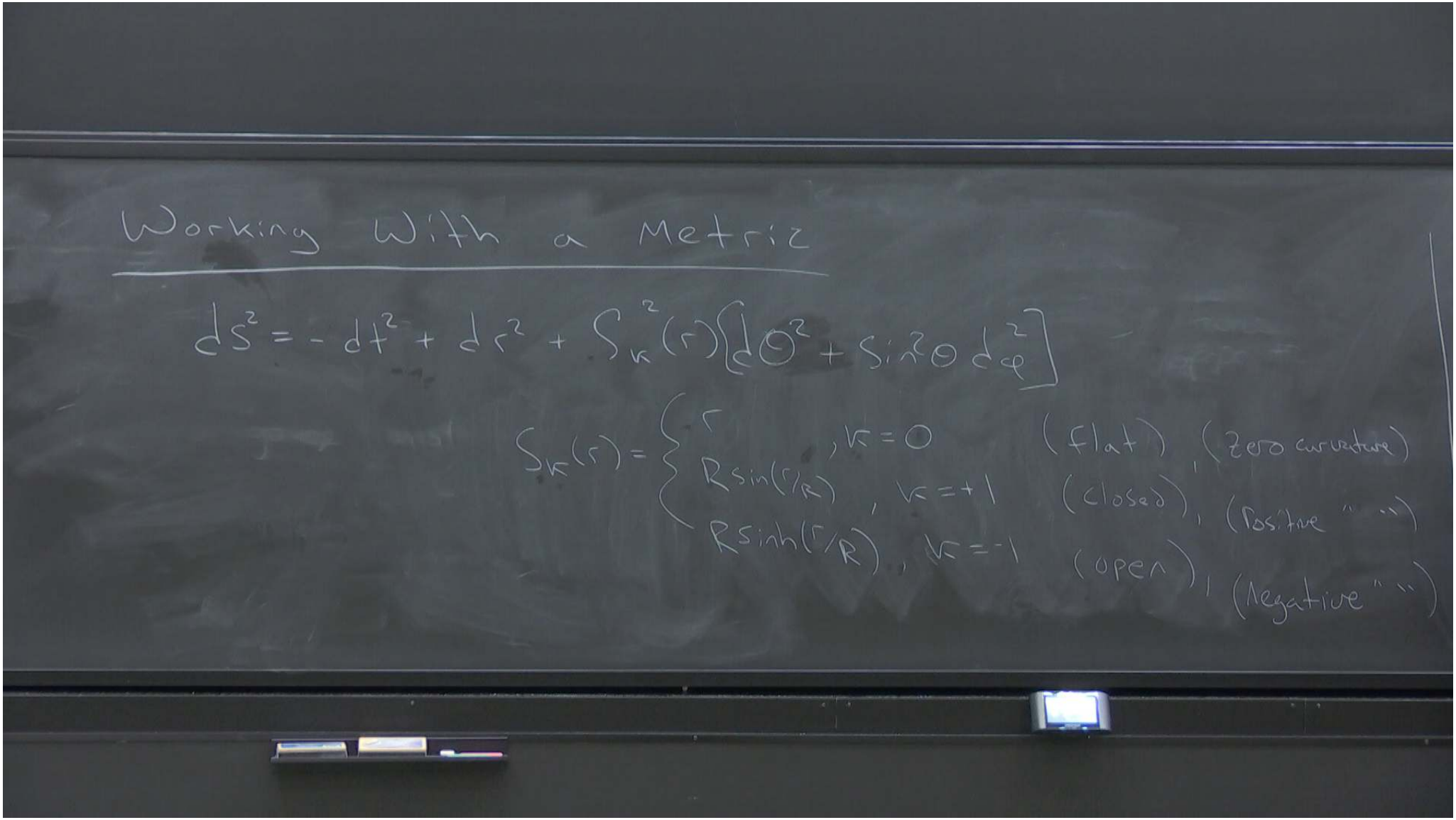
Title: PSI 2019/2020 - Cosmology Part 1 - Lecture 2

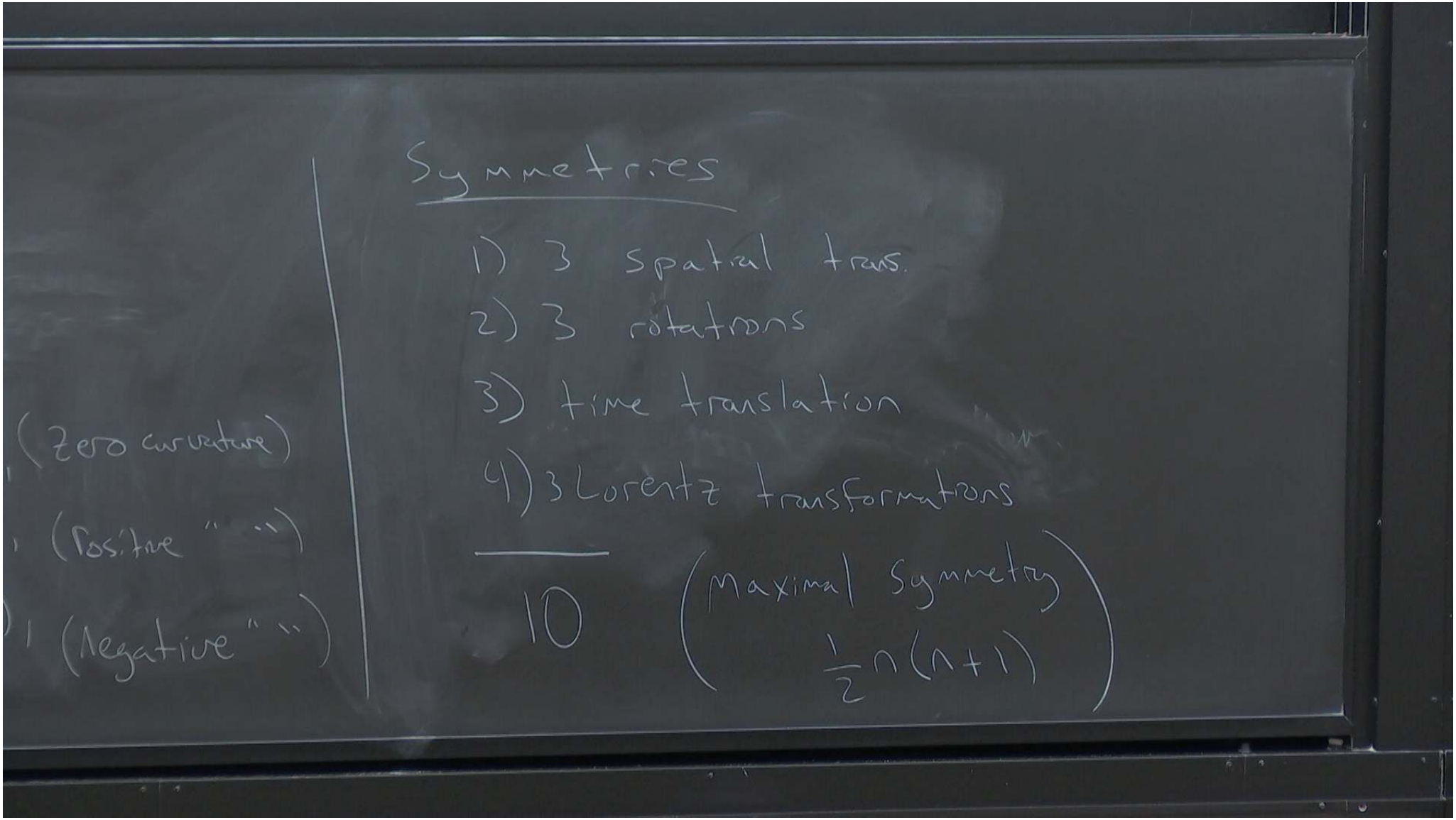
Speakers: Matthew Johnson

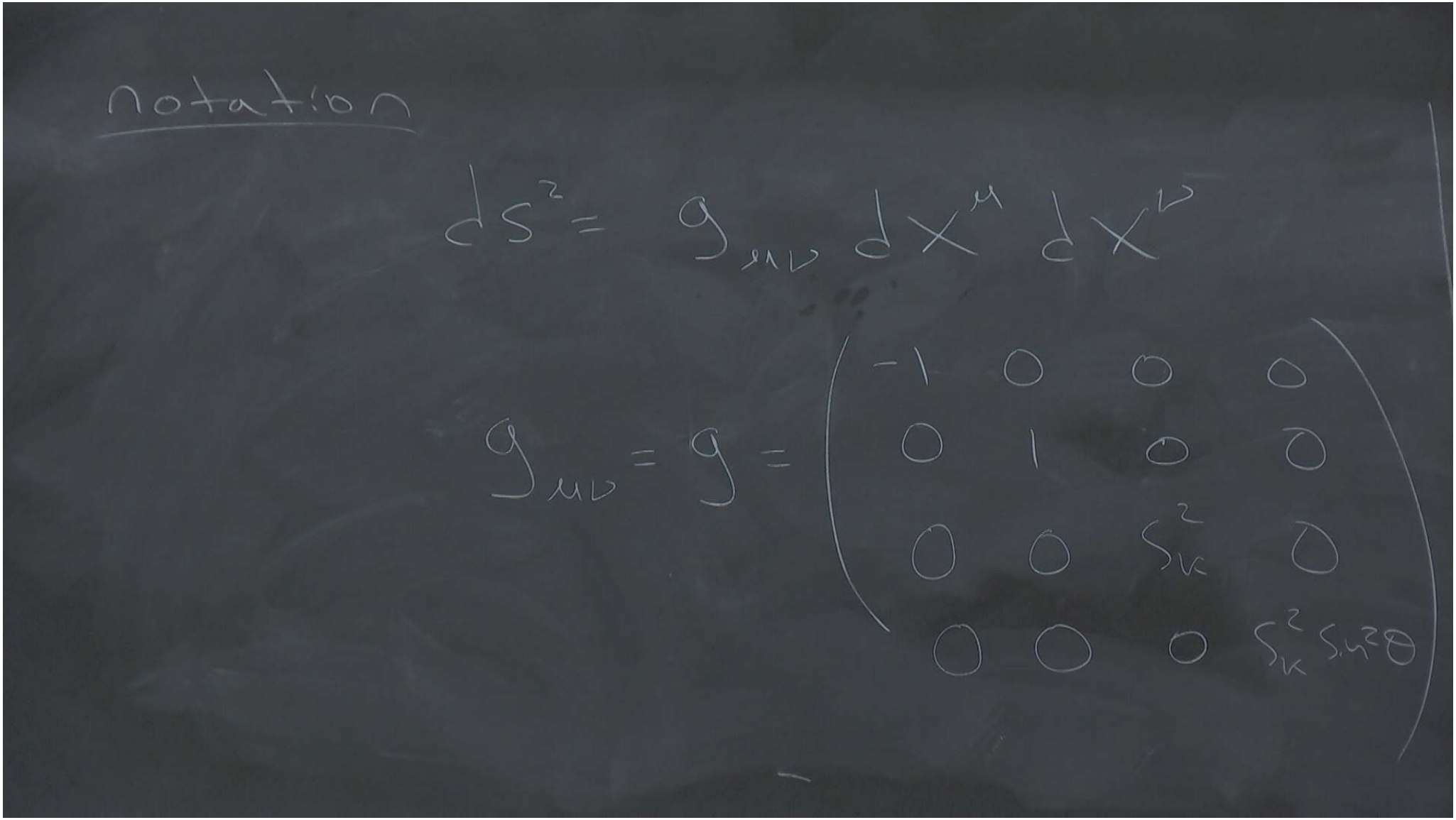
Collection: PSI 2019/2020 - Cosmology (Part 1)

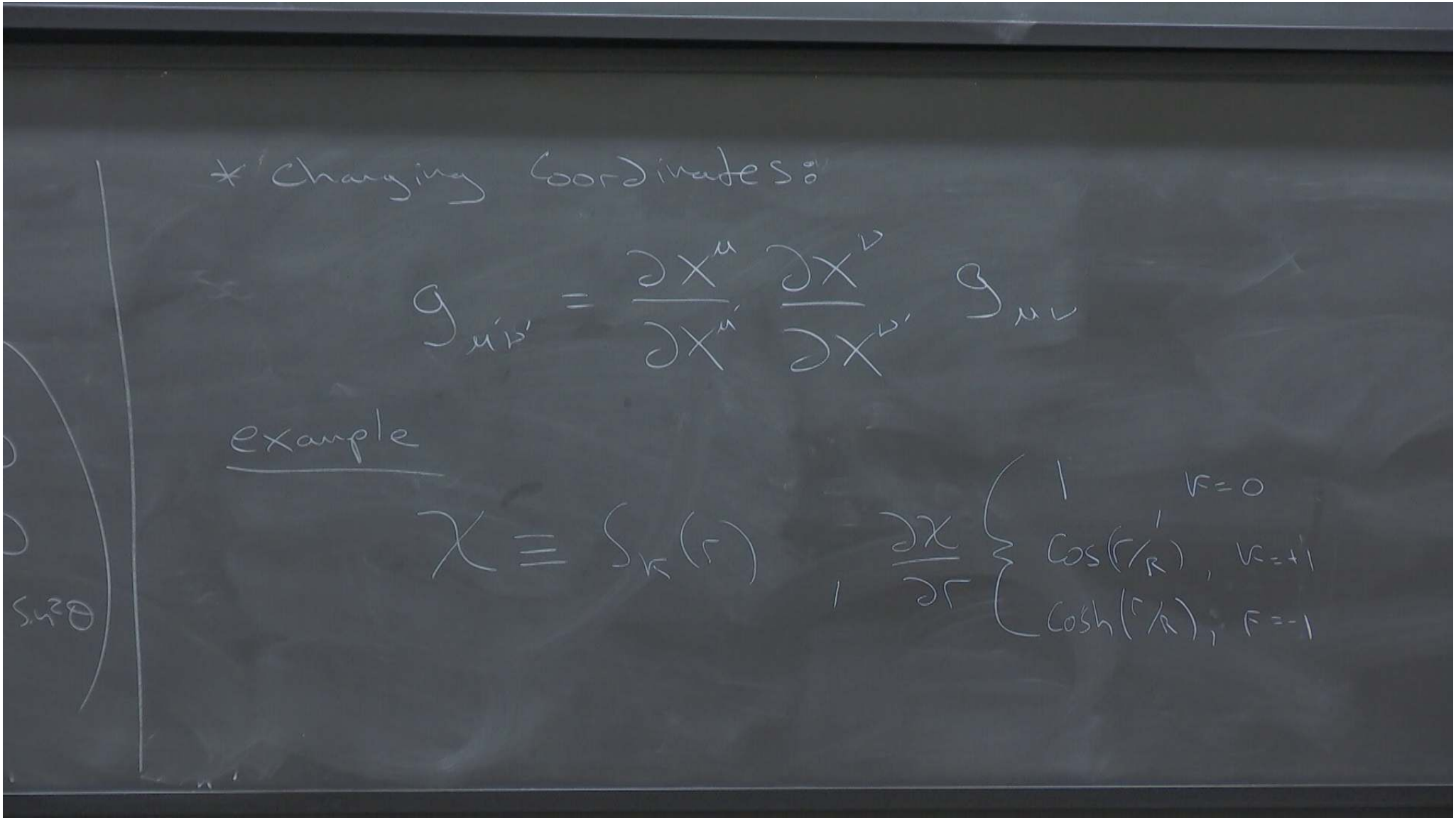
Date: February 13, 2020 - 1:00 PM

URL: http://pirsa.org/20020040


Working With a Metric
$ds^2 = -dt^2 + dr^2 + S_k^2(r)\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right]$
$S_k(r) = \begin{cases} r, & k=0 \quad (\text{flat}), (\text{zero curvature}) \\ R\sin(r/R), & k=+1 \quad (\text{closed}), (\text{positive } " \ ") \\ R\sinh(r/R), & k=-1 \quad (\text{open}), (\text{negative } " \ ") \end{cases}$


(zero curvature)
(positive " ")
(negative " ")
Symmetries
1) 3 spatial trans.
2) 3 rotations
3) time translation
4) 3 Lorentz transformations
10
(Maximal Symmetry
$\frac{1}{2}n(n+1)$)


notation
$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
$g_{\mu\nu} = g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & S_k^2 & 0 \\ 0 & 0 & 0 & S_k^2 \sin^2\theta \end{pmatrix}$


* Changing coordinates:
$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$
example
$\chi \equiv S_K(r)$, $\frac{\partial \chi}{\partial r} \begin{cases} 1, & K=0 \\ \cos(r/R), & K=+1 \\ \cosh(r/R), & K=-1 \end{cases}$
$S_K^2\theta$

$$g_{\mu\nu} = g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & S_k^2 & 0 \\ 0 & 0 & 0 & S_k^2 \sin^2\theta \end{pmatrix}$$

example

$$\chi \equiv S_k(r) \quad , \quad \frac{\partial \chi}{\partial r} \begin{cases} 1 & k=0 \\ \cos(r/R) & k=+1 \\ \cosh(r/R) & k=-1 \end{cases}$$

Component-by-component:

$$g_{\chi\chi} = \left(\frac{\partial r}{\partial \chi}\right)^2 g_{rr} + \left(\frac{\partial t}{\partial \chi}\right)^2 g_{tt} + \left(\frac{\partial \theta}{\partial \chi}\right)^2 g_{\theta\theta} + \left(\frac{\partial \varphi}{\partial \chi}\right)^2 g_{\varphi\varphi}$$

$$= \frac{1}{\left(1 - \frac{k\chi^2}{R^2}\right)} \quad (1)$$

as matrix mult.:

$$g' = \Lambda^T g \Lambda$$

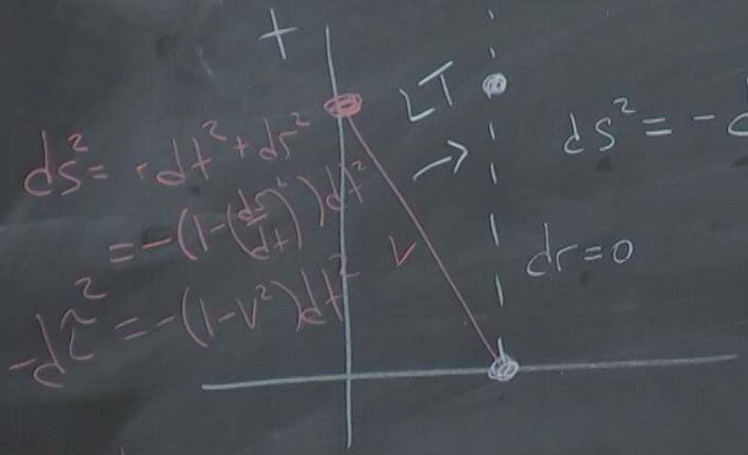
work w/ differentials:

$$dr = \left(\frac{\partial r}{\partial \chi}\right) d\chi$$

$$ds^2 = -dt^2 + \left(\frac{\partial r}{\partial \chi}\right)^2 d\chi^2 + \chi^2 d\Omega_2^2$$

$$= -dt^2 + \frac{d\chi^2}{\left(1 - \frac{k\chi^2}{R^2}\right)} + \chi^2 d\Omega_2^2$$

* Use metric to define distances/time i



$ds^2 = -dt^2 + dr^2$

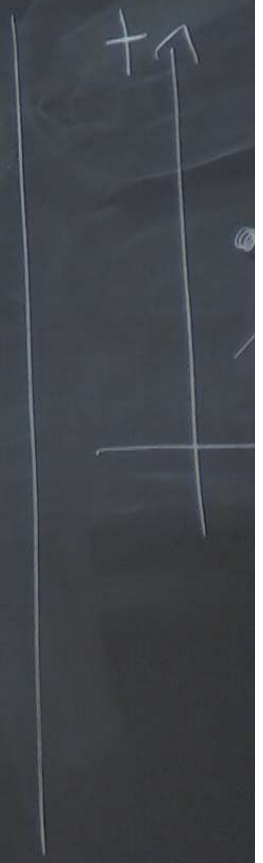
$= -(1 - \frac{dr^2}{dt})dt^2$

$-d\tau^2 = -(1-v^2)dt^2$

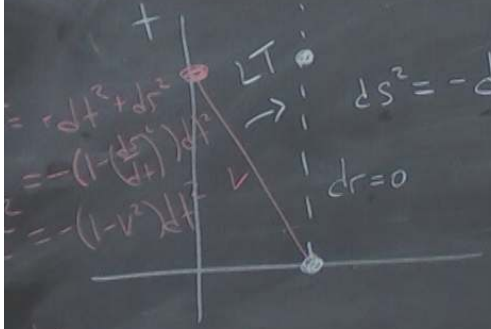
$\frac{d\tau}{dt} = (1-v^2)^{1/2}$

$ds^2 = -dt^2 = -d\tau^2$ "Proper time"

time-like separated



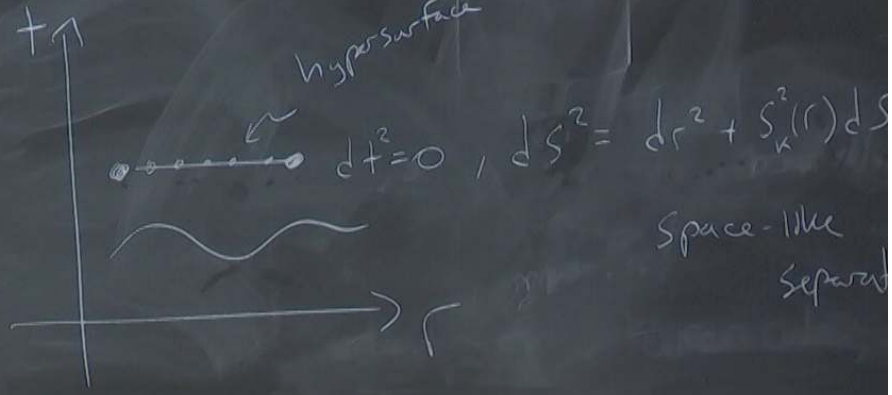
* Use metric to define distances/time intervals between events.



$ds^2 = -dt^2 = -d\tau^2$ "Proper time"

time-like separated

$\frac{d\tau}{dt} = (1-v^2)^{1/2}$

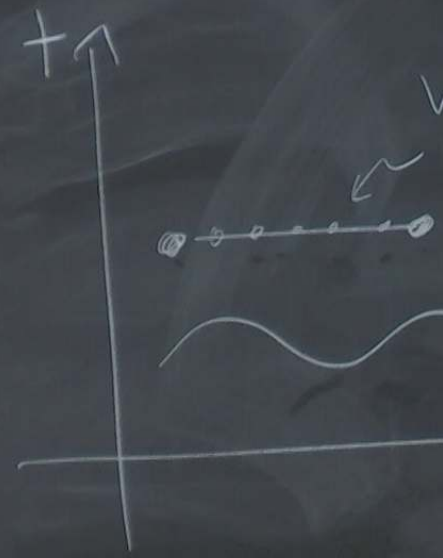


$dt^2 = 0$, $dS^2 = dr^2 + S_\kappa^2(r)d$

space-like

separat

time intervals between events.



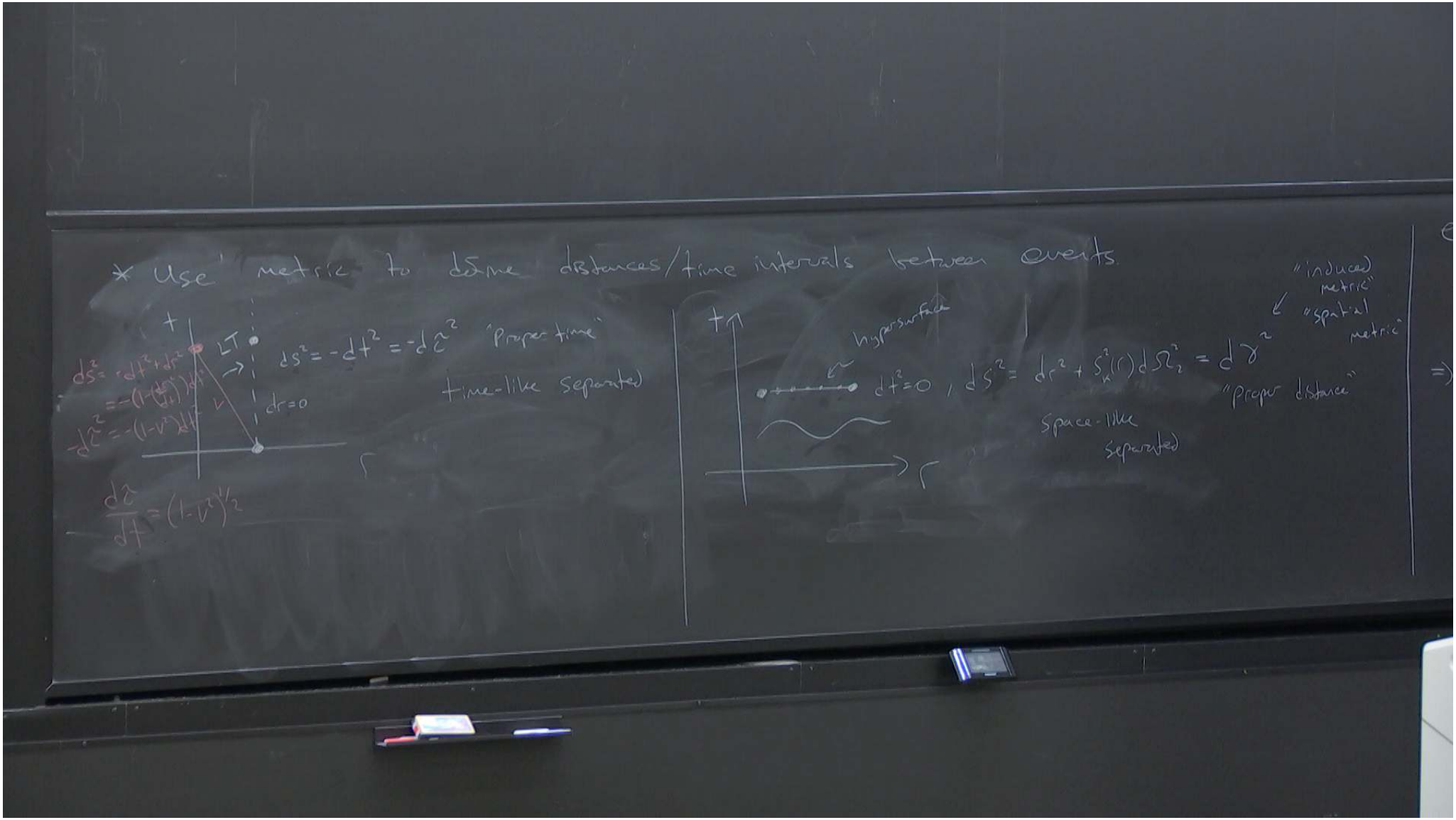
$dt^2 = 0$, $ds^2 = dr^2 + S_K^2(r)\, d\Omega_2^2 = d\gamma^2$

"induced metric"

"spatial metric"

"proper distance"

space-like separated


* Use metric to define distances/time intervals between events.
$ds^2 = -dt^2 + dr^2$
$= -(1-(\frac{dr}{dt})^2)dt^2$
$-d\tau^2 = -(1-v^2)dt^2$
LT
$ds^2 = -dt^2 = -d\tau^2$ 'Proper time'
time-like separated
dr=0
$\frac{d\tau}{dt} = (1-v^2)^{1/2}$
t
r
hypersurface
$dt^2 = 0$, $ds^2 = dr^2 + S_k^2(r)d\Omega_2^2 = d\gamma^2$
"induced metric"
"spatial metric"
"Proper distance"
space-like separated

...ween events

"induced metric" / "spatial metric"

$ds^2 = dr^2 + S_\kappa^2(r)\, d\Omega_2^2 = d\gamma^2$

"proper distance"

Space-like separated

e.g. Circumference of circles.

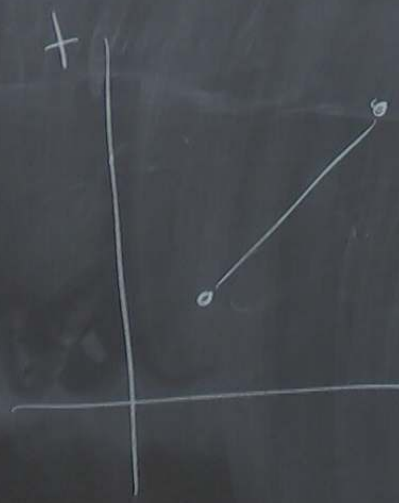
$dt^2 = 0,\ dr^2 = 0,\ d\theta = 0$

$\Rightarrow d\gamma^2 = S_\kappa^2(r_*)\, d\varphi^2$

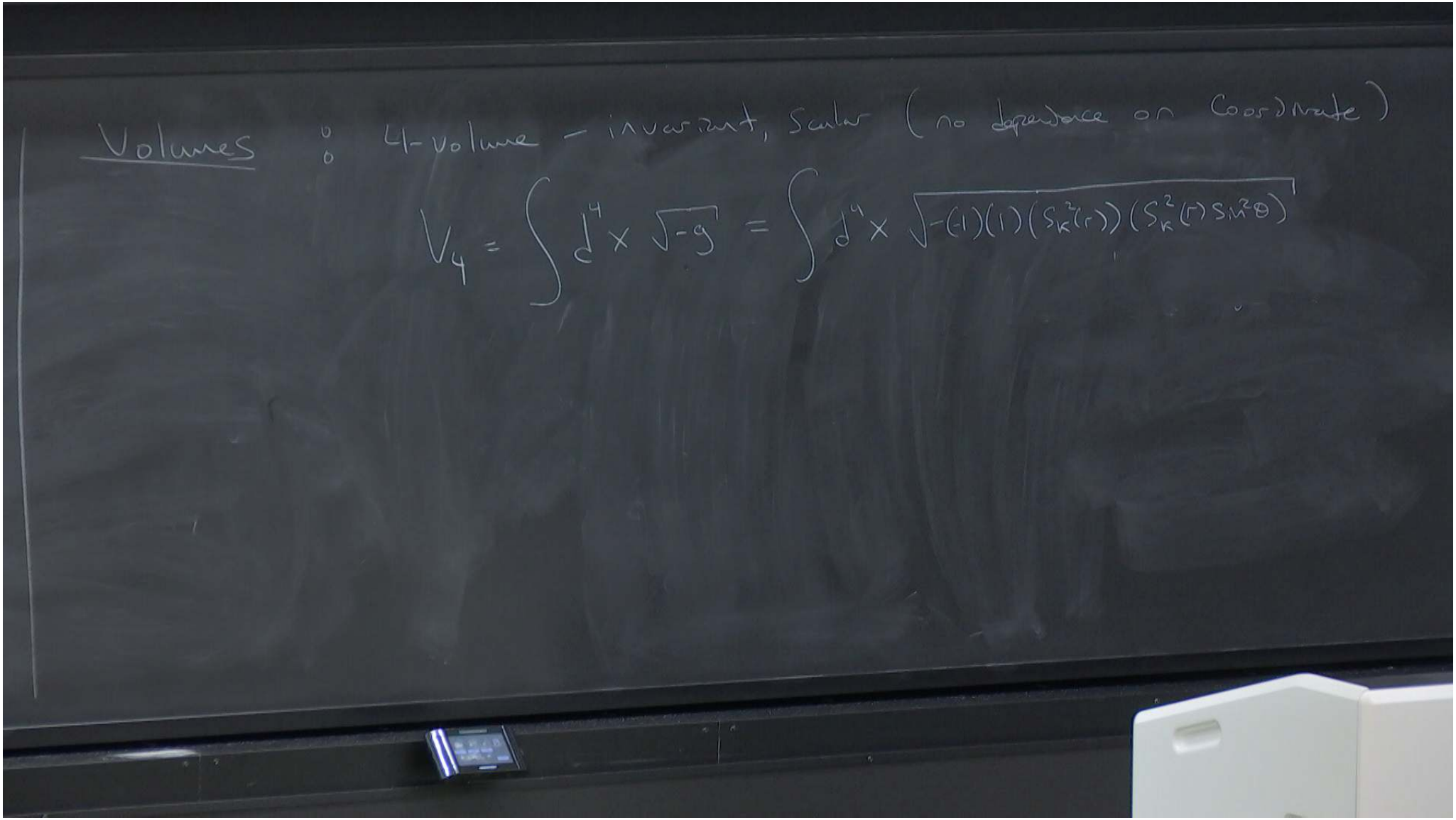
$$\int_0^\gamma d\gamma = \int_0^{2\pi} S_\kappa(r_*)\, d\varphi$$

$$\gamma = 2\pi S_\kappa(r_*)$$

Null Separation



$$ds^2 = 0 = -dt^2 + dr^2 \Rightarrow \frac{dr}{dt} = 1$$


Volumes : 4-volume – invariant, scalar (no dependence on coordinate)
$V_4 = \int d^4x \sqrt{-g} = \int d^4x \sqrt{-(-1)(1)(S_K^2(r))(S_K^2(r) S_M^2\theta)}$

Volumes : 4-volume – invariant, scalar (no dependence on coordinate)

$$V_4 = \int d^4x \sqrt{-g} = \int d^4x \sqrt{-(-1)(1)(S_k^2(r))(S_k^2(r)\sin^2\theta)}$$

$$= \int_0^{\pi} dt\, dr\, d\theta\, d\varphi\, S_k^2(r)\sin\theta$$

3-Volume – coordinate dependent

$$V_3 = \int d^3x \sqrt{g}$$

* Causal Structure

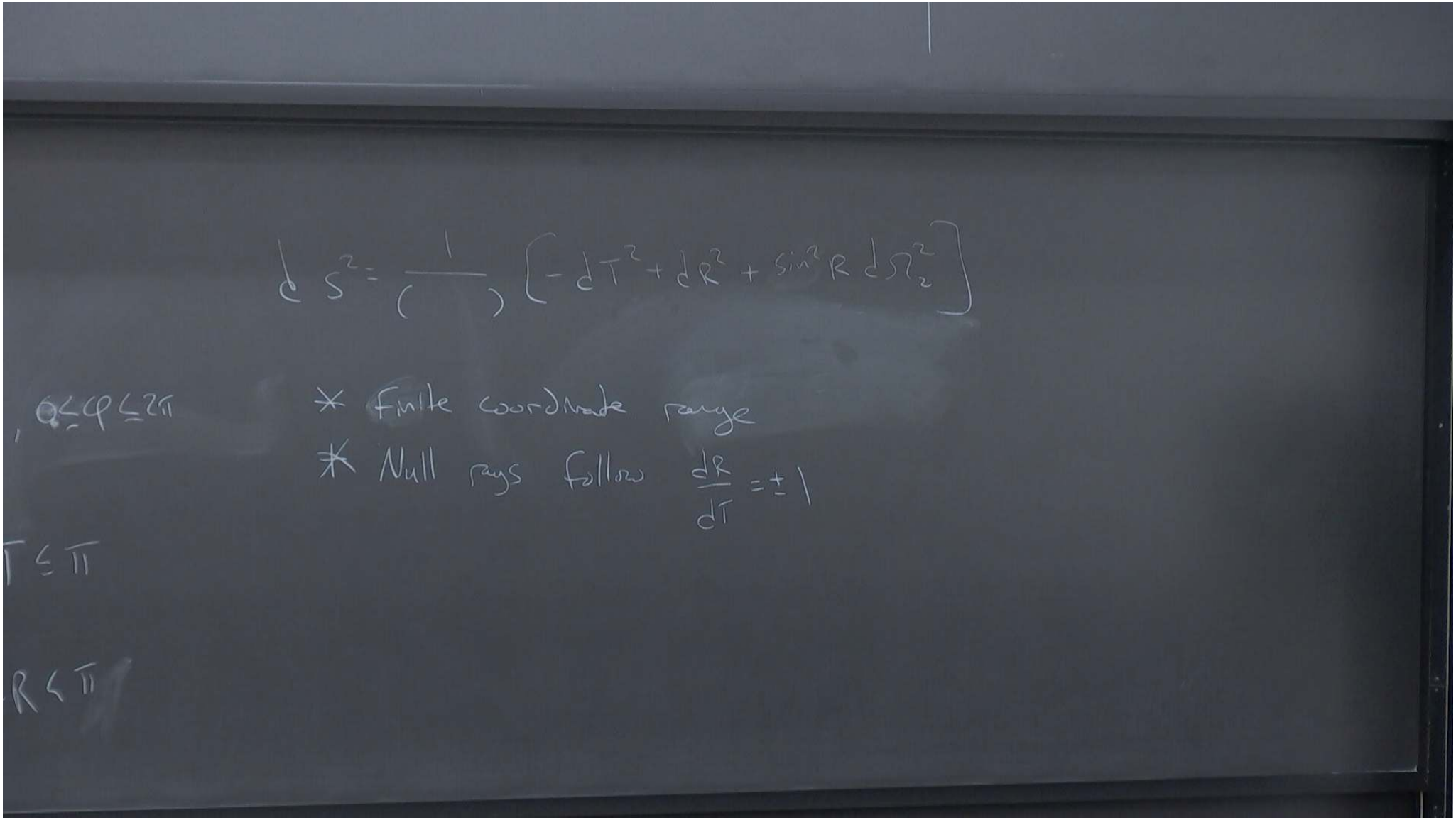
$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$$

$$-\infty < t < \infty \;,\; 0 \le r < \infty \;,\; 0 \le \theta \le \pi \;,\; 0 \le \varphi \le 2\pi$$

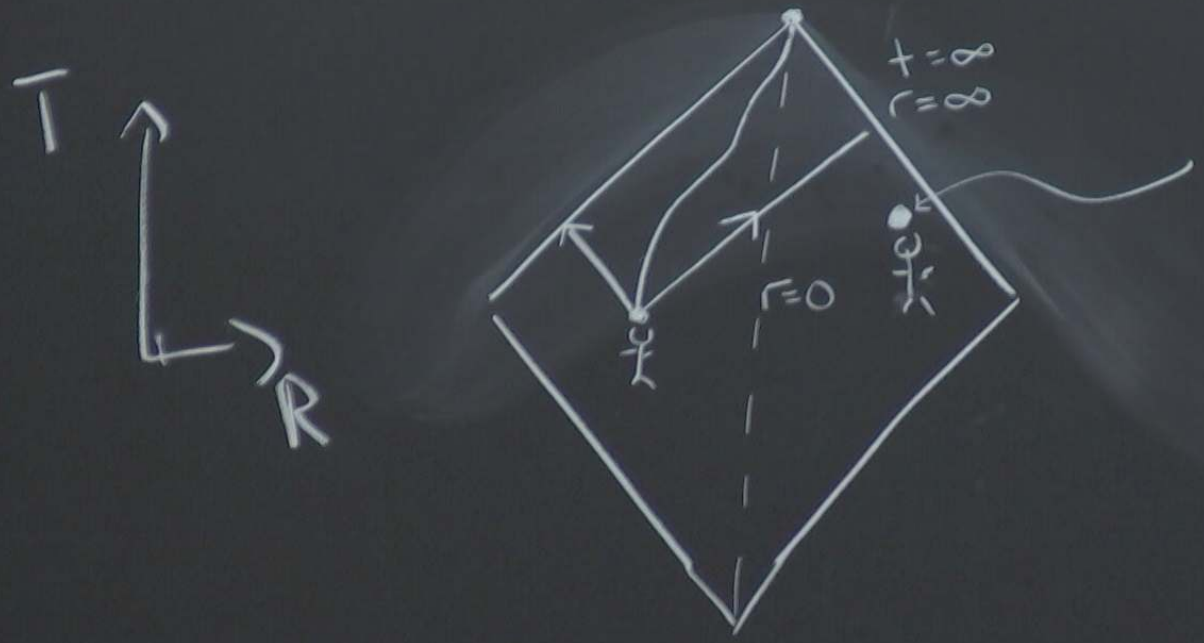
Compression Scheme:

$$T = \tan^{-1}(t+r) + \tan^{-1}(t-r) \;,\; 0 \le T \le \pi$$

$$R = \tan^{-1}(t+r) - \tan^{-1}(t-r) \;,\; |T| + R < \pi$$


$ds^2 = \frac{1}{(\quad)}\left[-dT^2 + dR^2 + \sin^2 R\, d\Omega_2^2\right]$
$0 \leq \varphi \leq 2\pi$
* Finite coordinate range
* Null rays follow $\frac{dR}{dT} = \pm 1$
$T \leq \pi$
$R < \pi$

Conformal diagram



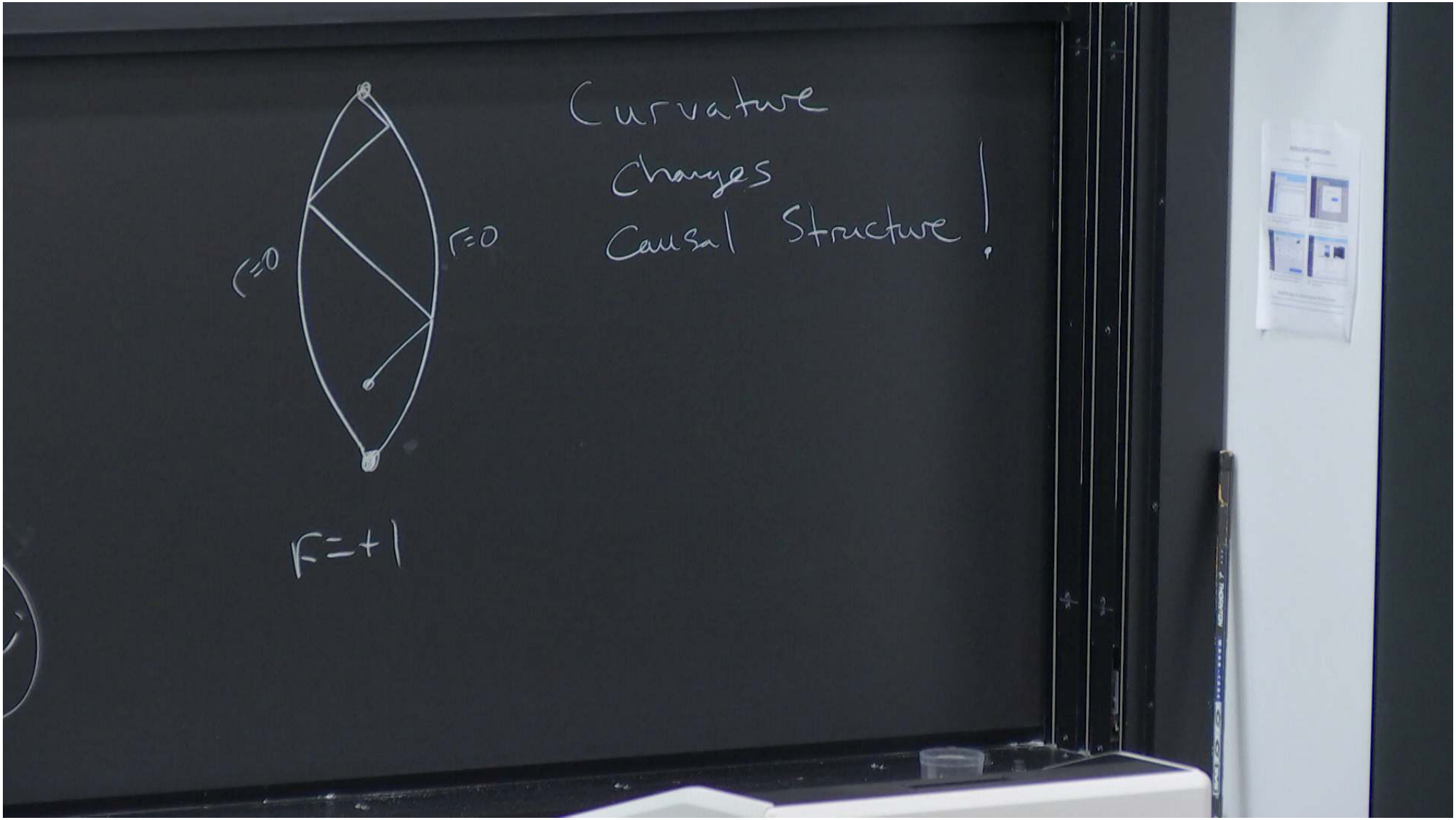
t,

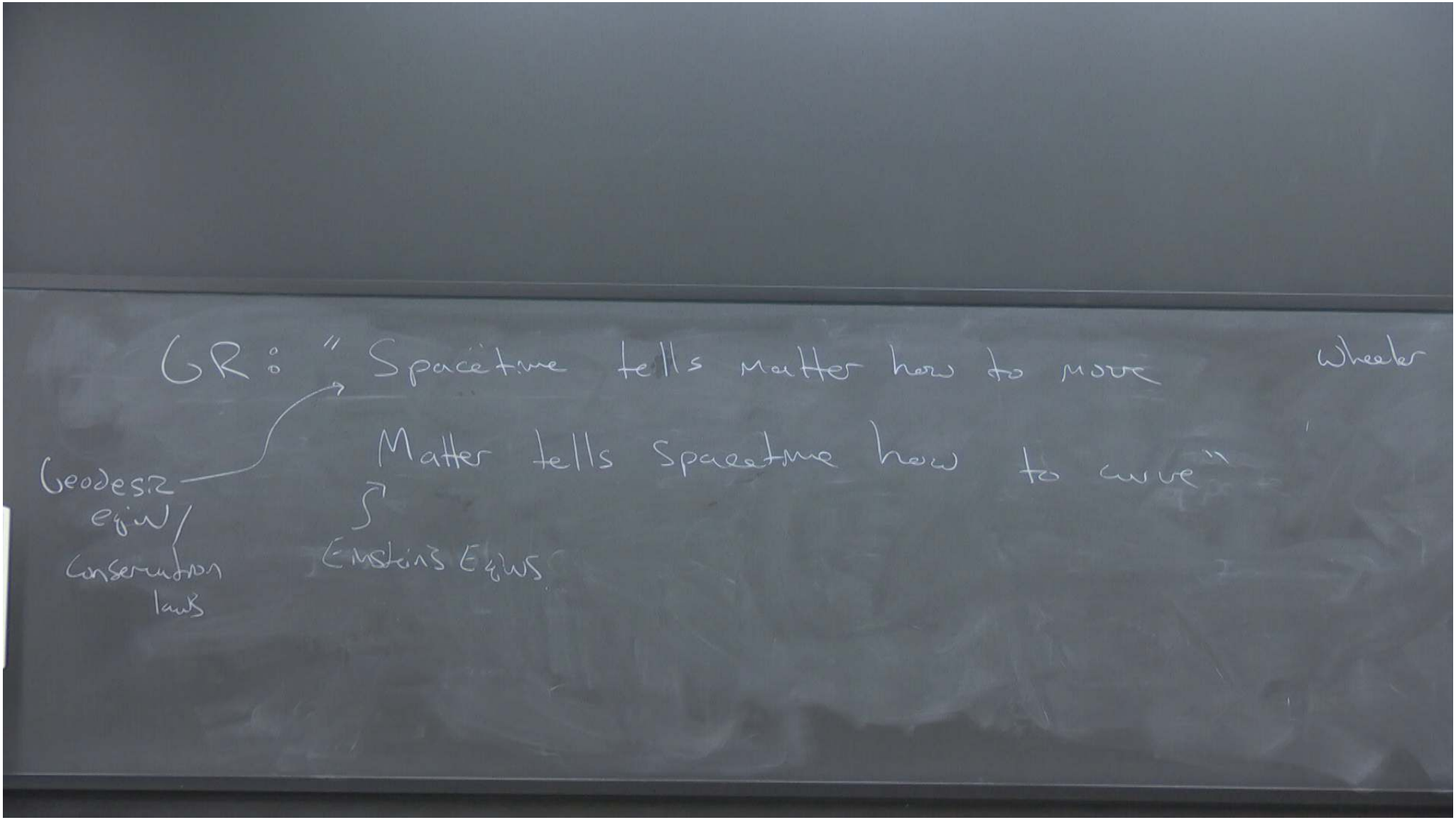
$k = 0, -1$

$K=+1$, $\quad S_K = R \sin(r/R)$, $\quad 0 \le r \le \pi$

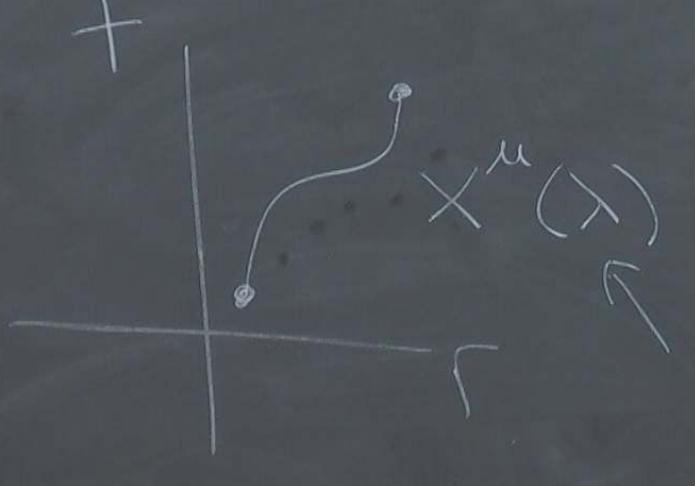
$$ds^2 = -dt^2 + \underbrace{dr^2 + R^2 \sin^2(r/R)\, d\Omega_2^2}_{d\gamma^2 = d\Omega_3^2}$$

$\Rightarrow$ light rays can return to their origin.


r=0
r=0
Curvature
changes
Causal Structure!
k=+1


GR: " Spacetime tells matter how to move
Wheeler
Matter tells Spacetime how to curve "
Geodesic eqn / Conservation laws
Einstein Eqns

Geodesic Eqn : Maximizing Pro



$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu$$

Maximizing Proper time between time-like separated events.

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

comes from $\rightarrow$ $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$

$\downarrow$ Christoffel Symbol

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$$

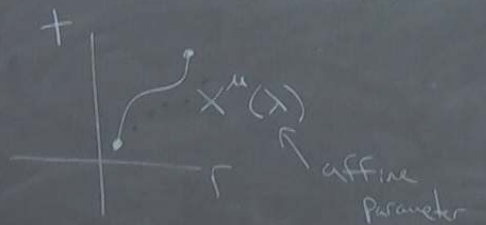
GR: "Spacetime tells Matter how to move — Wheeler

Matter tells Spacetime how to curve"

↑ Einstein Eqns

↑ motion laws

Geodesic Eqn: Maximizing proper time between time-like separated events.



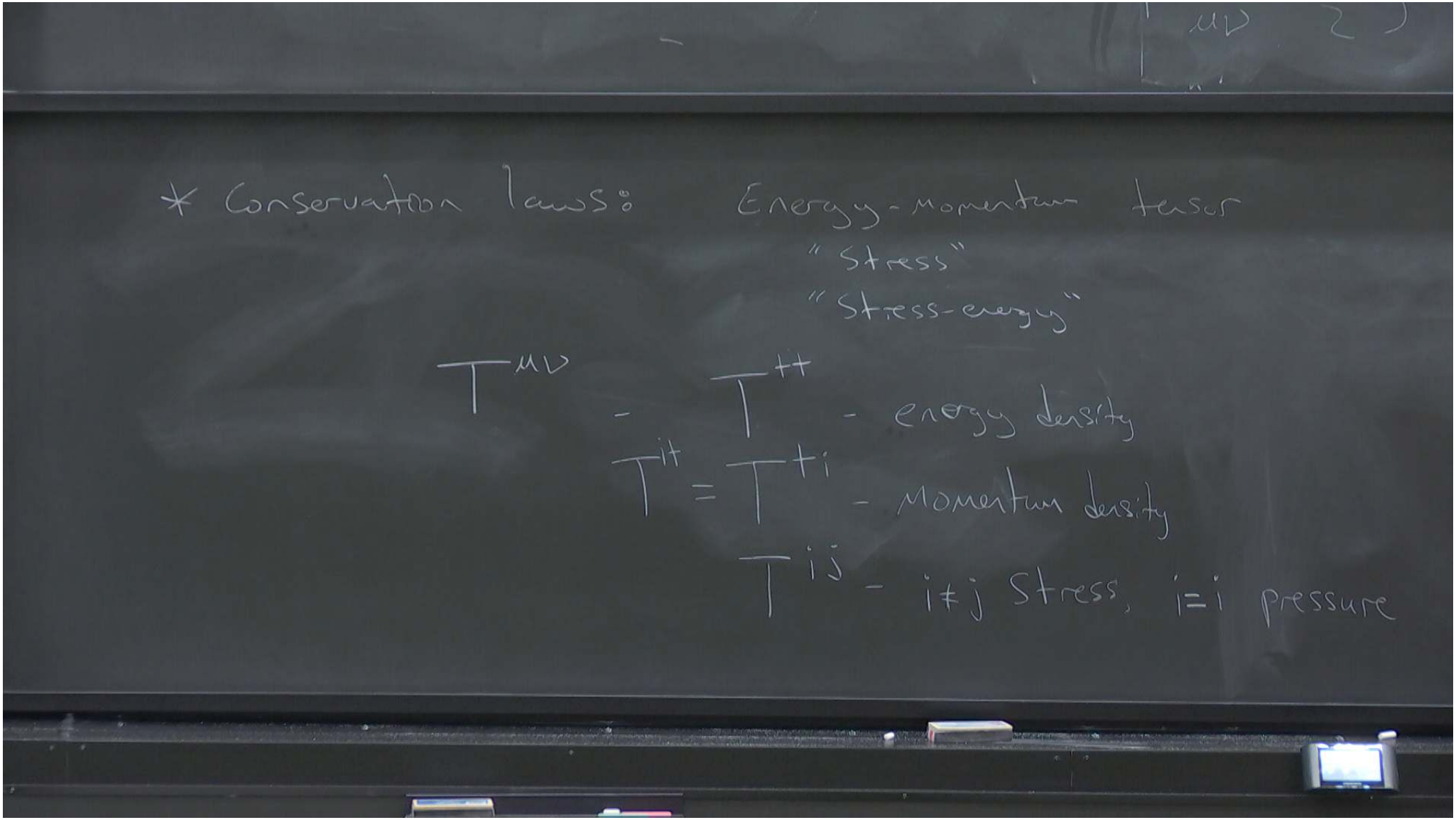
$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

comes from

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

↓ Christoffel Symbol

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$$


* Conservation laws: Energy-momentum tensor
"stress"
"stress-energy"
$T^{\mu\nu}$
T^{tt} - energy density
$T^{it} = T^{ti}$ - momentum density
T^{ij} - i≠j Stress, i=i pressure

Conservation laws:

$$\nabla_\mu T^{\mu\nu} = 0$$

T^{ij} - $i \neq j$ Stress, $i = i$ pressure

* Einstein Eq'ns:

* Curvature:

$$R^{\rho}_{\mu\sigma\nu} = \partial_\sigma \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}$$

$$\sim \partial^2 g \,, (\partial g)^2$$

Ricci tensor:

$$R^{\rho}_{\mu\rho\nu} = R_{\mu\nu} = R_{\nu\mu}$$

4×4 Symmetric - 10

Ricci

i=i pressure

$\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}$

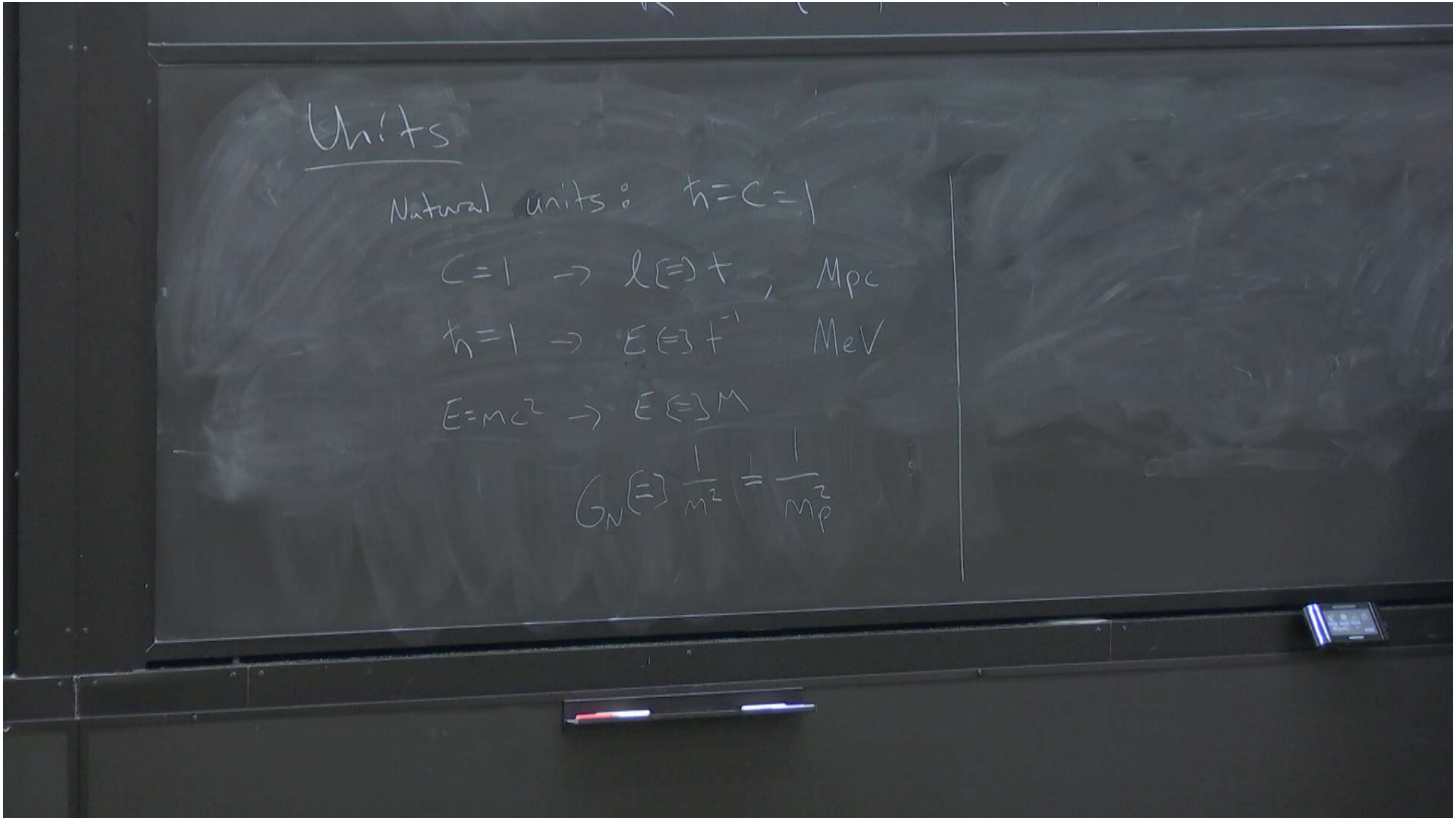
4×4 Symmetric - 10

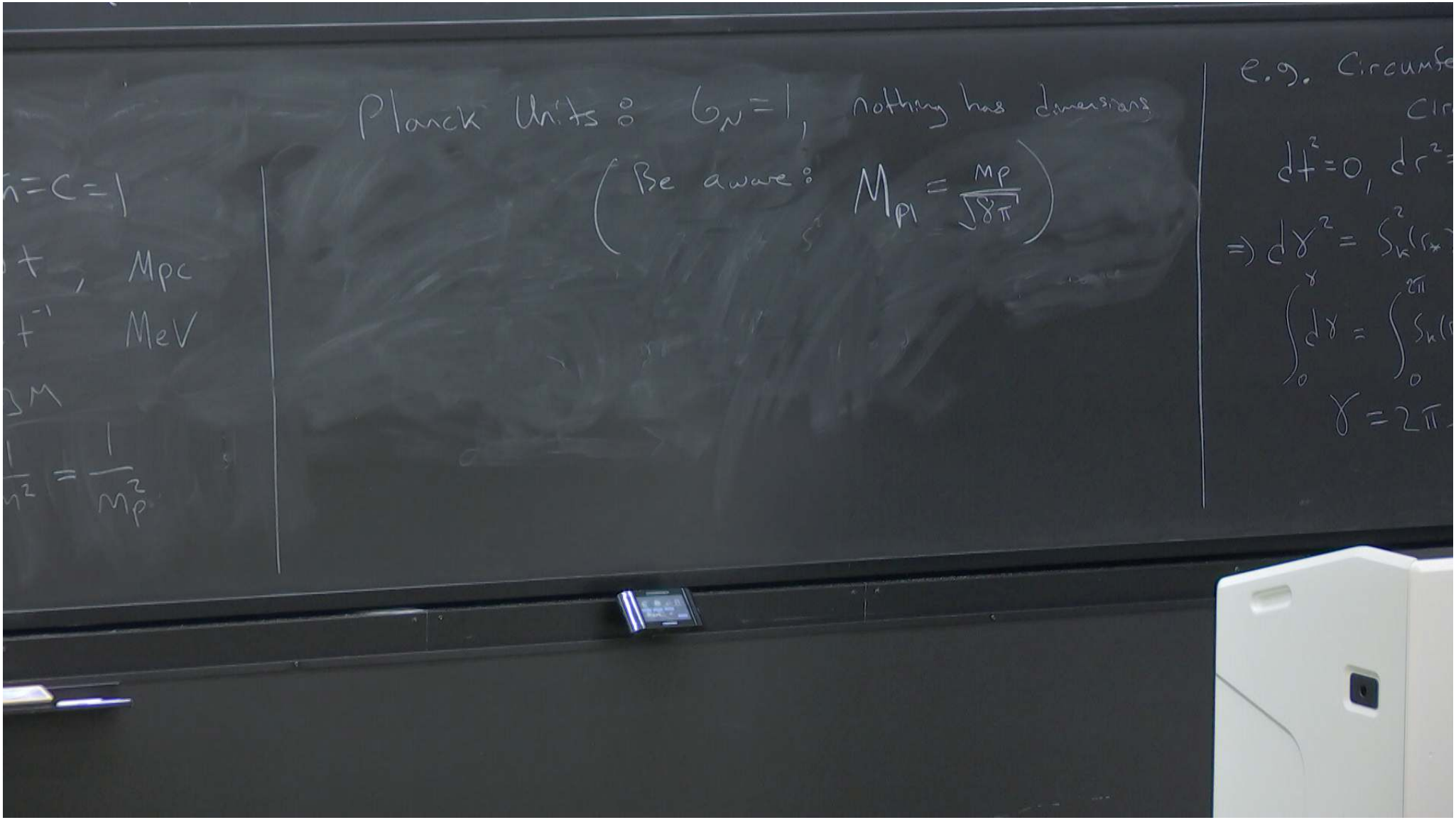
Ricci scalar - $R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}$

Einstein Eqns

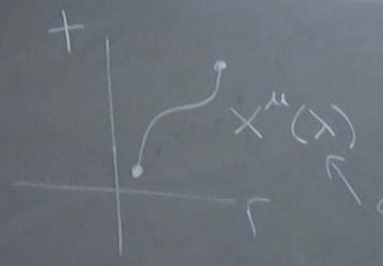
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

10 PDE, non-linear, coupled ....


Units
Natural units: $\hbar = c = 1$
$c = 1 \rightarrow l \sim t$, Mpc
$\hbar = 1 \rightarrow E \sim t^{-1}$ MeV
$E = mc^2 \rightarrow E \sim M$
$G_N \sim \frac{1}{M^2} = \frac{1}{M_p^2}$


Planck Units: $G_N = 1$, nothing has dimensions
(Be aware: $M_{Pl} = \frac{M_P}{\sqrt{8\pi}}$)
$\hbar = c = 1$
t, Mpc
t^{-1} MeV
$\frac{1}{M^2} = \frac{1}{M_P^2}$
e.g. Circumfe
$dt^2 = 0$, dr^2
$\Rightarrow d\gamma^2 = S_k^2(r$
$\int_0^{\gamma} d\gamma = \int_0^{2\pi} S_k($
$\gamma = 2\pi$

- Maximizing proper time between time-like separa



$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

comes from $\nabla_\mu V^\nu$

$\Gamma^\mu_{\rho\sigma}$ ↓ Christoffel Symbol

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$$

FRW Universes – homogeneous + isotropic, evolves in time

spatial trans. 3 · rotations 3 · ~~time trans.~~ −4, ↙T

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + S_k^2(r)\,d\Omega_2^2\right]$$

dt: coordinate time; $a(t)$: scale factor

$$= a^2(\tau)\left[-d\tau^2 + dr^2 + S_k^2(r)\,d\Omega_2^2\right] \qquad \left(\tau = \int \frac{dt}{a(t)}\right)$$

$d\tau$: conformal time

Conformal time

Cartesian Coordinates:

$$ds^2 = a^2(\tau)\left[-d\tau^2 + \frac{dx^2+dy^2+dz^2}{\left(1+\frac{k(x^2+y^2+z^2)}{4R^2}\right)^2}\right]$$