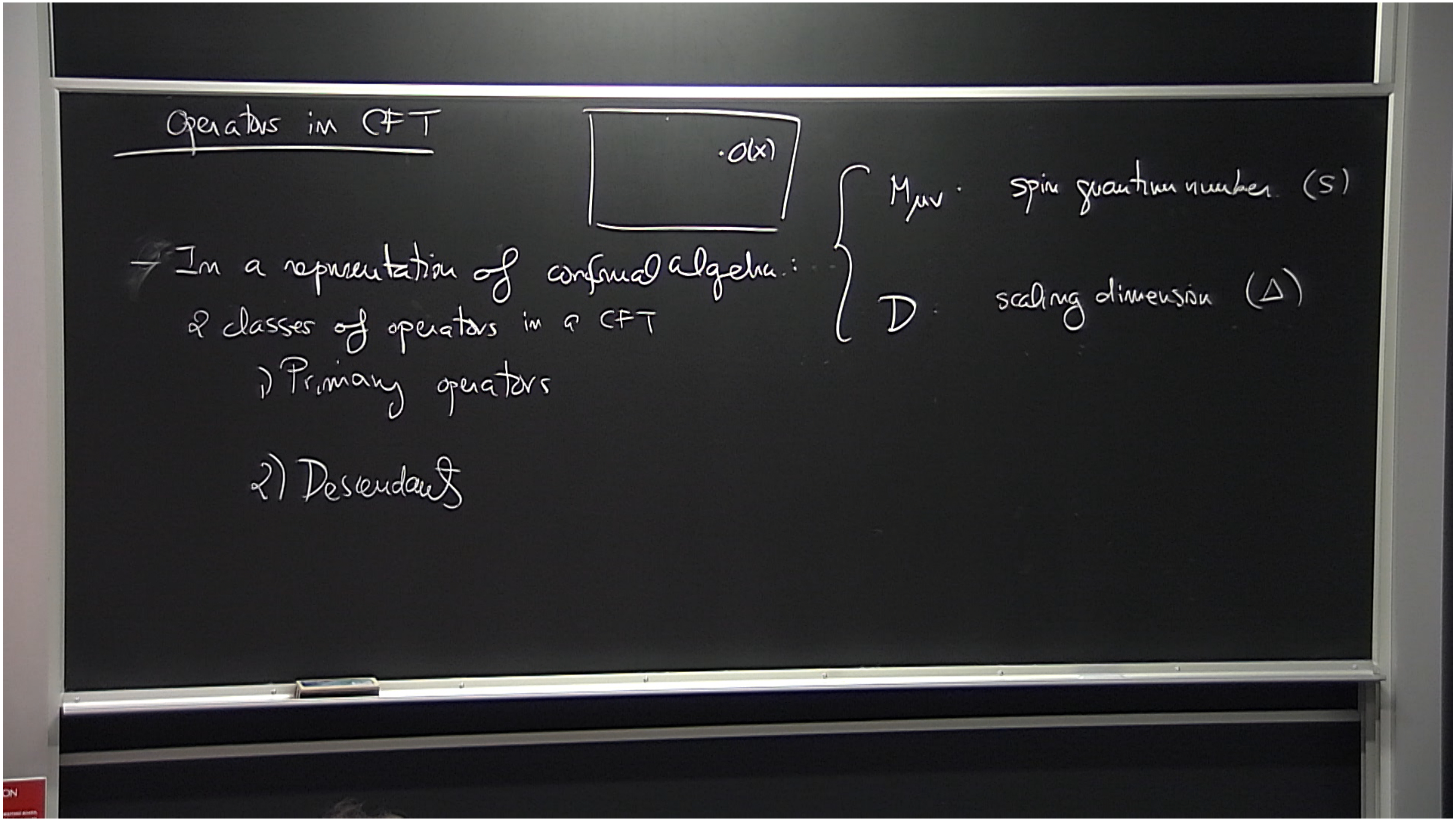
Finite special transformations:
$\frac{1}{1+\epsilon} = 1 - \epsilon$
$\tilde{x}^\mu = \frac{x^\mu + b^\mu x^2}{(1 + 2b\cdot x + b^2 x^2)} = x^\mu + b^\mu x^2 - 2b x\, x^\mu$
∞'al
$I P_\mu I = K_\mu$
$x^\mu \to x^\mu + a^\mu$

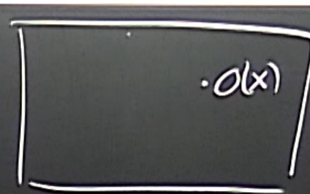

2) Z2 conformal transformation, I : x^μ → x^μ/x^2
(P_μ ⋊ I) ⇒ Full conformal group.
Finite special transformations: x̃^μ = (x^μ + b^μ x^2)/(1 + 2b·x + b^2x^2) = x^μ + b^μ x^2 − 2b·x x^μ
∞'al
1/(1+ε) = 1 − ε
(I P_μ I = K_μ)
x^μ → x^μ + a^μ
How conformal symmetry constraints a QFT?
QFT
CFT
Poincare'
ISO(1,D-1)
SO(2,D)
How to represent conformal symmetry on operators?


Any QFT, ∃ a canonical operator: ① Tμν ⟸ translational invariance
∂μ Tμν = 0
δS = ½ ∫dx δhμν Tμν
x → x + ξ
(∂μξν + ∂νξμ)
CKV
2/D (∂·ξ) ημν
~ ∫dx Tμν ημν (∂·ξ)
Tμμ
trace of EM.
② Tμν = Tνμ ⟸ Lorentz symmetry.


Show that:
D ⟹
⇑ ⇓?
K_μ ⟹
T^μ_μ = ∂_μ L^μ
T^μ_μ = ∂_μ∂_ν L^{μν} ⟹ T̂^μ_μ = 0 ⟸ CFT.
∂_μ j^μ = 0   j̃^μ = j^μ + ∂_ν θ^{[μν]}
∂_μ j̃^μ = 0


Operators in CFT
O(x)
→ In a representation of conformal algebra:
$M_{\mu\nu}$: spin quantum number (S)
D : scaling dimension (Δ)
2 classes of operators in a CFT
1) Primary operators
2) Descendants

Operators in CFT



→ In a representation of conformal algebra:

- $M_{\mu\nu}$: spin quantum number (S)
- D : scaling dimension (Δ)

2 classes of operators in a CFT

1) Primary operators. transforms as tensors under a conformal transformations. (linear representation).

2) Descendants.

SU(2) angular momentum spin j $(2j+1)$ state

$|j,m\rangle$ $m=-j,\dots,j$ J_+, J_-, J_3

$|j,j\rangle$ primary $\downarrow J_-$

$|j,m\rangle$ descendant

$1+\epsilon \quad (1+2b\cdot x+b^2x^2) = x^\mu + b^\mu x^2 - 2b\cdot x\, x^\mu$

$[D,\mathcal{O}] = \Delta\,\mathcal{O}$

$[D,[P_\mu,\mathcal{O}]] = (\Delta+1)\,[P_\mu,\mathcal{O}] \quad \uparrow$

$[D,[K_\mu,\mathcal{O}]] = (\Delta-1)\,[K_\mu,\mathcal{O}] \quad \downarrow$

Primary operator $[K_\mu,\mathcal{O}] = 0 \iff J_+|j,j\rangle = 0$

highest weight vector

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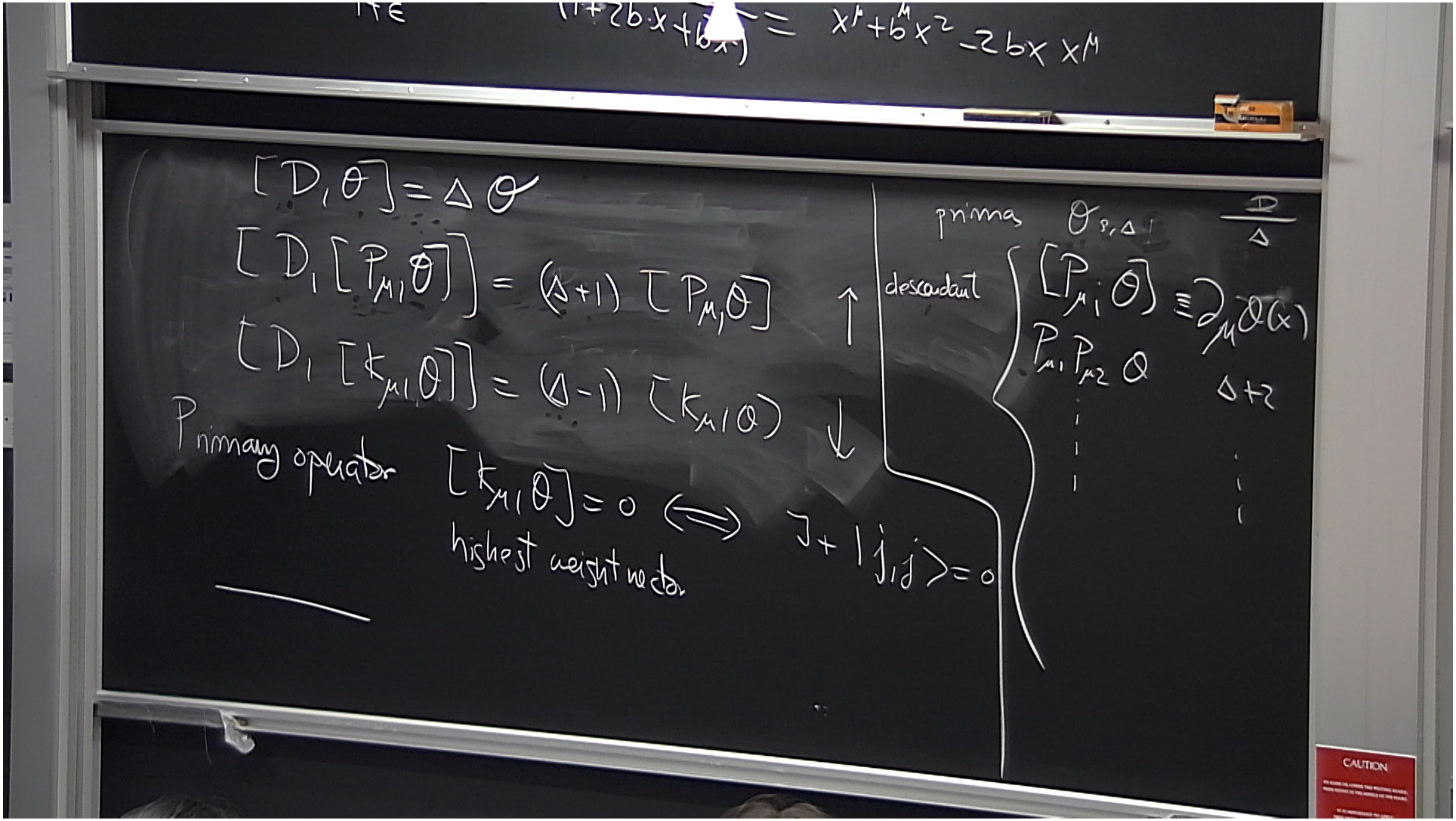
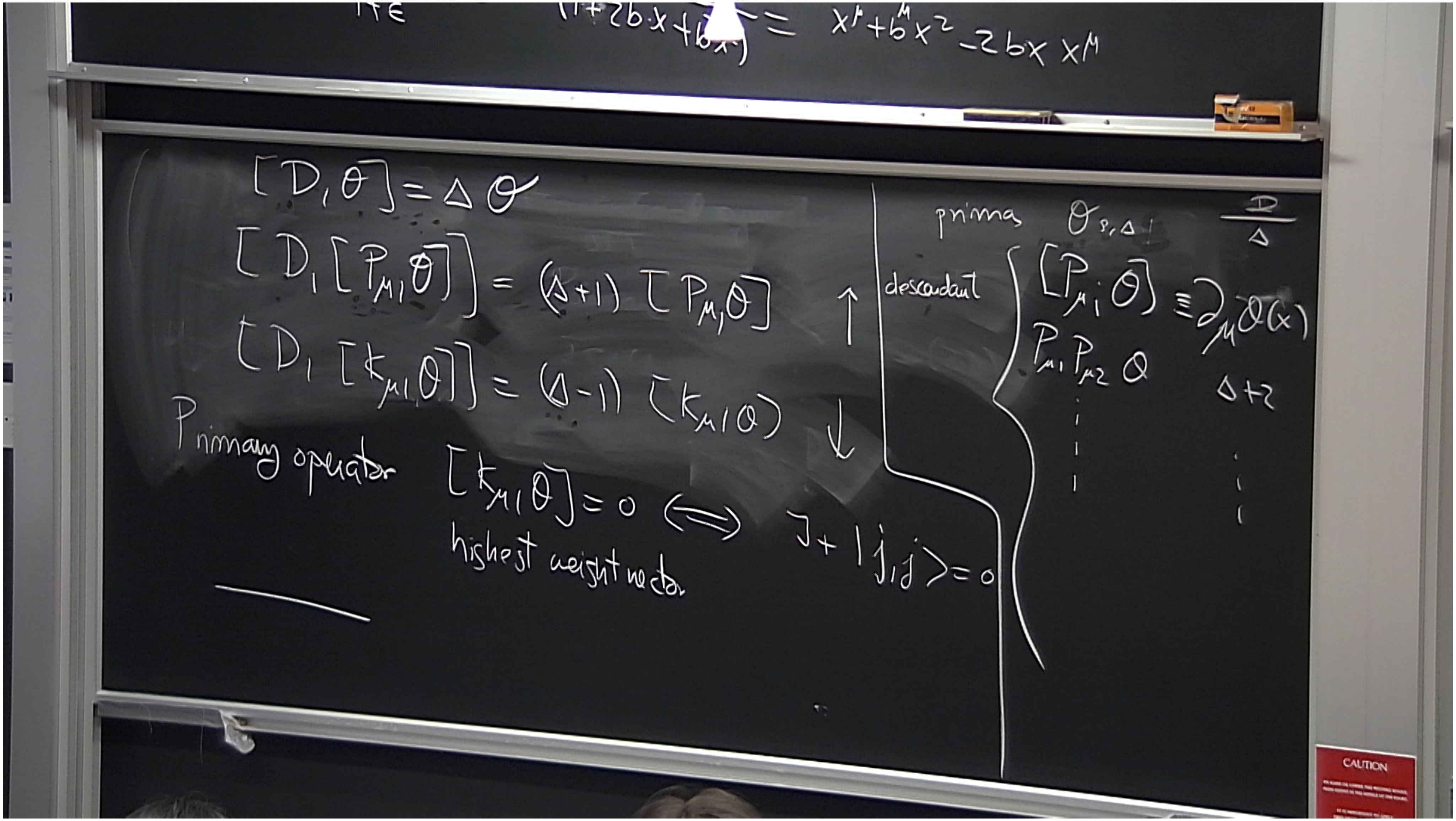
primary $\mathcal{O}_{s,\Delta}$ — Δ

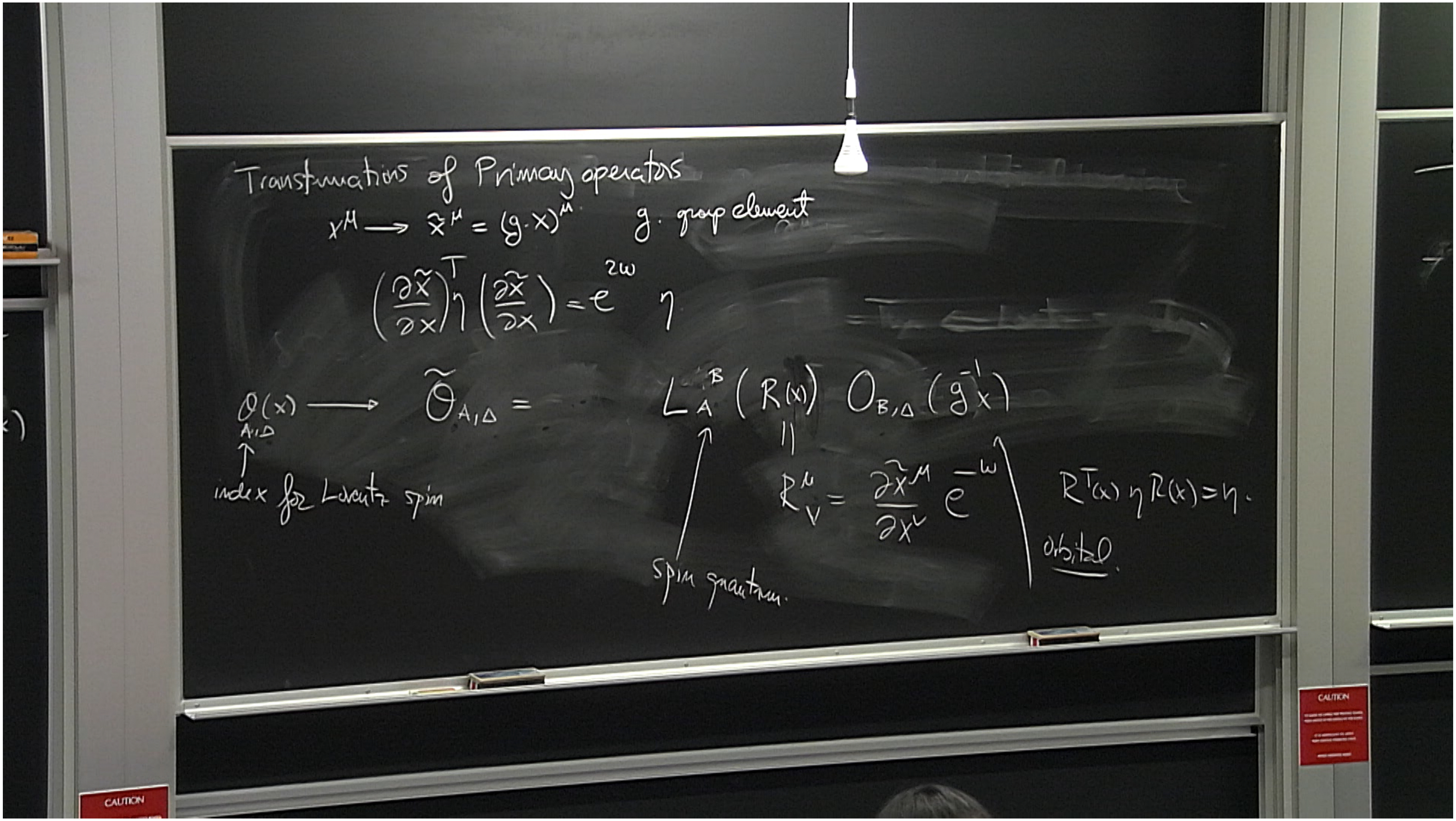
descendant: $[P_\mu,\mathcal{O}] =$ — $\Delta+1$

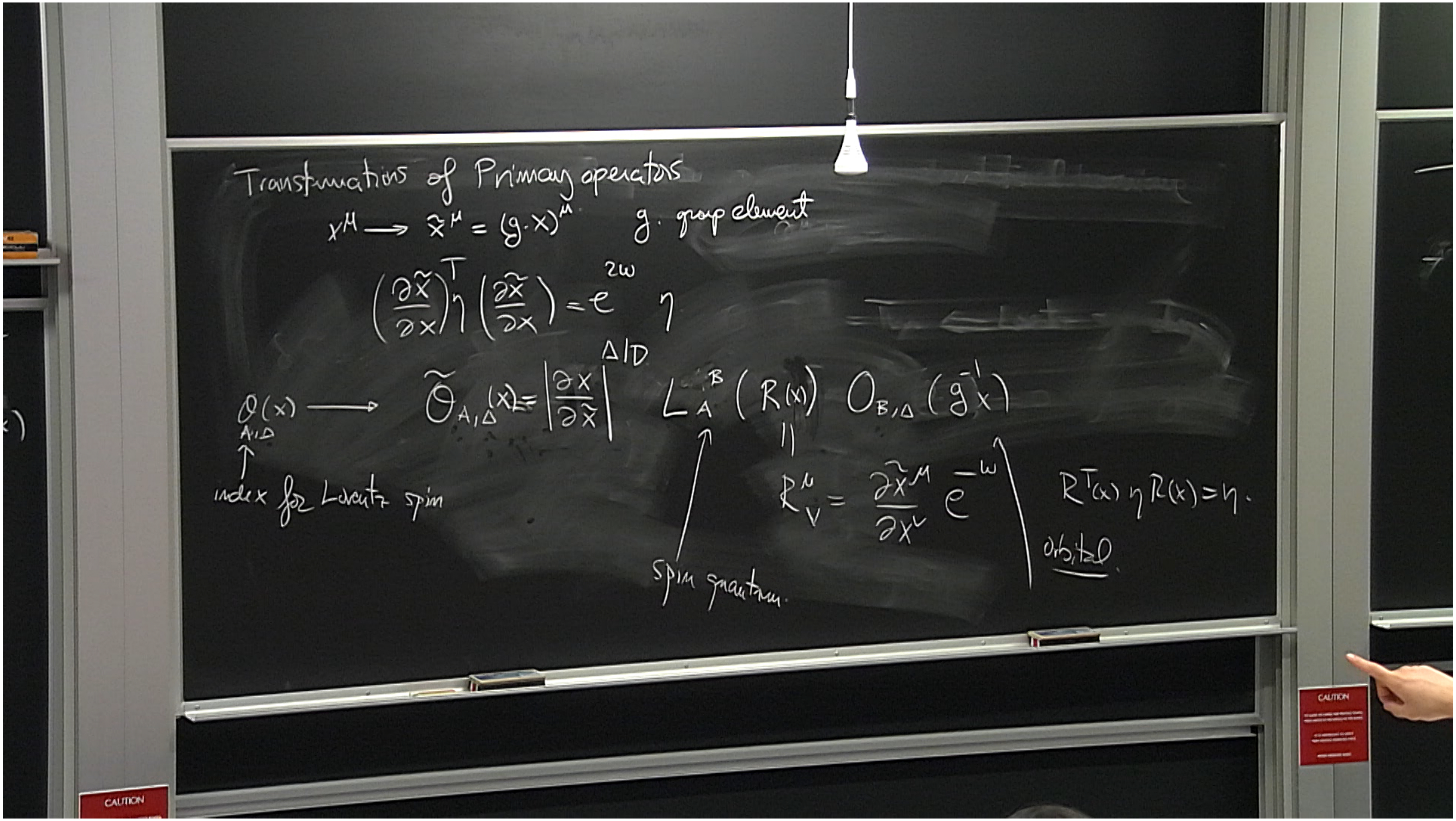
$P_{\mu_1}P_{\mu_2}\mathcal{O}$ — $\Delta+2$

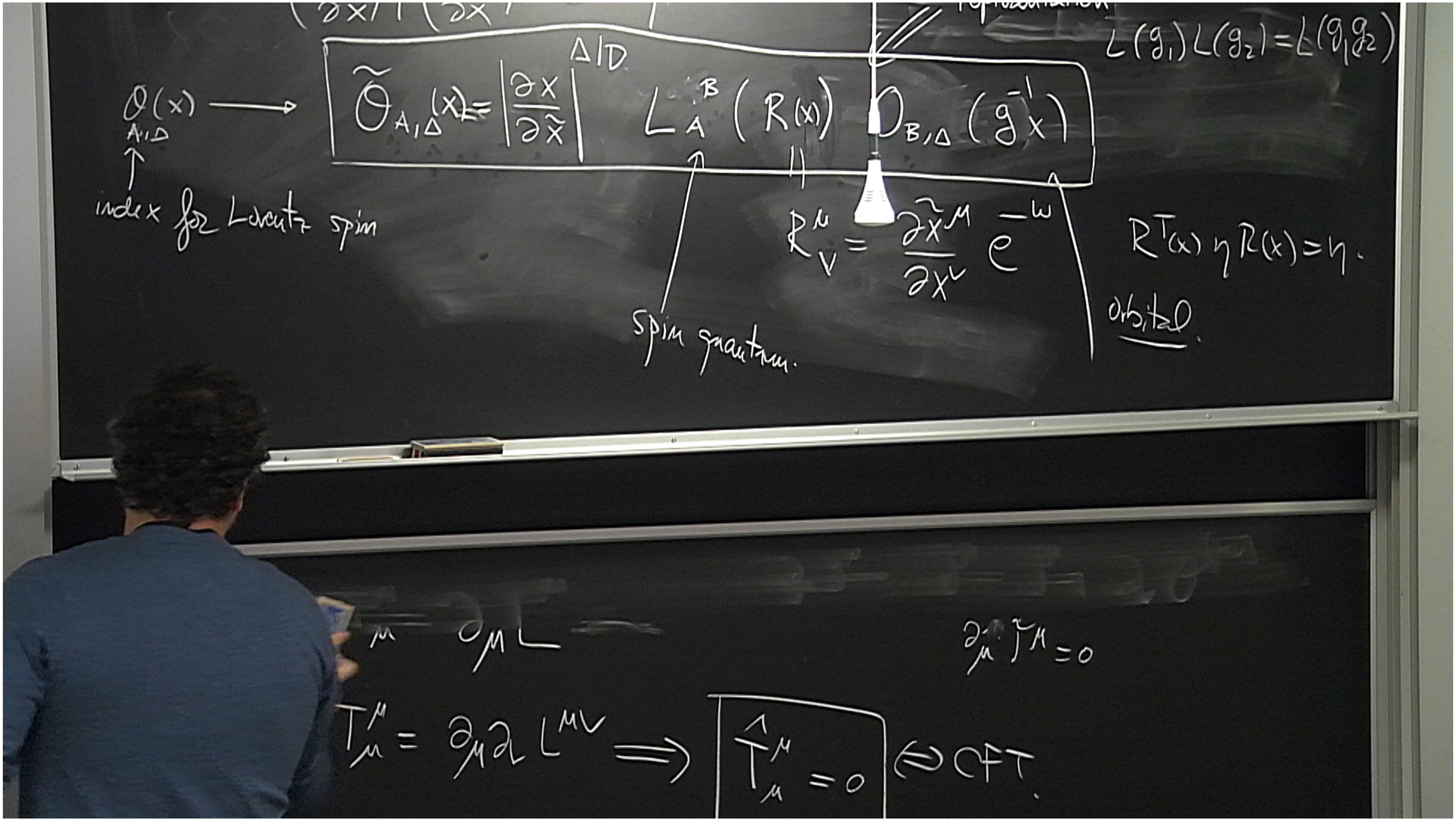
$\vdots$ — $\vdots$

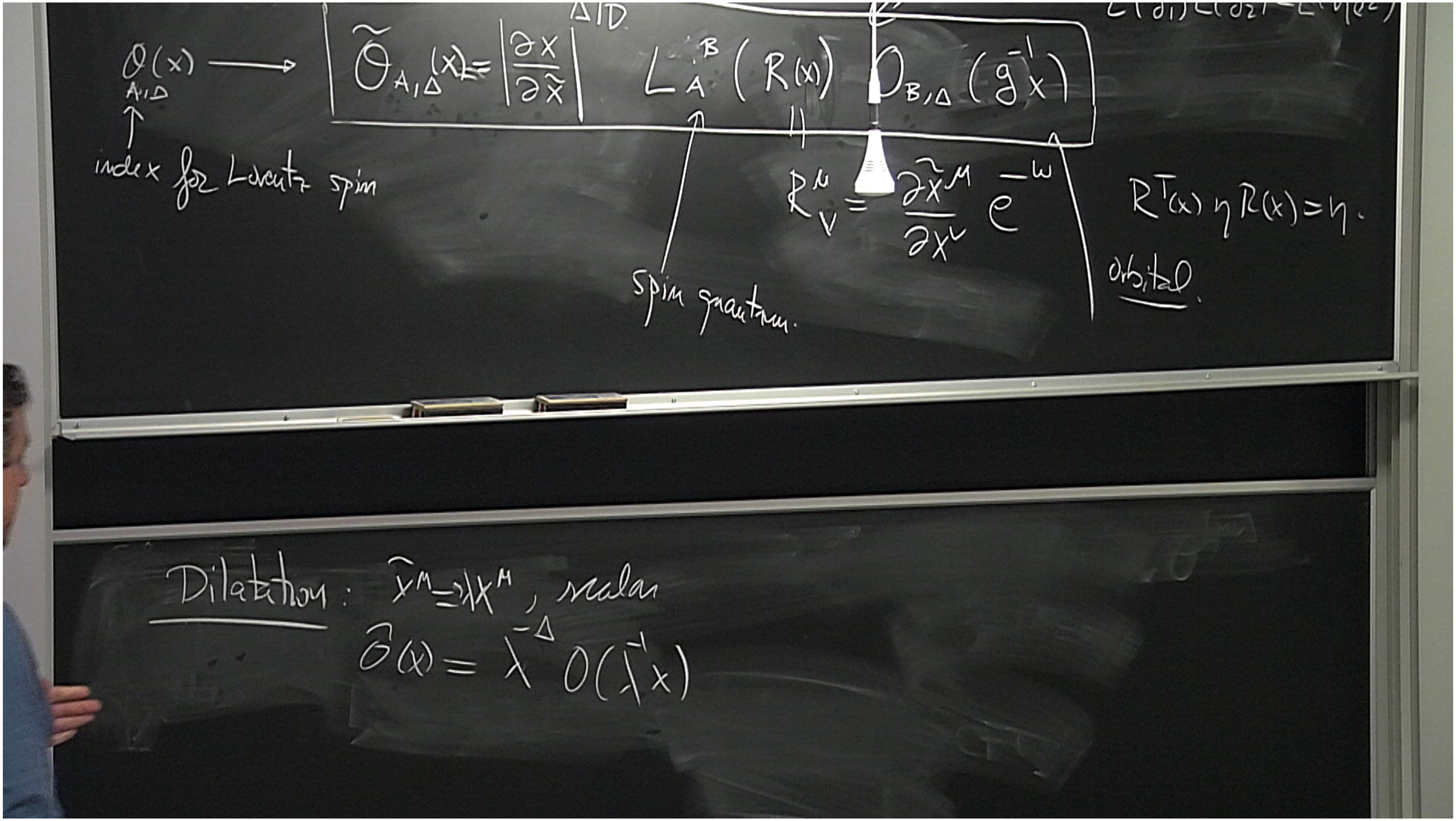

[D, O] = Δ O
[D, [P_μ, O]] = (Δ+1) [P_μ, O]
[D, [K_μ, O]] = (Δ-1) [K_μ, O]
Primary operator [K_μ, O] = 0 ⟺ J+ |j, j> = 0
highest weight vector
primary O_{s,Δ}
descendant
[P_μ, O] ≡ ∂_μ O(x)
P_μ1 P_μ2 O
D
Δ
Δ+2


$[D, \mathcal{O}] = \Delta \mathcal{O}$
$[D, [P_\mu, \mathcal{O}]] = (\Delta+1)[P_\mu, \mathcal{O}]$
$[D, [K_\mu, \mathcal{O}]] = (\Delta-1)[K_\mu, \mathcal{O}]$
Primary operator $[K_\mu, \mathcal{O}] = 0 \iff J_+ |j, j\rangle = 0$
highest weight vector
primary $\mathcal{O}_{s,\Delta}$
descendant
$[P_\mu, \mathcal{O}] \equiv \partial_\mu \mathcal{O}(x)$
$P_{\mu_1} P_{\mu_2} \mathcal{O}$
D
Δ
$\Delta+2$


Transformations of Primary operators
$x^\mu \longrightarrow \tilde{x}^\mu = (g \cdot x)^\mu$ g. group element
$\left(\frac{\partial \tilde{x}}{\partial x}\right)^T \eta \left(\frac{\partial \tilde{x}}{\partial x}\right) = e^{2\omega} \eta$
$\mathcal{O}_{A,\Delta}(x) \longrightarrow \tilde{\mathcal{O}}_{A,\Delta} = L_A^{\ B}(R(x))\, \mathcal{O}_{B,\Delta}(g^{-1}x)$
index for Lorentz spin
$R^\mu_{\ \nu} = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} e^{-\omega}$
$R^T(x)\,\eta\, R(x) = \eta$.
orbital.
spin quantum.


Transformations of Primary operators
x^μ → x̃^μ = (g·x)^μ   g. group element
(∂x̃/∂x)^T η (∂x̃/∂x) = e^{2ω} η
O_{A,Δ}(x) → Õ_{A,Δ}(x̃) = |∂x/∂x̃|^{Δ/D} L_A^B (R(x)) O_{B,Δ}(g^{-1}x)
index for Lorentz spin
spin quantum
R^μ_ν = ∂x̃^μ/∂x^ν e^{-ω}
R^T(x) η R(x) = η.
orbital


L(g1) L(g2) = L(g1g2)
O(x) A,Δ → Õ A,Δ (x̃) = |∂x/∂x̃|^Δ/D L_A^B (R(x)) O_B,Δ (g⁻¹x)
index for Lorentz spin
R^μ_ν = ∂x̃^μ/∂x^ν e^−ω
R^T(x) η R(x) = η.
spin quantum.
orbital
∂_μ J^μ = 0
T^μ_μ = ∂_μ ∂_ν L^μν ⟹ T^μ_μ = 0 ⟺ CFT.


index for Lorentz spin
spin quantum.
Orbital.
Dilatation : , scalar

index for Lorentz spin

spin quantum.

$$R^\mu{}_\nu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\, e^{-\omega}$$

$R^T(x)\,\eta\, R(x) = \eta$.

orbital.

Dilatation: $\tilde{x}^\mu = \lambda x^\mu$, scalar

$$\tilde{O}(x) = \lambda^{-\Delta}\, O(\lambda^{-1} x)$$

$$\begin{cases} \Omega_{\mu\nu}(x) = \omega_{\mu\nu} + 2(x_\mu b_\nu - x_\nu b_\mu) \\ \omega(x) = \lambda - 2x\cdot b \end{cases}$$

Under ∞'al transf: $\tilde{x}^\mu = x^\mu + \xi^\mu$

orbital — representation of Lorentz algebra.

$$\delta O_{A,\Delta}(x) = \tilde{O}_{A,\Delta}(x) - O_{A,\Delta}(x) = -\xi^\mu \partial_\mu O_{A,\Delta}(x) + \frac{1}{2}\Omega_{\mu\nu}(x)\left(M^{\mu\nu}_R\right)_A{}^B O_B(x)$$

$$-\Delta\,\omega(x)\, O_{A,\Delta}(x)$$

$R^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}\, e$ … $R^{T}(x)\,\eta\, R(x) = \eta$.

spin quantum. | orbital.

Dilatation: $\tilde{x}^{\mu} = \lambda x^{\mu}$, scalar

$$\tilde{O}(x) = \lambda^{-\Delta}\, O(\lambda^{-1} x)$$

$$\begin{cases} \Omega_{\mu\nu}(x) = \omega_{\mu\nu} + 2(x_{\mu} b_{\nu} - x_{\nu} b_{\mu}) \\ \omega(x) = \lambda - 2 x \cdot b \end{cases}$$

Under ∞'al transf: $\tilde{x}^{\mu} = x^{\mu} + \xi^{\mu}$

orbital — representation of Lorentz algebra.

$$\delta O_{A,\Delta}(x) = \tilde{O}_{A,\Delta}(x) - O_{A,\Delta}(x) = -\xi^{\mu}\partial_{\mu} O_{A,\Delta}(x) + \frac{1}{2}\Omega_{\mu\nu}(x)\,(M^{\mu\nu}_{R})_{A}{}^{B}\, O_{B}(x)$$

$$-\Delta\,\omega(x)\, O_{A,\Delta}(x)$$

representation of D

δ_1, δ_2 are CKV

$$[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{\xi_3} \leftarrow \text{CKV}$$

$$[\delta_{\xi_1}, \delta_{\xi_2}]\, \Theta_{A,\Delta} = \delta_{\xi_3} \cdot \Theta_{A,\Delta}$$

Dilatation: $\tilde{x}^\mu = \lambda x^\mu$, scalar

$$\tilde{O}(x) = \lambda^{-\Delta}\, O(\lambda^{-1} x)$$

$$\begin{cases} \Omega_{\mu\nu}(x) = \omega_{\mu\nu} + 2(x_\mu b_\nu - x_\nu b_\mu) \\ \omega(x) = \lambda - 2x\cdot b \end{cases}$$

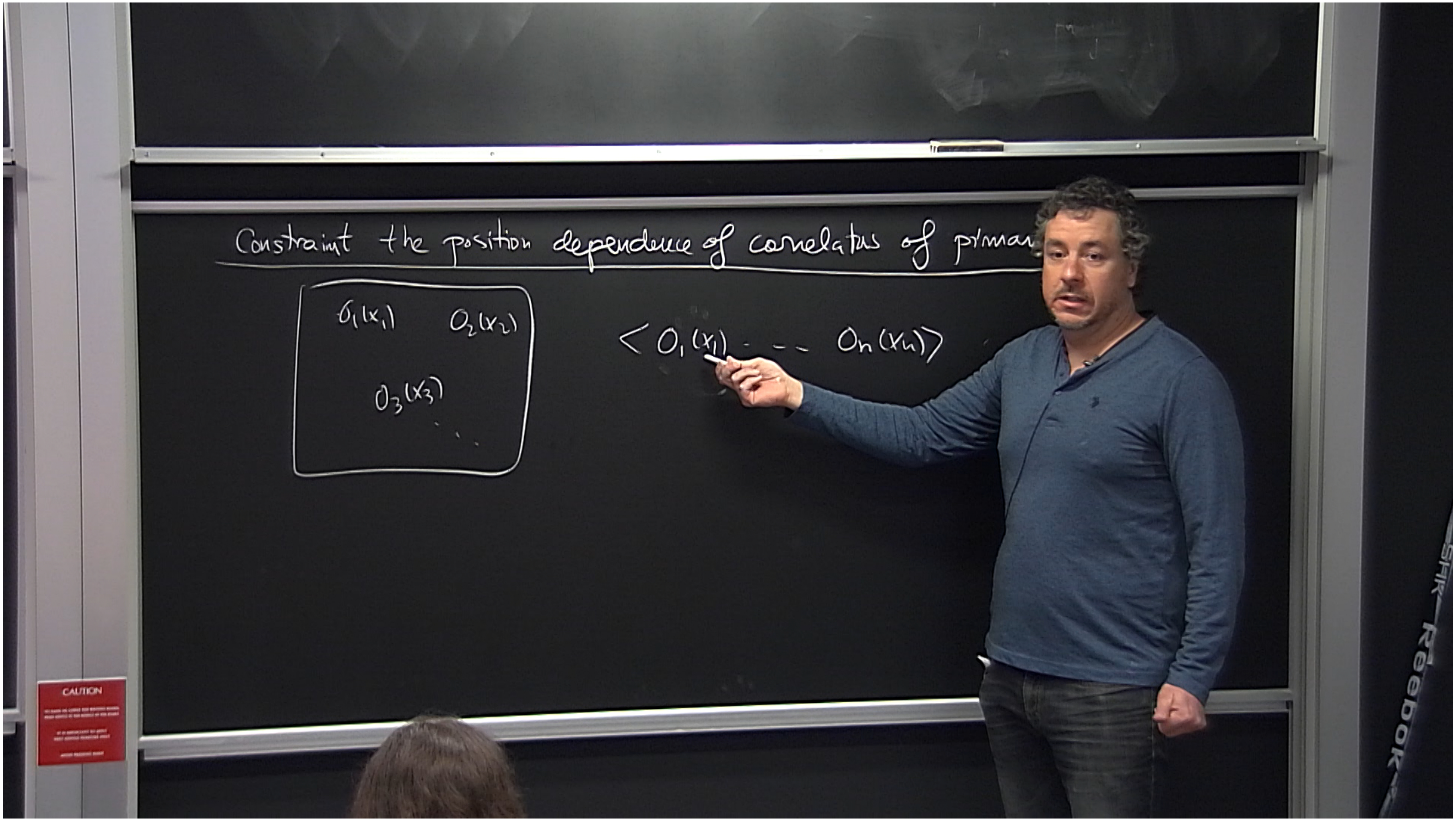
Under ∞'al transf: $\tilde{x}^\mu = x^\mu + \xi^\mu$

$$\delta\Theta_{A,\Delta}(x) = \tilde{\Theta}_{A,\Delta}(x) - \Theta_{A,\Delta}(x) = -\xi^\mu \partial_\mu \Theta_{A,\Delta}(x) + \frac{1}{2}\Omega_{\mu\nu}(x)\left(M^{\mu\nu}_R\right)_A{}^B\, \Theta_{B,\Delta}(x) - \Delta\, \omega(x)\, \Theta_{A,\Delta}(x)$$

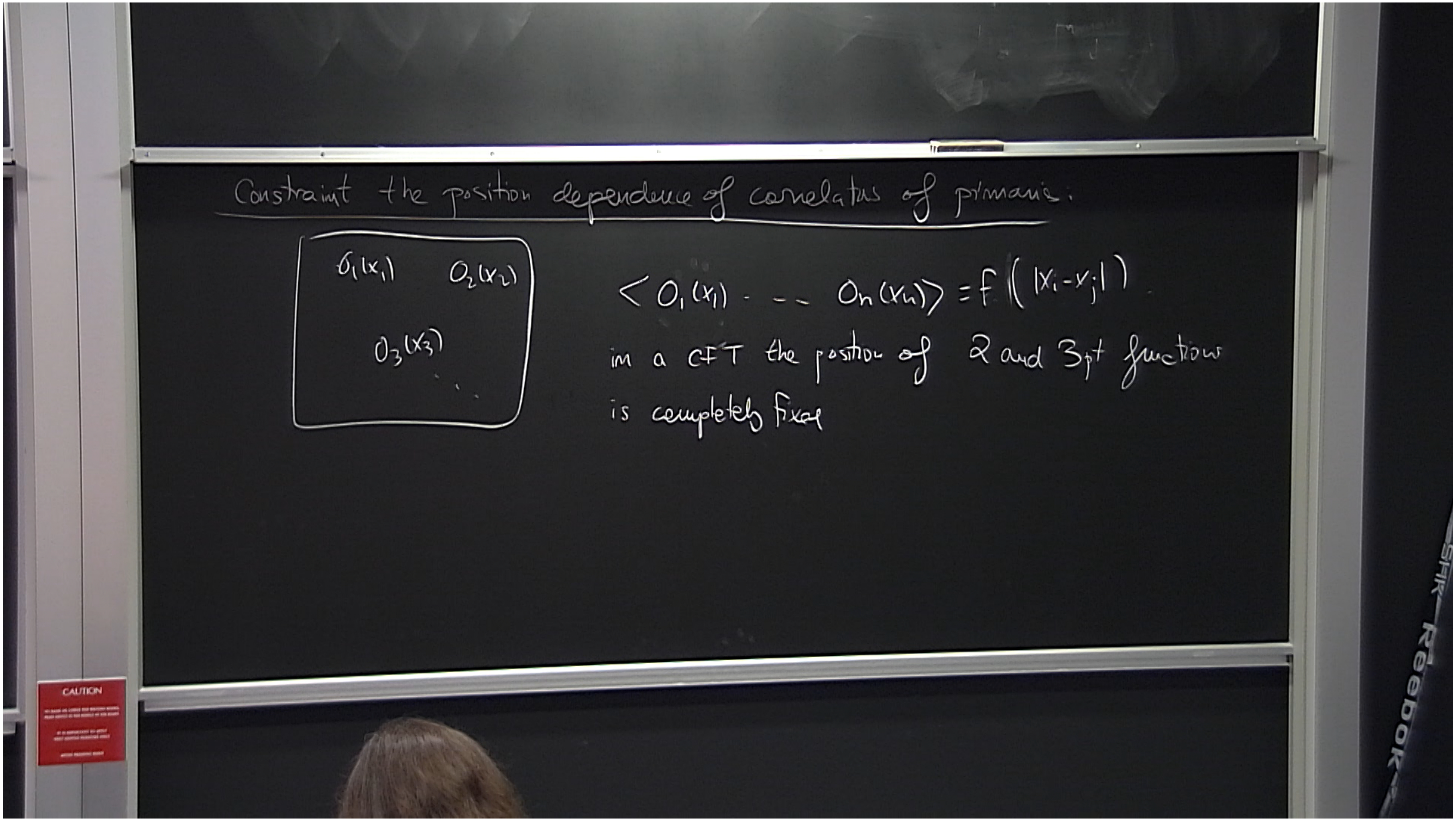
orbital

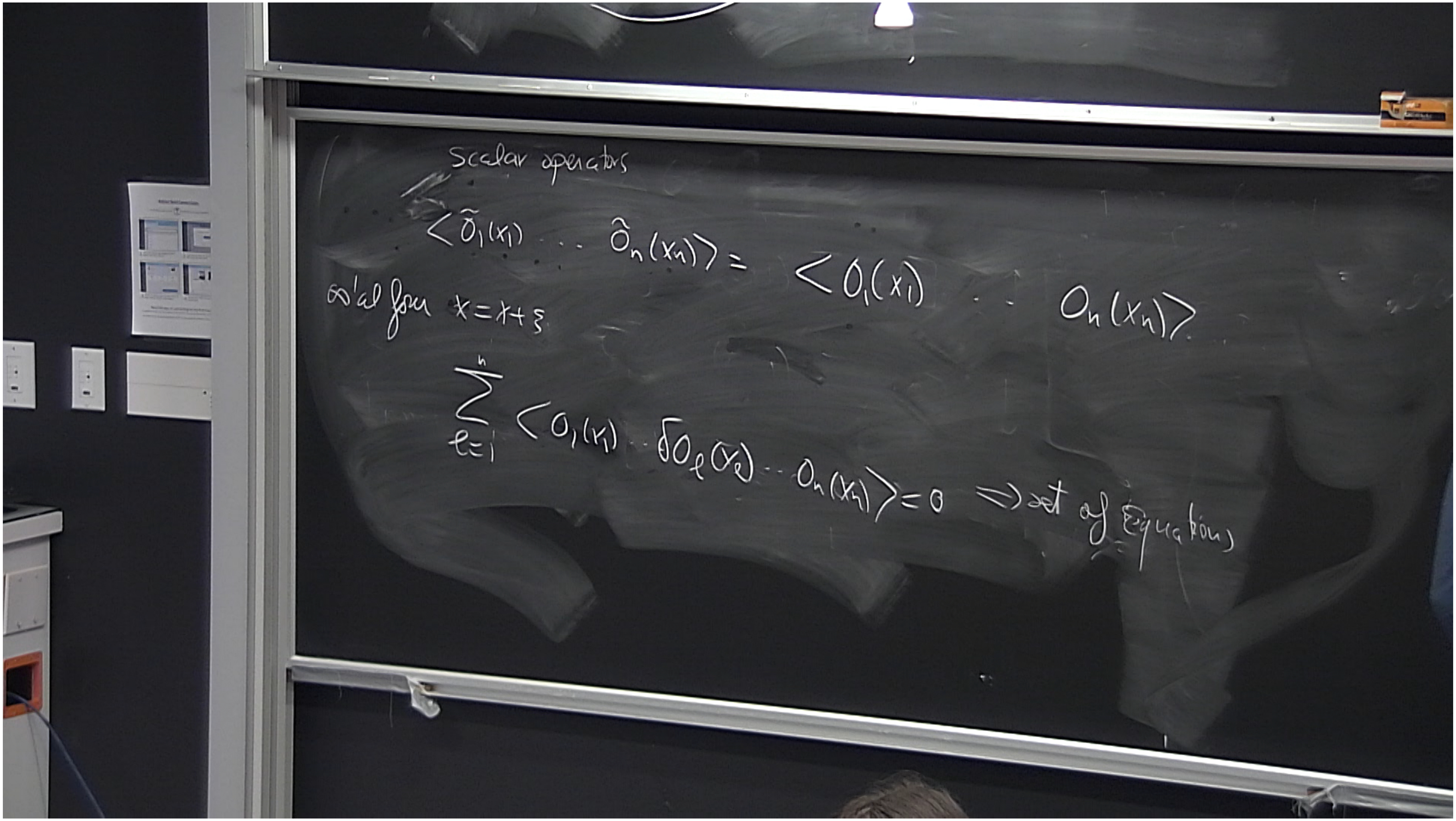
representation of Lorentz algebra

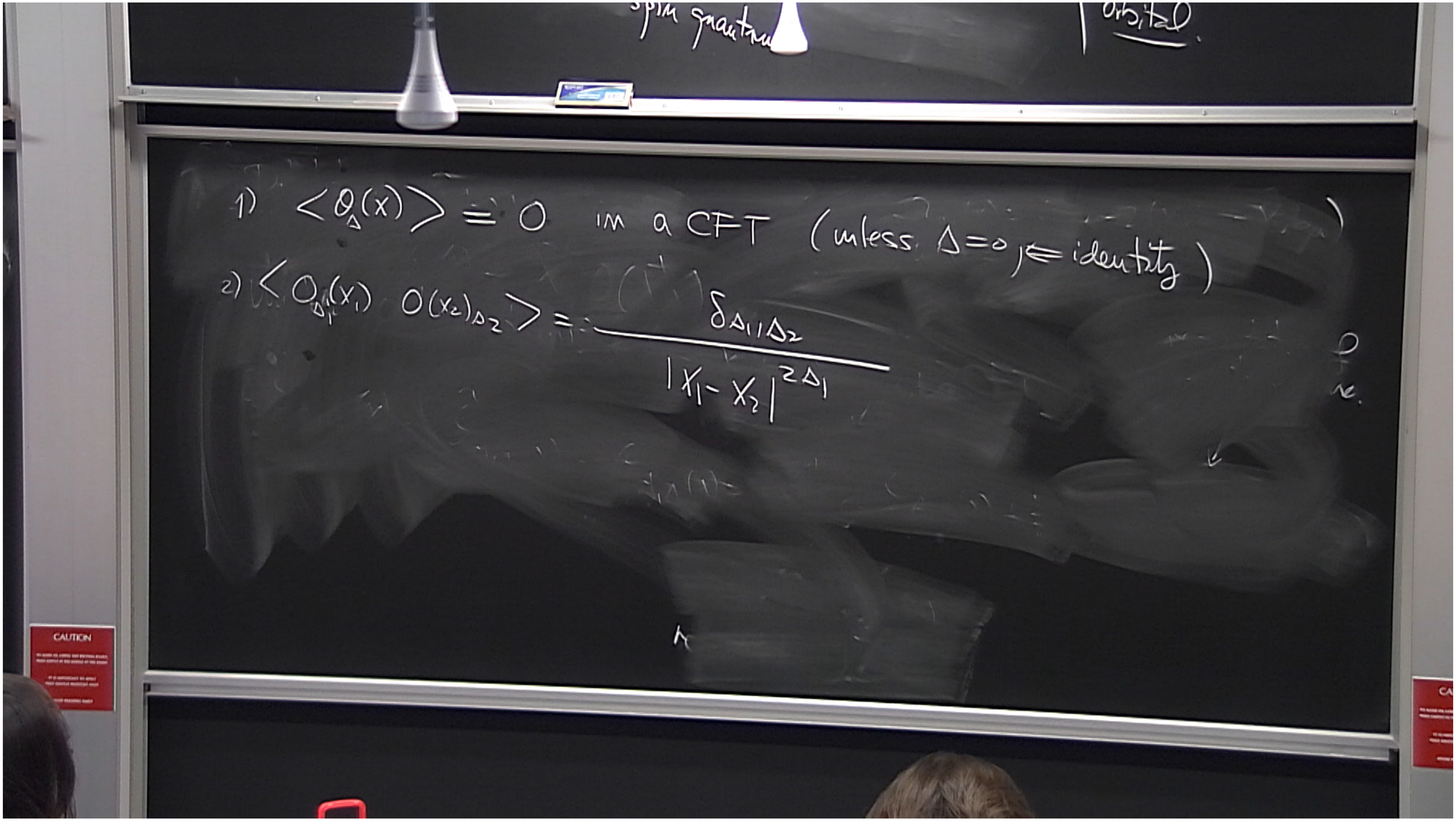
representation of D

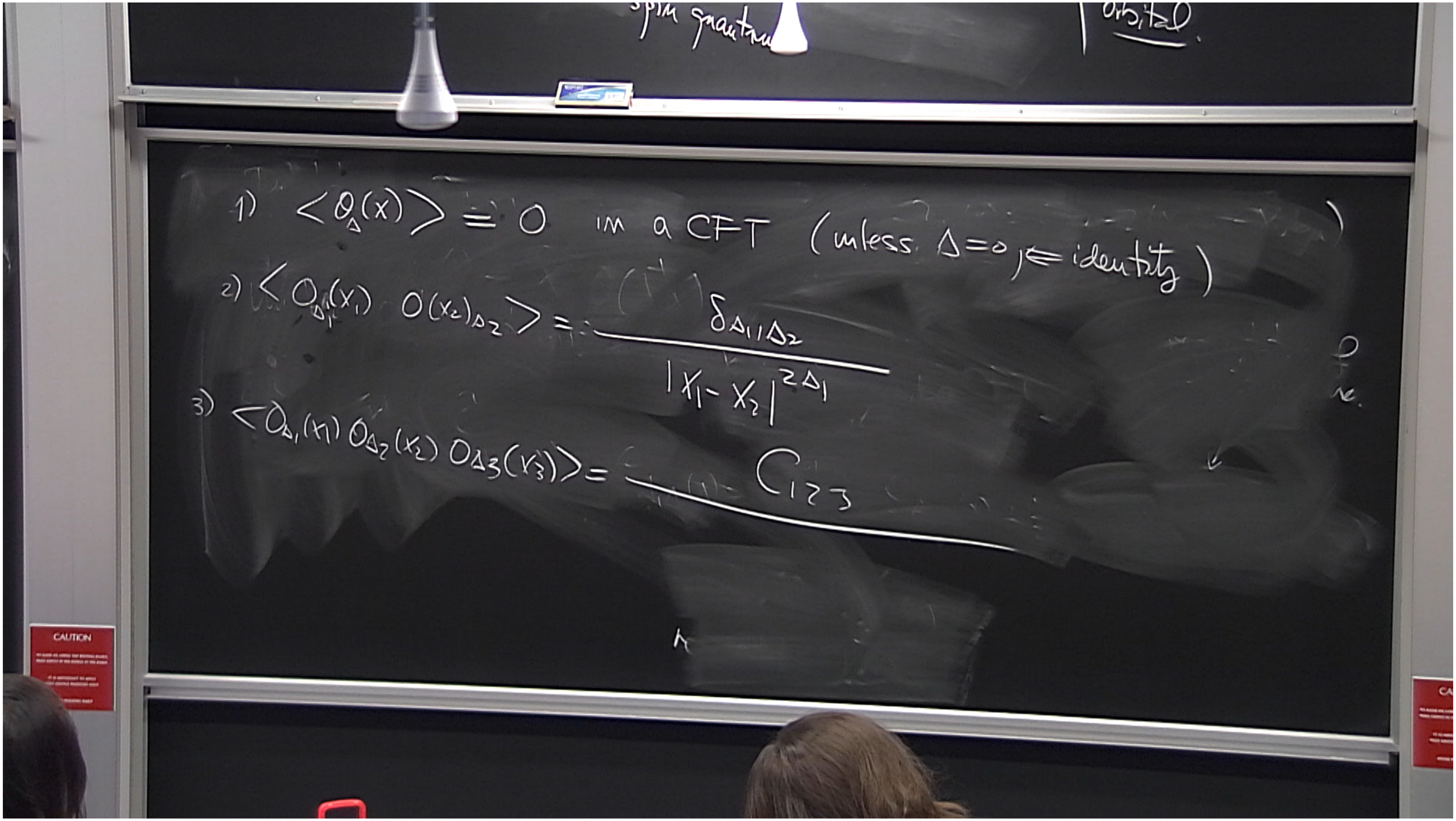


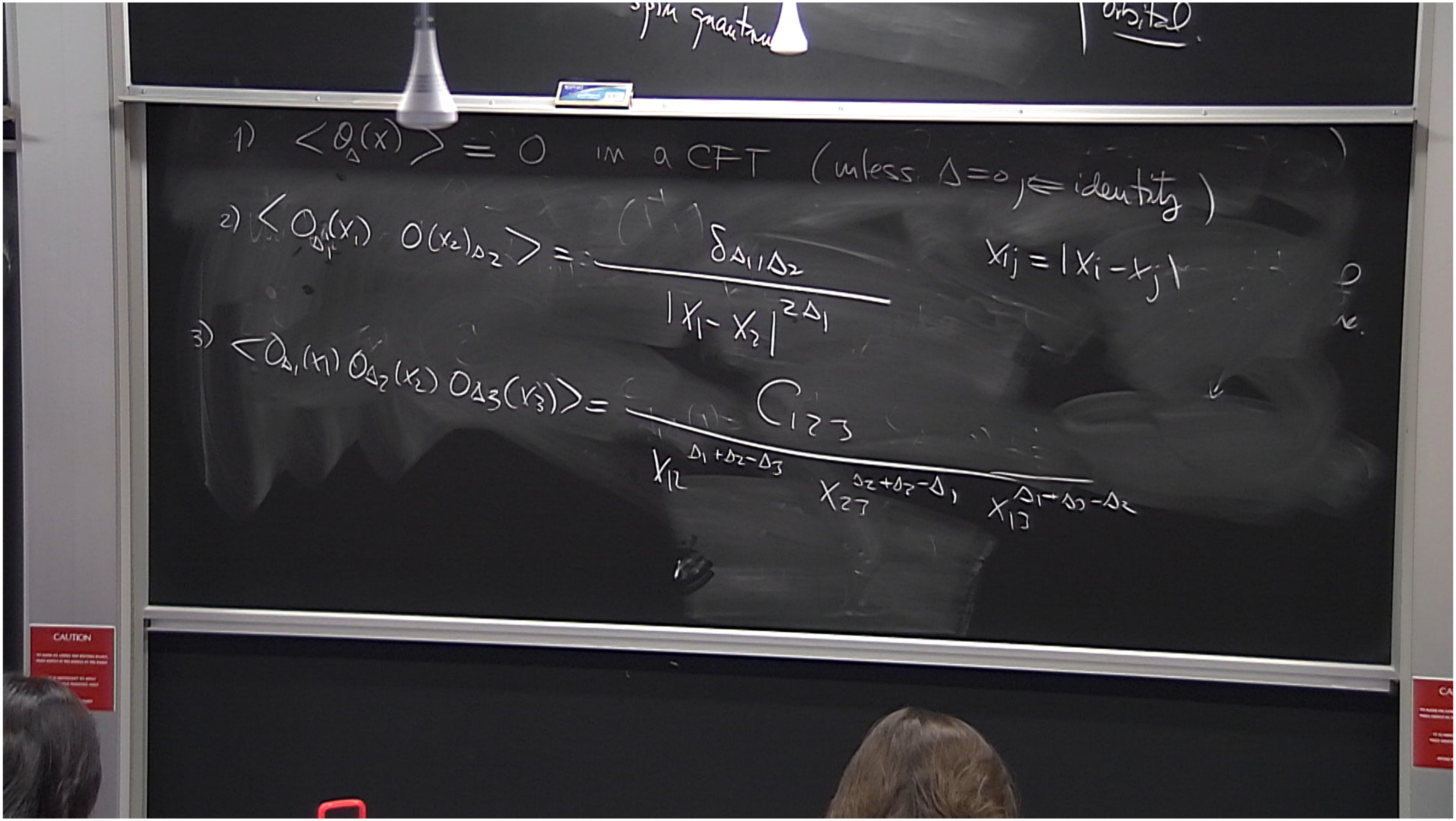
Constraint the position dependence of correlators of primar
$O_1(x_1)$ $O_2(x_2)$
$O_3(x_3)$ . . .
$\langle O_1(x_1) \ldots O_n(x_n) \rangle$
CAUTION
Reebok

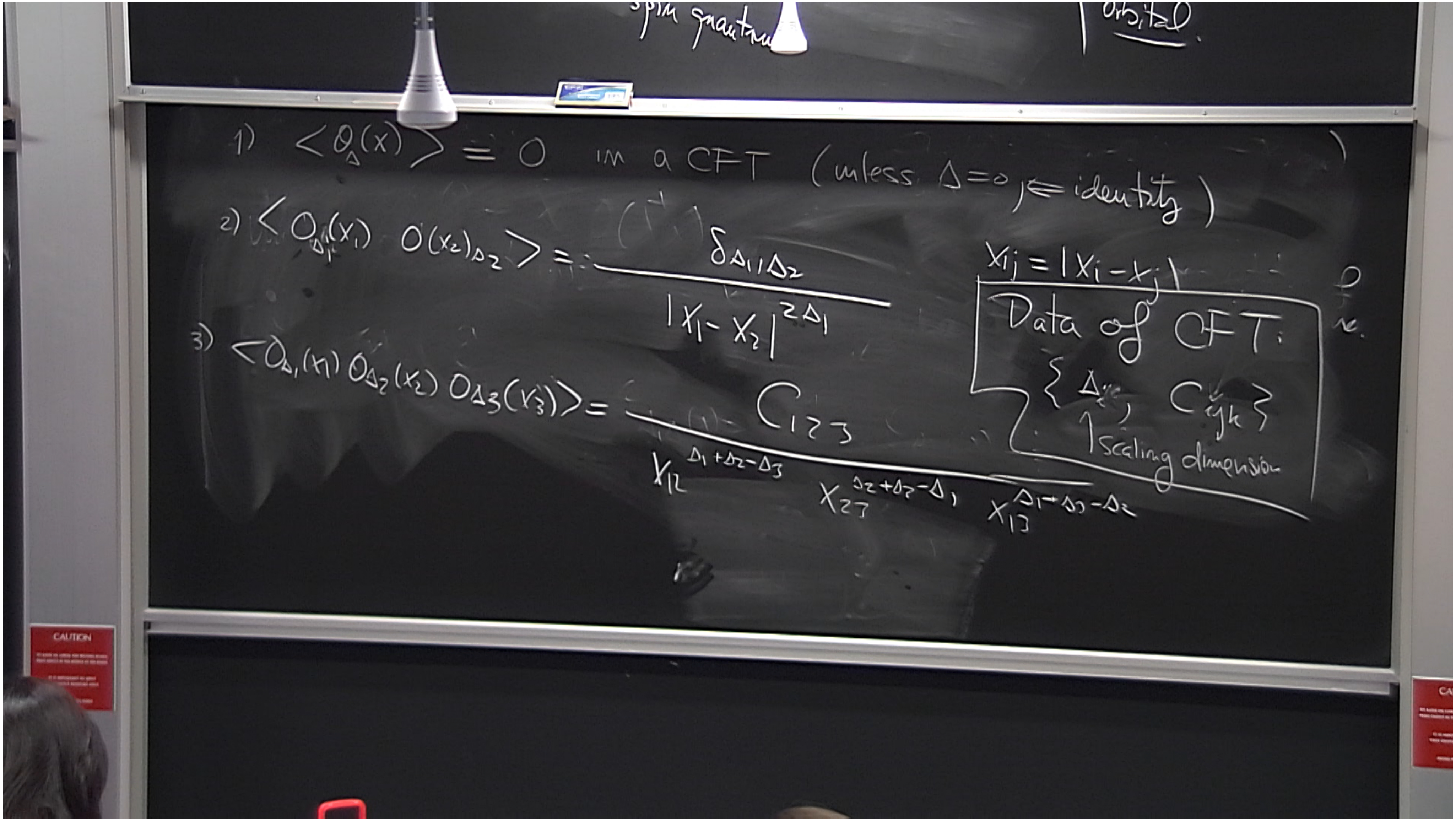

Constraint the position dependence of correlators of primaries:
O1(x1) O2(x2)
O3(x3) . . .
< O1(x1) . . . On(xn) > = f (|xi - xj|)
in a CFT the position of 2 and 3pt function
is completely fixed
CAUTION
Reebok


Scalar operators
$\langle \tilde{O}_1(x_1) \ldots \hat{O}_n(x_n) \rangle = \langle O_1(x_1) \ldots O_n(x_n) \rangle$
∞'al form $x = x + \xi$
$\sum_{\ell=1}^{n} \langle O_1(x_1) \ldots \delta O_\ell(x_\ell) \ldots O_n(x_n) \rangle = 0 \Rightarrow$ set of equations


spin quantum
orbital
1) $\langle O_\Delta(x) \rangle = 0$ in a CFT (unless $\Delta = 0$, $\Leftarrow$ identity)
2) $\langle O_{\Delta_1}(x_1)\, O(x_2)_{\Delta_2} \rangle = \frac{\delta_{\Delta_1 \Delta_2}}{|x_1 - x_2|^{2\Delta_1}}$


spin quantum
orbital
1) <O_Δ(x)> = 0 in a CFT (unless Δ=0, ⇐ identity)
2) <O_Δ1(x1) O(x2)_Δ2> = δ_Δ1Δ2 / |x1 - x2|^2Δ1
3) <O_Δ1(x1) O_Δ2(x2) O_Δ3(x3)> = C_123


spin quantum
orbital
1) ⟨O_Δ(x)⟩ = 0 in a CFT (unless Δ=0, ⇐ identity)
2) ⟨O_Δ1(x1) O(x2)_Δ2⟩ = δ_Δ1Δ2 / |x1 − x2|^2Δ1
x_ij = |x_i − x_j|
3) ⟨O_Δ1(x1) O_Δ2(x2) O_Δ3(x3)⟩ = C_123 / x12^(Δ1+Δ2−Δ3) x23^(Δ2+Δ3−Δ1) x13^(Δ1+Δ3−Δ2)
CAUTION


spin quantum
orbital
1) <O_Δ(x)> = 0 in a CFT (unless Δ=0 ⇐ identity)
2) <O_Δ1(x1) O(x2)_Δ2> = δ_Δ1,Δ2 / |x1 - x2|^2Δ1
3) <O_Δ1(x1) O_Δ2(x2) O_Δ3(x3)> = C_123 / x12^(Δ1+Δ2-Δ3) x23^(Δ2+Δ3-Δ1) x13^(Δ1+Δ3-Δ2)
x_ij = |x_i - x_j|
Data of CFT:
{Δ_i, C_ijk}
scaling dimension