Title: PSI 2019/2020 - Cosmology Part 1 - Lecture 4

Speakers: Matthew Johnson

Collection: PSI 2019/2020 - Cosmology (Part 1)

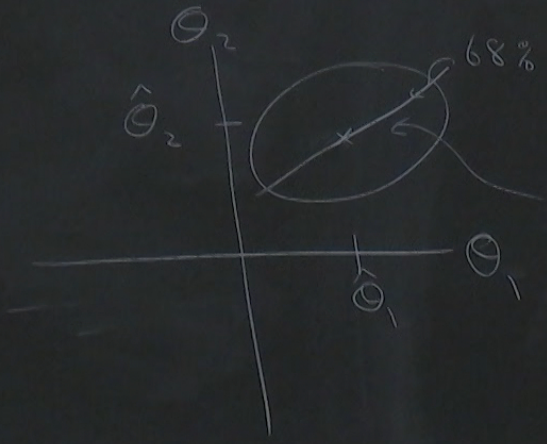
Date: February 20, 2020 - 1:00 PM

URL: http://pirsa.org/20020042

Approximate likelihood function:

$$\mathcal{L} \propto \exp\left[-\frac{1}{2}(\theta_i - \hat{\theta}_i)\, F_{ij}\, (\theta_j - \hat{\theta}_j)\right]$$

e.g.
2-D parameter space



Off-diagonal elements of Fisher encode parameter degeneracies

e.g.

* Fisher Matrix:

$$F_{\alpha\beta} = -\left\langle \frac{\partial^2 \ln \mathcal{L}}{\partial\theta_\alpha \partial\theta_\beta} \right\rangle$$

$\theta = (\theta_1, \theta_2, \theta_3 \ldots)$

marginalized error.

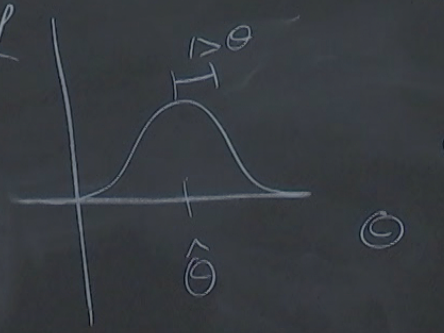
$$\Delta_{\theta_\alpha} = \sqrt{(F^{-1})_{\alpha\alpha}}$$

(Cramer-Rao Bound - Can't do better than this!

Likelihood function / Fisher Matrix

Gaussian Likelihoods: $\mathcal{L} \propto \exp\left[-\frac{(d-\mu)^2}{\Delta^2}\right]$

$$P_r(\theta|d) = \frac{P_r(\theta)\mathcal{L}(d|\theta)}{P_r(d)}$$



$$\mathcal{L} \propto \exp\left[-\frac{(\theta-\hat{\theta})^2}{2\Delta_\theta^2}\right]$$

$$\Delta_\theta = \left(-\frac{\partial^2 \ln \mathcal{L}}{\partial \theta^2}\bigg|_{\theta=\hat{\theta}}\right)^{-1/2}$$

e.g. Assume $\hat{\theta}_1 = \hat{\theta}_2 = 0$

$$\mathcal{L} = \propto e\left[-\theta_i \frac{F_{ij}}{2} \theta_j\right)$$

$$\propto e\left\{-\frac{1}{2}(\theta_1, \theta_2)\begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}\right\}$$

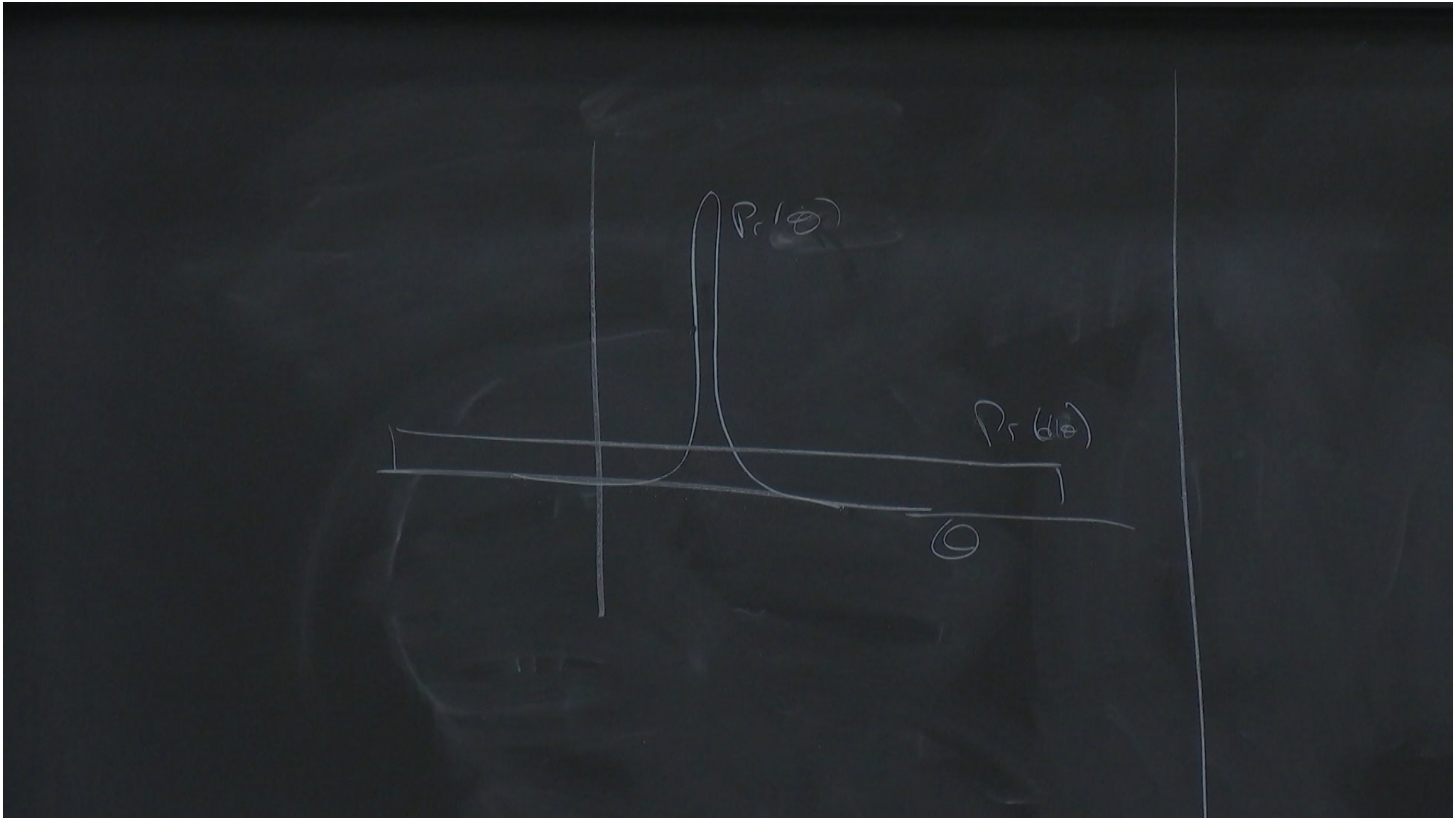
$$\mathcal{L}(\theta_1, \theta_2) \propto e\left[-\frac{1}{2}\left(F_{11}\theta_1^2 + 2F_{12}\theta_1\theta_2 + F_{22}\theta_2^2\right)\right)$$

* Marginalized likelihood:

$$\mathcal{L}(\theta_1) = \int d\theta_2 \mathcal{L}(\theta_1, \theta_2)$$

$$\propto \exp\left[-\frac{\theta_1^2}{2}\left(\frac{F_{11}F_{22} - F_{12}^2}{F_{22}}\right)\right]$$

$$\propto \exp\left[-\frac{\theta_1^2}{2(F^{-1})_{11}}\right]$$


Pr(θ)
Pr(bθ)
0

FRW Universe

$$ds^2 = -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right]$$

* Matter in FRW - geodesics of particles

e.g. null geodesics - $ds^2 = 0 = -dt^2 + a^2(t)\left[d\vec{x}^2\right]$

$$\int d\vec{x} = \pm \int \frac{dt}{a(t)}$$

More generally: Geodesic eqn

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

In Flat FRW:

$$\Gamma^0_{ij} = a\dot{a}\delta_{ij} \quad , \quad \Gamma^i_{j0} = \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j$$

$(+)\left[dx^2\right]$

$$\frac{d^2t}{d\lambda^2} + a\dot{a}\left(\frac{dx^i}{d\lambda}\right)\left(\frac{dx^i}{d\lambda}\right) = 0$$

$$\frac{d}{d\lambda}\left(\frac{dt}{d\lambda}\right) + a\dot{a}p^i p^i = 0$$

$$E\frac{dE}{dt} + a\dot{a}p^i p^i = 0$$

$$\frac{d^2x^i}{d\lambda^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\lambda}\frac{dx^i}{d\lambda} = 0$$

$$\frac{d}{d\lambda}\left(\frac{dx^i}{d\lambda}\right) + 2\frac{\dot{a}}{a}Ep^i = 0$$

$$E\frac{dp^i}{dt} + 2\frac{\dot{a}}{a}Ep^i = 0$$

$$\frac{dp^i}{p^i} = -2\frac{da}{a}$$

$$\frac{d^2x^i}{d\lambda^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\lambda}\frac{dx^i}{d\lambda} = 0$$

$$\frac{d}{d\lambda}\left(\frac{dx^i}{d\lambda}\right) + 2\frac{\dot{a}}{a}EP^i = 0$$

$$E\frac{dP^i}{dt} + 2\frac{\dot{a}}{a}EP^i = 0$$

$$\frac{dP^i}{P^i} = -2\frac{da}{a} \rightarrow \boxed{P^i = \frac{P_0^i}{a^2}}$$

Define 4-momentum of a particle:

$$P^\alpha = \frac{dx^\alpha}{d\lambda} = (E, P^i)$$

$$g_{\mu\nu}P^\mu P^\nu = -m^2$$

$$-E^2 + a^2 P^i P^i = -m^2$$

$$\frac{dt}{d\lambda} = E \rightarrow \frac{d}{d\lambda} = \frac{dt}{d\lambda}\frac{d}{dt} = E\frac{d}{dt}$$

Using Normalization of p^μ:

$$-E^2 + a^2 \frac{p_0^{i\,2}}{a^4} = -m^2$$

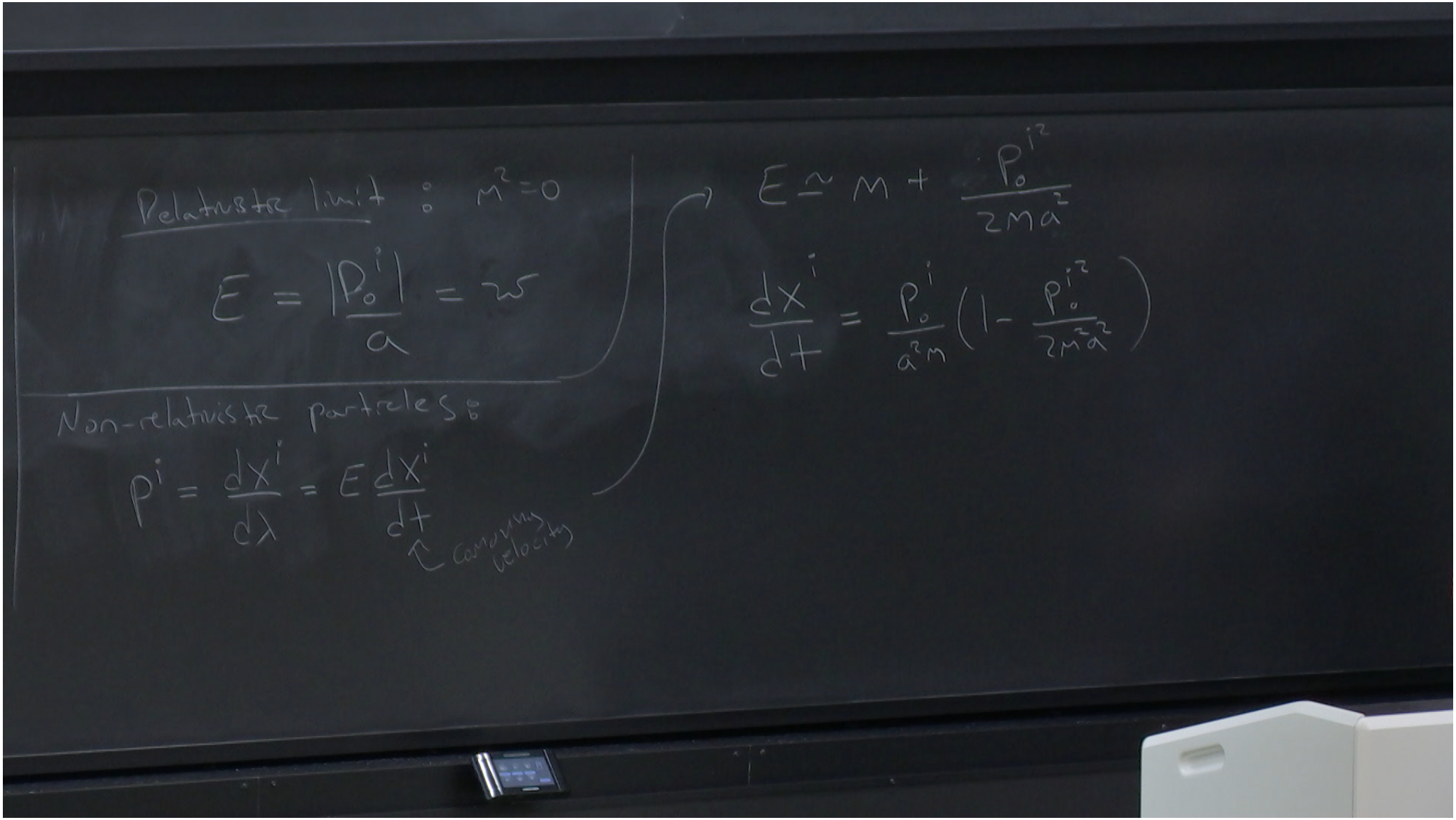
$$E = \sqrt{m^2 + \left(\frac{p_0^i}{a}\right)^2}$$

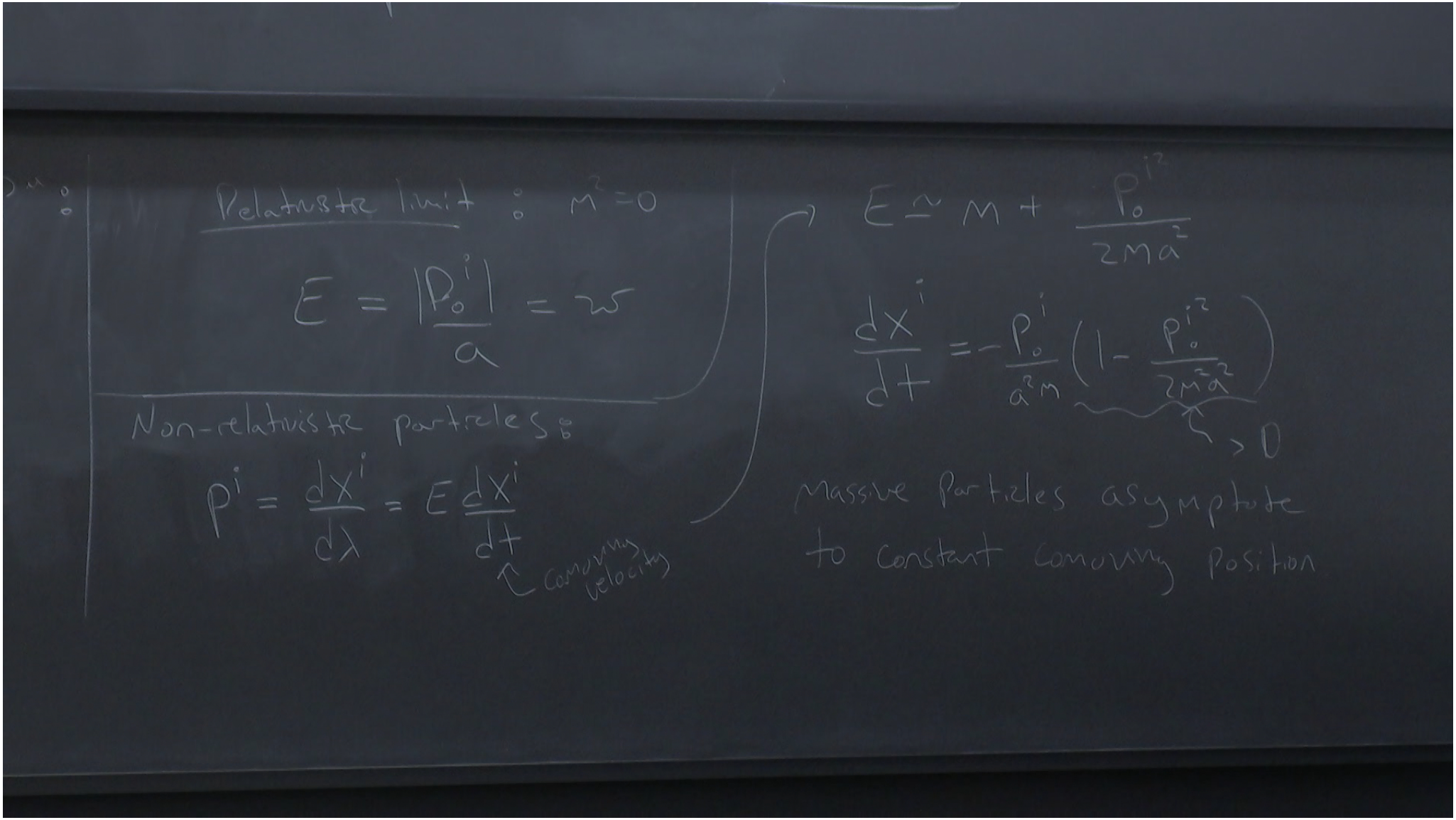
Relativistic limit: $m^2 = 0$

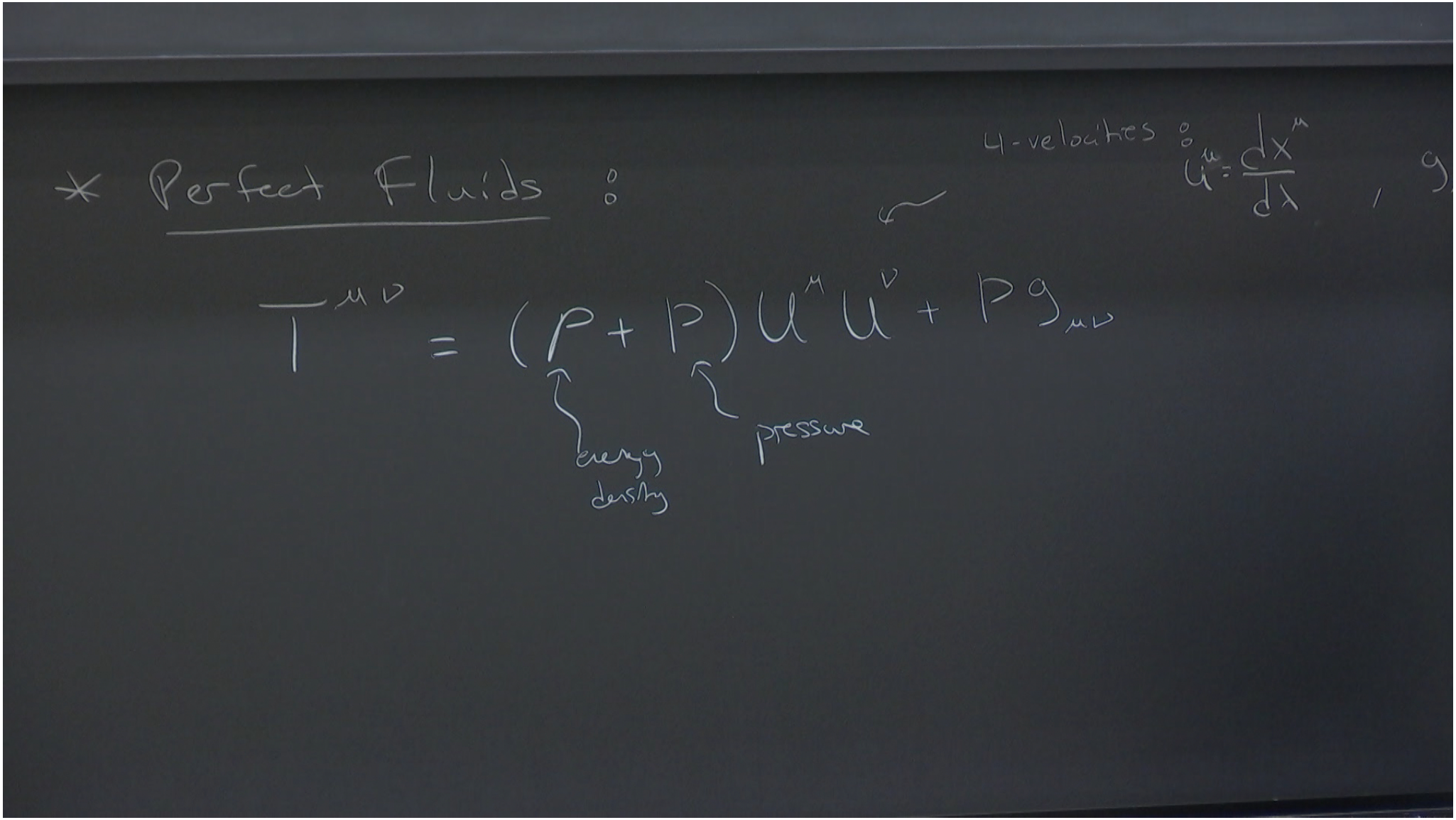
$$E = \frac{|p_0^i|}{a} = 2\nu$$

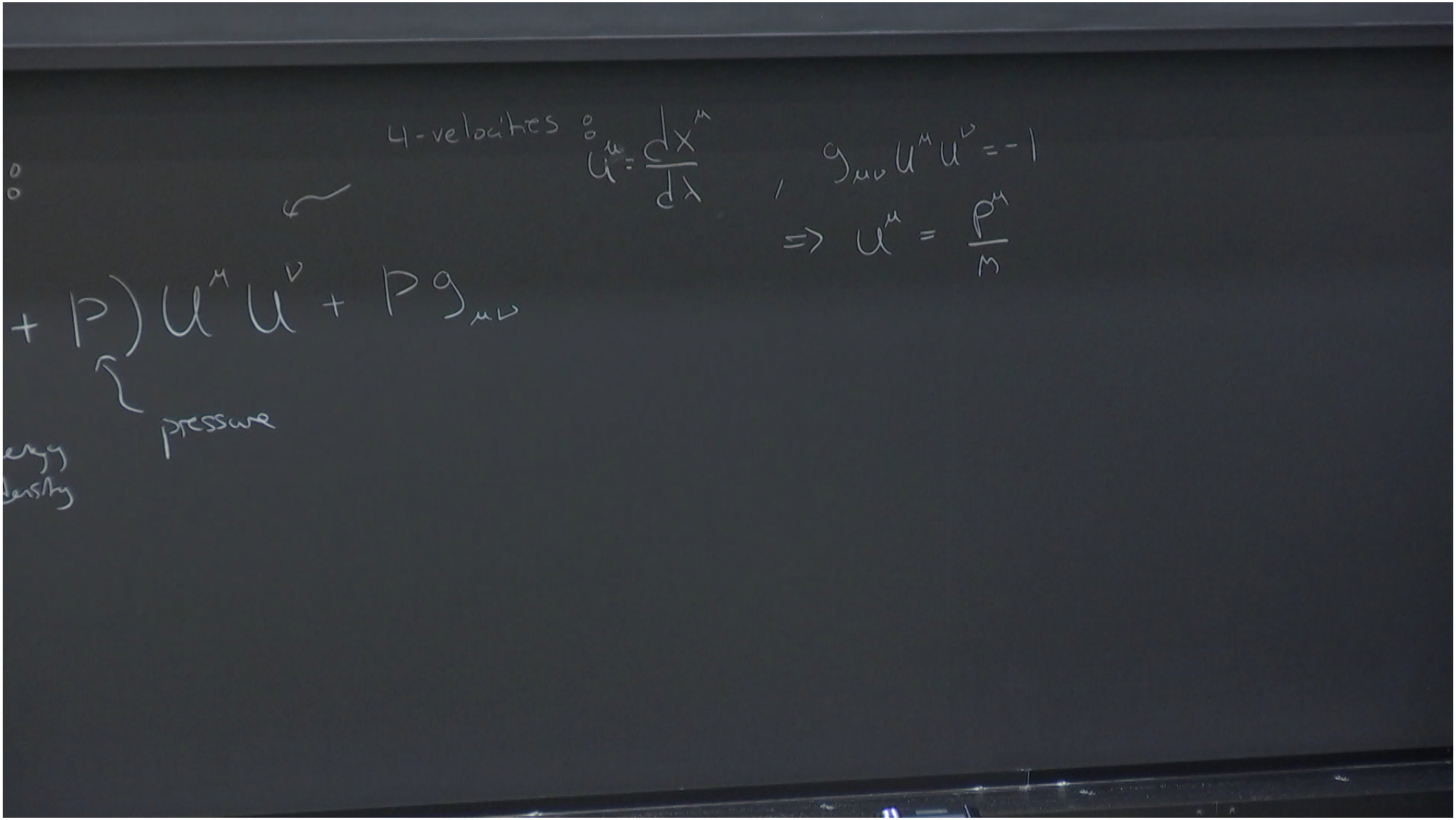
Non-relativistic particles:

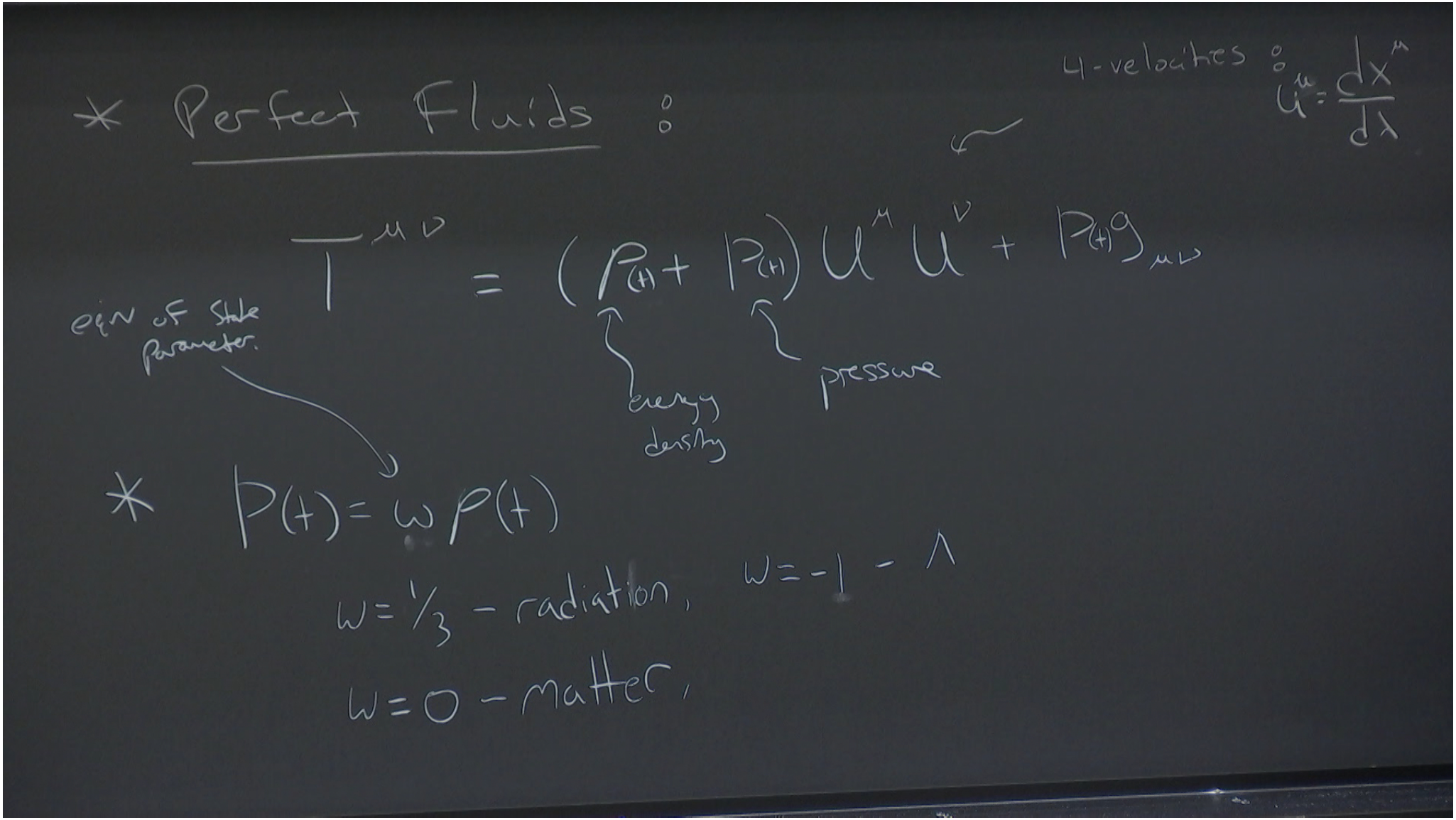
$$p^i = \frac{dx^i}{d\lambda} = E\frac{dx^i}{dt}$$


Relativistic limit : $m^2 = 0$
$E = \frac{|p_0^i|}{a} = w$
Non-relativistic particles:
$p^i = \frac{dx^i}{d\lambda} = E\frac{dx^i}{dt}$
↑ comoving velocity
$E \simeq m + \frac{p_0^{i\,2}}{2ma^2}$
$\frac{dx^i}{dt} = \frac{p_0^i}{a^2 m}\left(1 - \frac{p_0^{i\,2}}{2m^2a^2}\right)$


Relativistic limit : $m^2 = 0$
$E = \frac{|p_0^i|}{a} = w$
Non-relativistic particles:
$p^i = \frac{dx^i}{d\lambda} = E\frac{dx^i}{dt}$
Comoving velocity
$E \simeq m + \frac{p_0^{i2}}{2ma^2}$
$\frac{dx^i}{dt} = -\frac{p_0^i}{a^2 m}\left(1 - \frac{p_0^{i2}}{2m^2a^2}\right)$
> 0
Massive particles asymptote to constant comoving position


* Perfect Fluids :
4-velocities :
$u^\mu = \frac{dx^\mu}{d\lambda}$, g
$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g_{\mu\nu}$
energy density
pressure


4-velocities :
$u^\mu = \frac{dx^\mu}{d\lambda}$, $g_{\mu\nu} u^\mu u^\nu = -1$
$\Rightarrow u^\mu = \frac{p^\mu}{m}$
$+ P) u^\mu u^\nu + P g_{\mu\nu}$
pressure


* Perfect Fluids :
4-velocities : $u^\mu = \frac{dx^\mu}{d\lambda}$
$T^{\mu\nu} = (\rho(t) + p(t)) u^\mu u^\nu + p(t) g_{\mu\nu}$
eqn of State parameter.
energy density
pressure
* $p(t) = w\rho(t)$
$w = 1/3$ - radiation, $w = -1$ - Λ
$w = 0$ - matter,

$u^\mu u^\nu = -1$

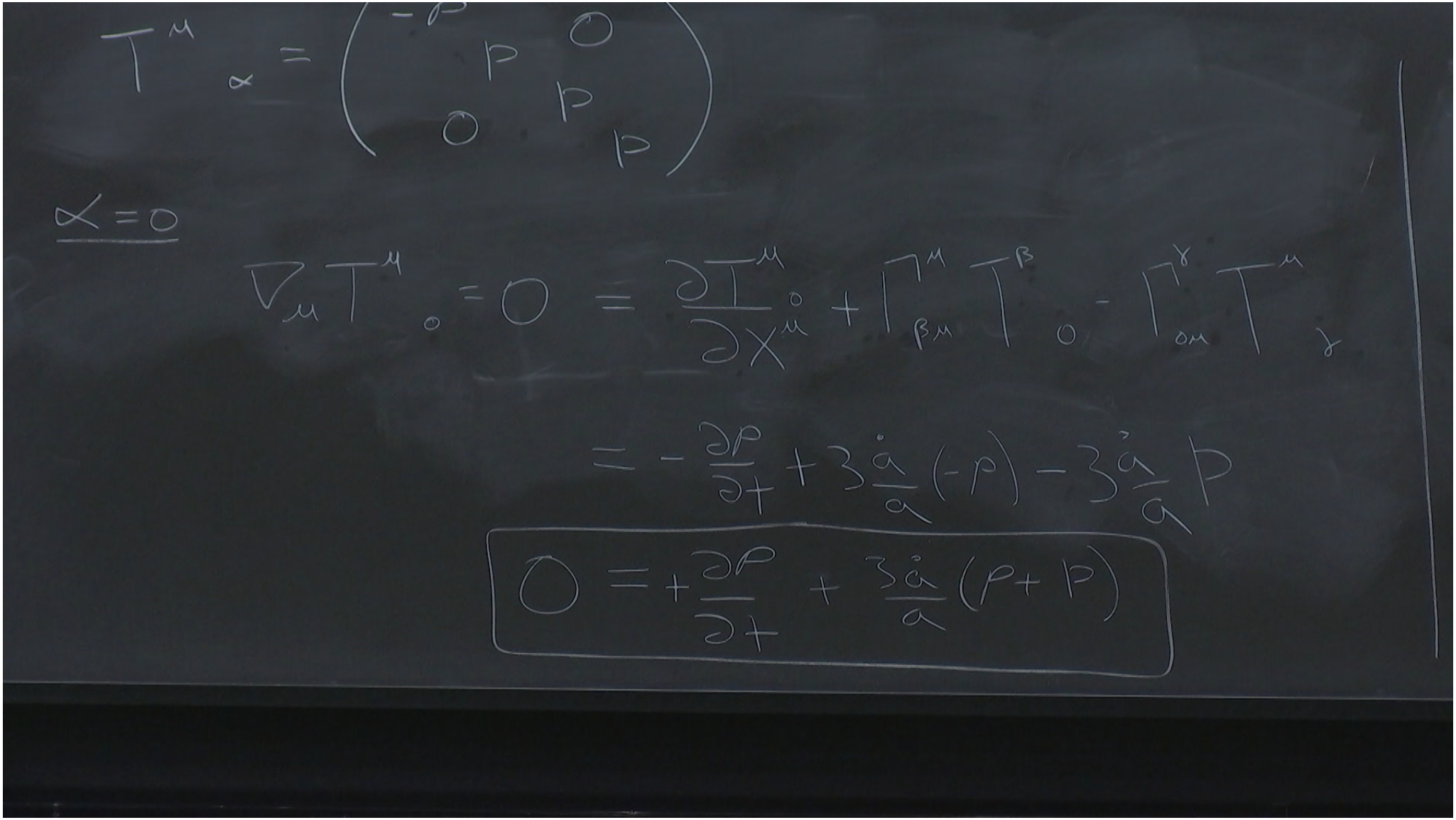
$= \frac{p^\mu}{m}$

* Homogeneity + isotropy → Frame where fluid at rest.

$$u^\mu = (1,0,0,0) \quad , \quad u_\nu = (-1,0,0,0)$$

* Conservation laws:

$$\nabla_\mu T^\mu{}_\alpha = 0 \qquad T^\mu{}_\alpha = (\rho + P) u^\mu u_\alpha + P \delta^\mu{}_\alpha$$


$T^{\mu}{}_{\alpha} = \begin{pmatrix} -\rho & 0 & & \\ & p & & \\ 0 & & p & \\ & & & p \end{pmatrix}$
$\alpha = 0$
$\nabla_\mu T^\mu{}_0 = 0 = \frac{\partial T^\mu_0}{\partial x^\mu} + \Gamma^\mu_{\beta\mu} T^\beta_0 - \Gamma^\gamma_{0\mu} T^\mu_\gamma$
$= -\frac{\partial \rho}{\partial t} + 3\frac{\dot{a}}{a}(-\rho) - 3\frac{\dot{a}}{a} p$
$0 = +\frac{\partial \rho}{\partial t} + \frac{3\dot{a}}{a}(\rho + p)$

$$\frac{\partial \rho}{\partial t} + 3\frac{\dot{a}}{a}\rho(1+\omega) = 0$$

$$\Rightarrow \frac{d\rho}{\rho} = -\frac{da}{a}3(1+\omega)$$

$$\Rightarrow \boxed{\rho = \rho_0 a^{-3(1+\omega)}}$$

$+ \frac{3\dot{a}}{a} \rho (1+w) = 0$

$\dot{\rho} = -\frac{da}{a} 3(1+w)$

$\boxed{\rho = \rho_0 a^{-3(1+w)}}$

Fluids dilute when

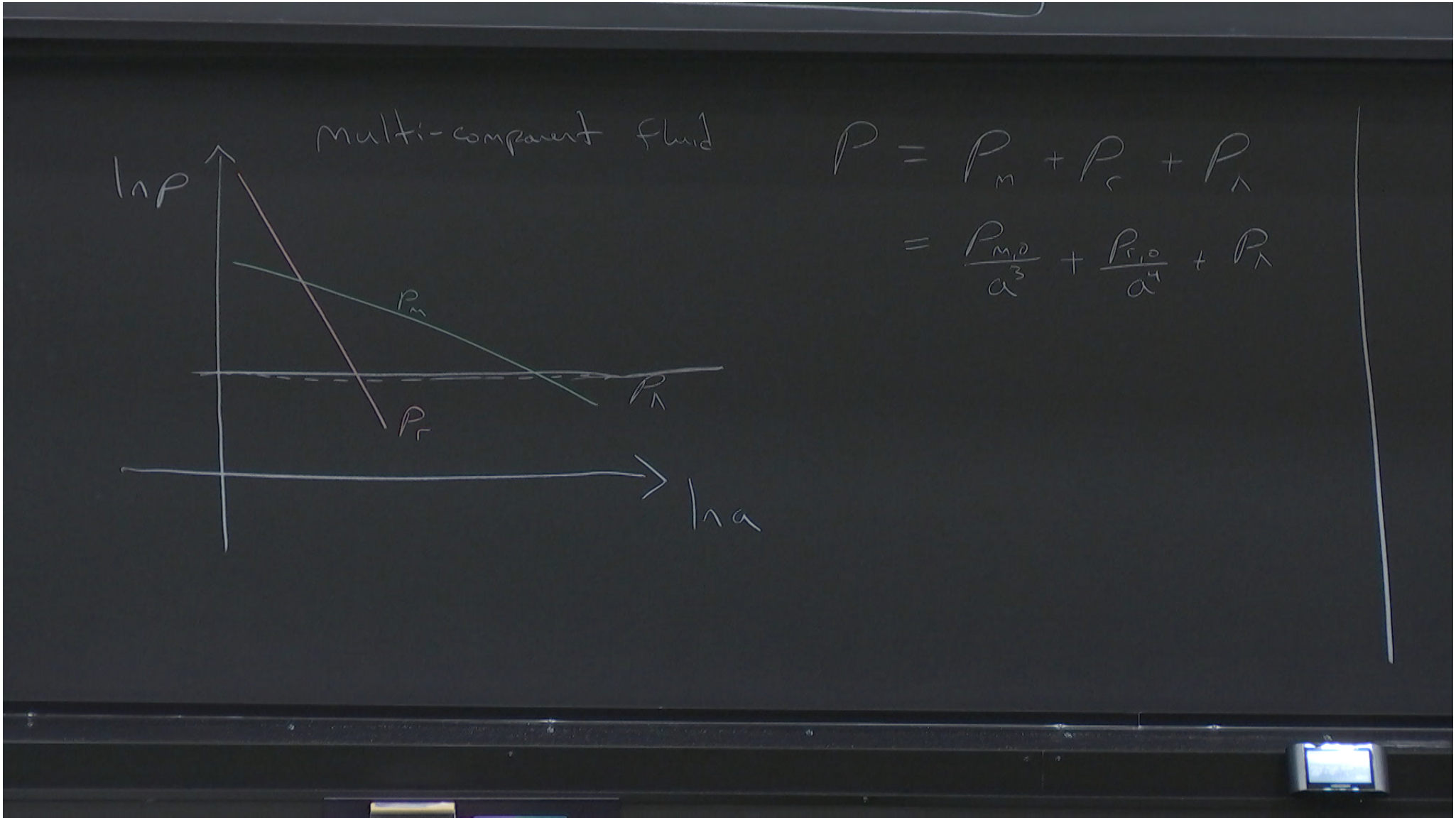
$\rho + P \geq 0$

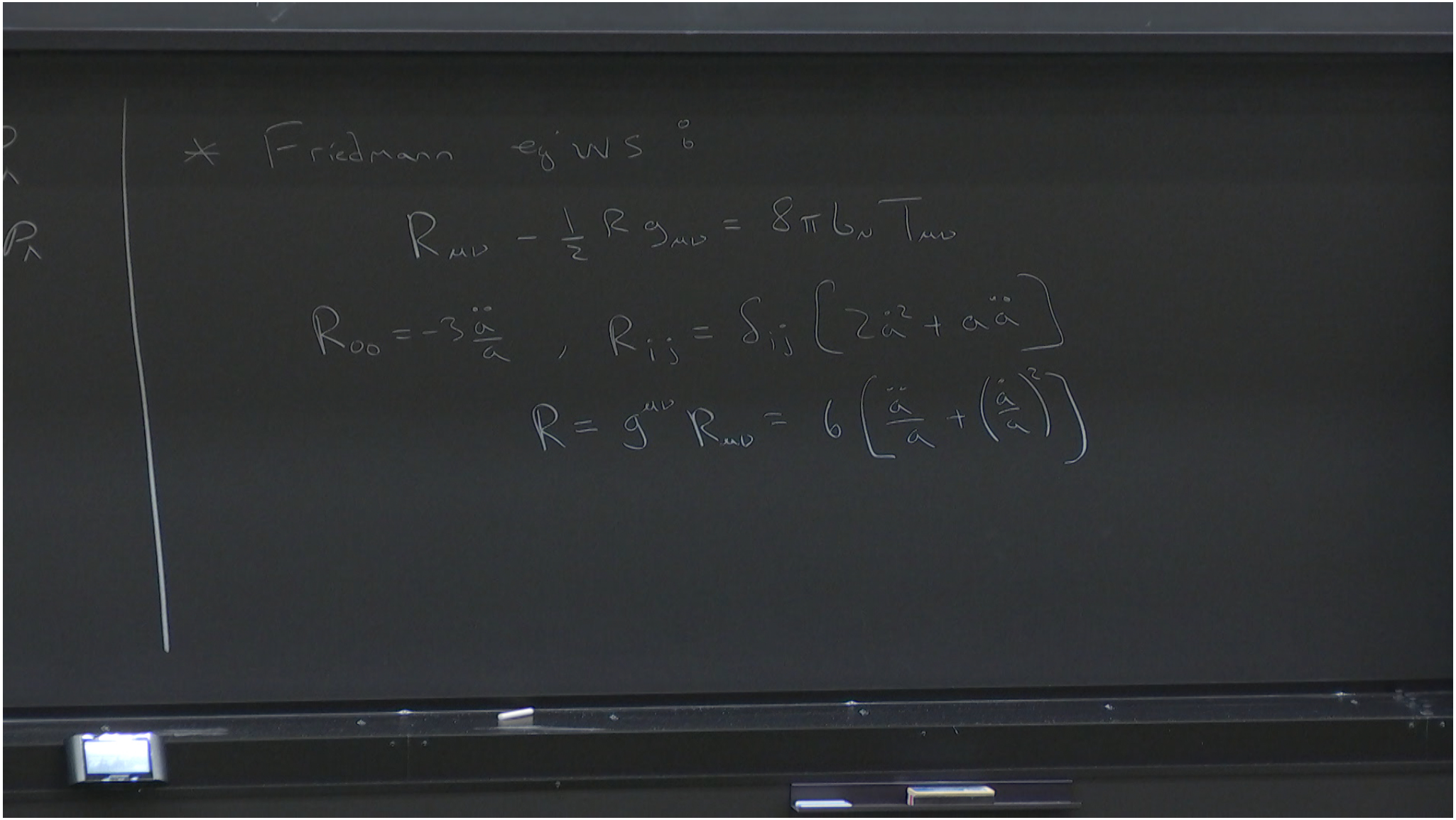
Energy conditions : Null energy condition

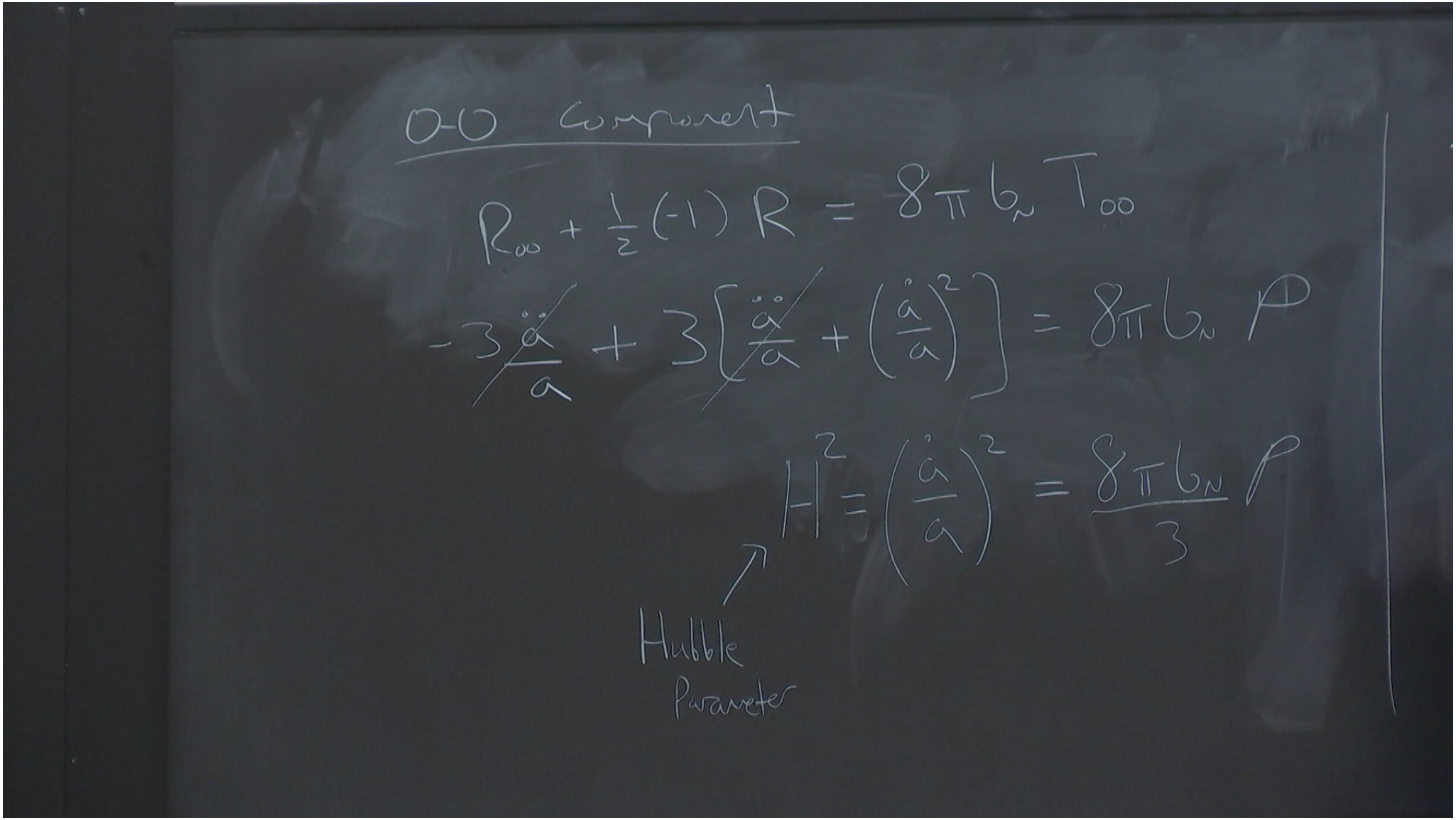
$w = -1, \Lambda \rightarrow \rho = \text{const.}$

$w = 0, \rho = \rho_0 / a^3$

$w = 1/3, \rho = \rho_0 / a^4$


multi-component fluid
$\ln \rho$
ρ_m
ρ_r
ρ_Λ
$\ln a$
$\rho = \rho_m + \rho_r + \rho_\Lambda$
$= \frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4} + \rho_\Lambda$


* Friedmann eqⁿs :
$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$
$R_{00} = -3\frac{\ddot{a}}{a}$, $R_{ij} = \delta_{ij}\left[2\dot{a}^2 + a\ddot{a}\right]$
$R = g^{\mu\nu} R_{\mu\nu} = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)$


0-0 Component
$R_{00} + \frac{1}{2}(-1) R = 8\pi G_N T_{00}$
$-3\frac{\ddot{a}}{a} + 3\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = 8\pi G_N \rho$
$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho$
Hubble Parameter

$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p)$

$\pi G_N \rho$

$\frac{\pi G_N}{3} \rho$

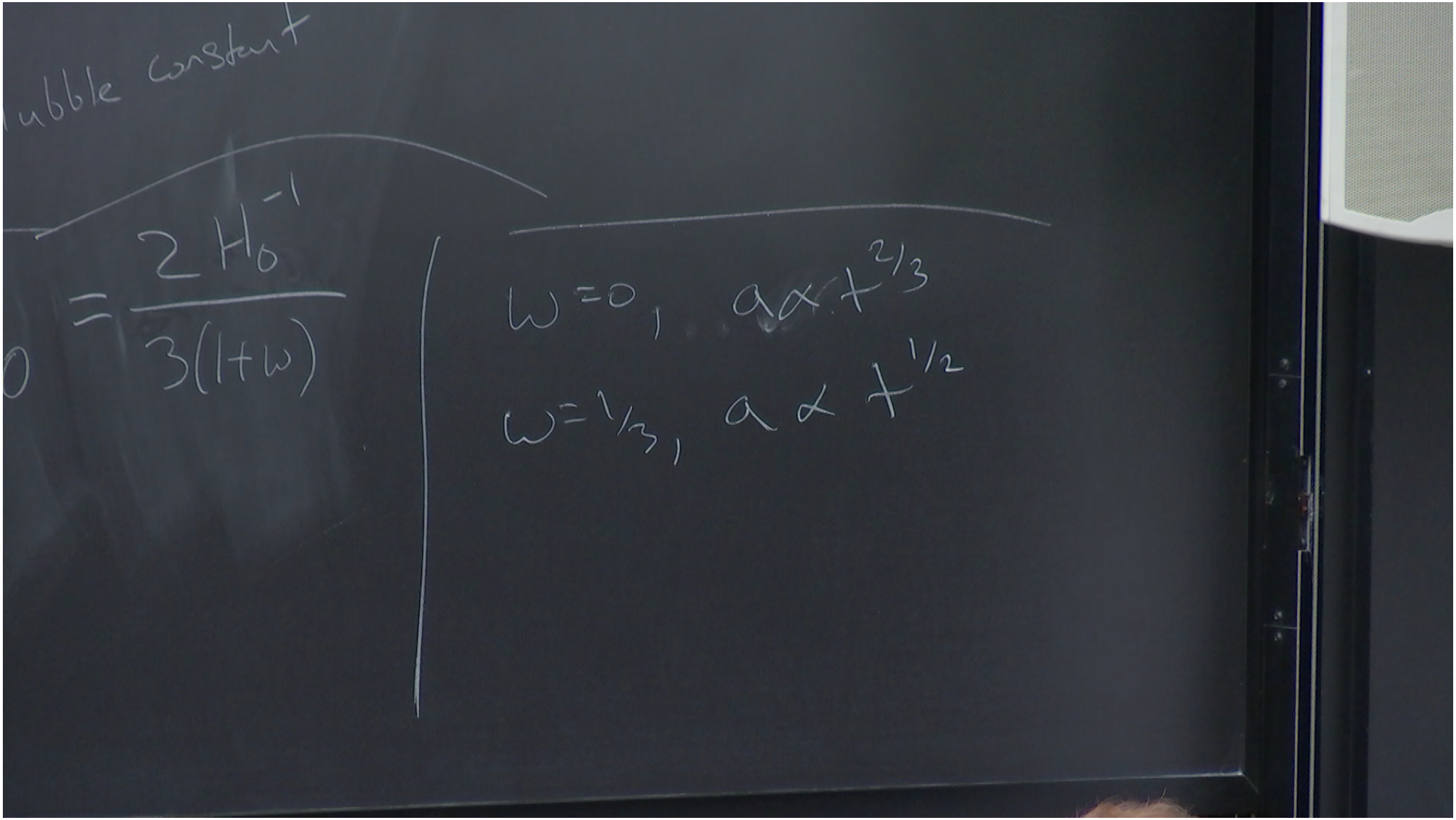
$$p = w\rho \quad , \quad \rho = \rho_0 a^{-3(1+w)}$$

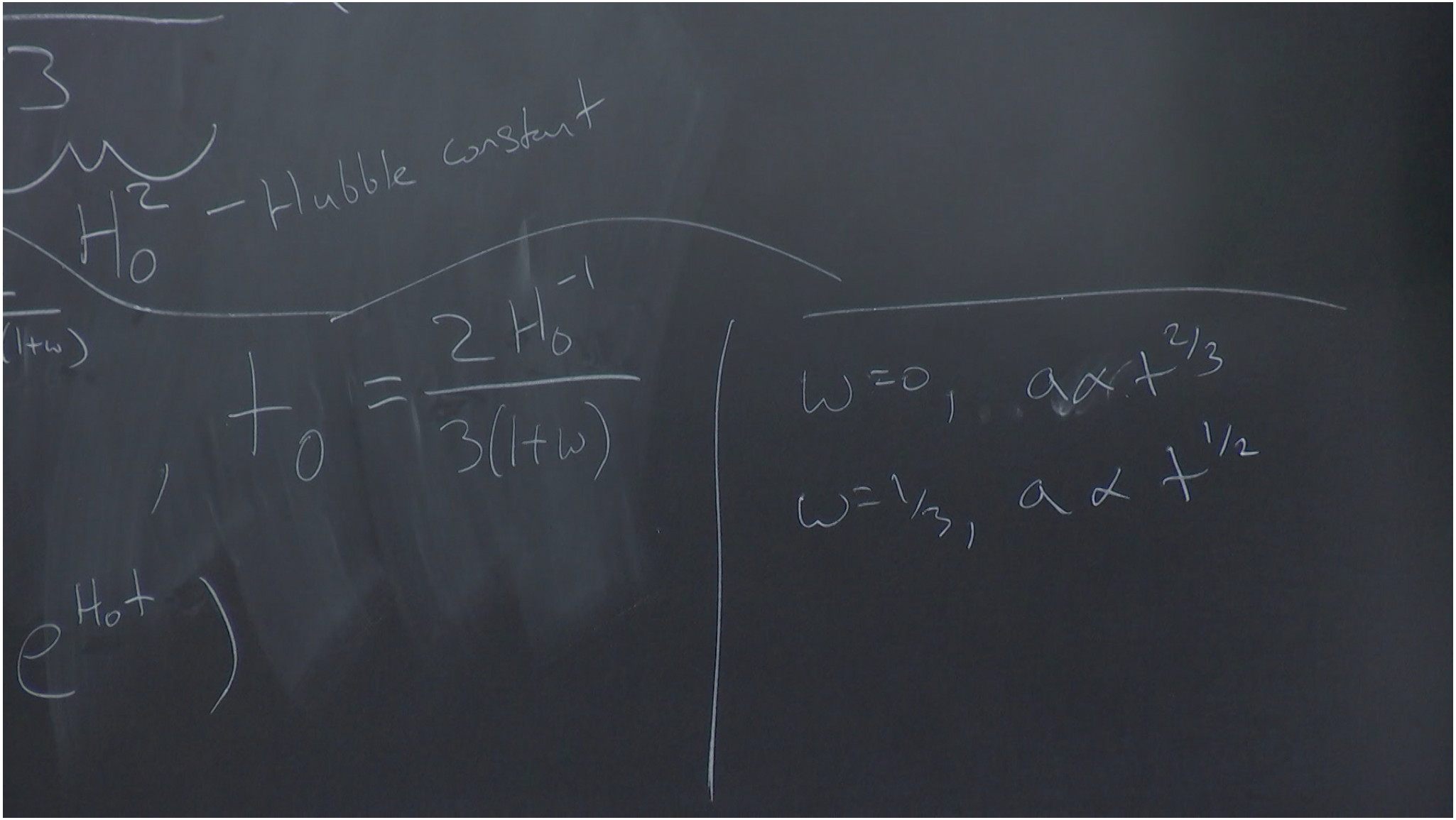
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N \rho_0}{3} a^{-3(1+w)}$$

H_0^2 – Hubble constant

$$a = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \quad , \quad t_0 = \frac{2 H_0^{-1}}{3(1+w)}$$

$$\left(w = -1 \quad , \quad a = e^{H_0 t}\right)$$


ubble constant
$= \frac{2 H_0^{-1}}{3(1+w)}$
$w=0, \; a \propto t^{2/3}$
$w=1/3, \; a \propto t^{1/2}$


3
H_0^2 - Hubble constant
$t_0 = \frac{2 H_0^{-1}}{3(1+w)}$
$w=0, \; a \propto t^{2/3}$
$w=1/3, \; a \propto t^{1/2}$
$e^{H_0 t}$

$$\Omega = \sum \frac{\rho_i}{\rho_{crit.}} = 1$$

$$\rho_{crit} = \frac{3H^2}{8\pi G_N}$$

$$\Rightarrow \quad a = \frac{1}{M}$$