Title: PSI 2019/2020 - QFT III - Lecture 12

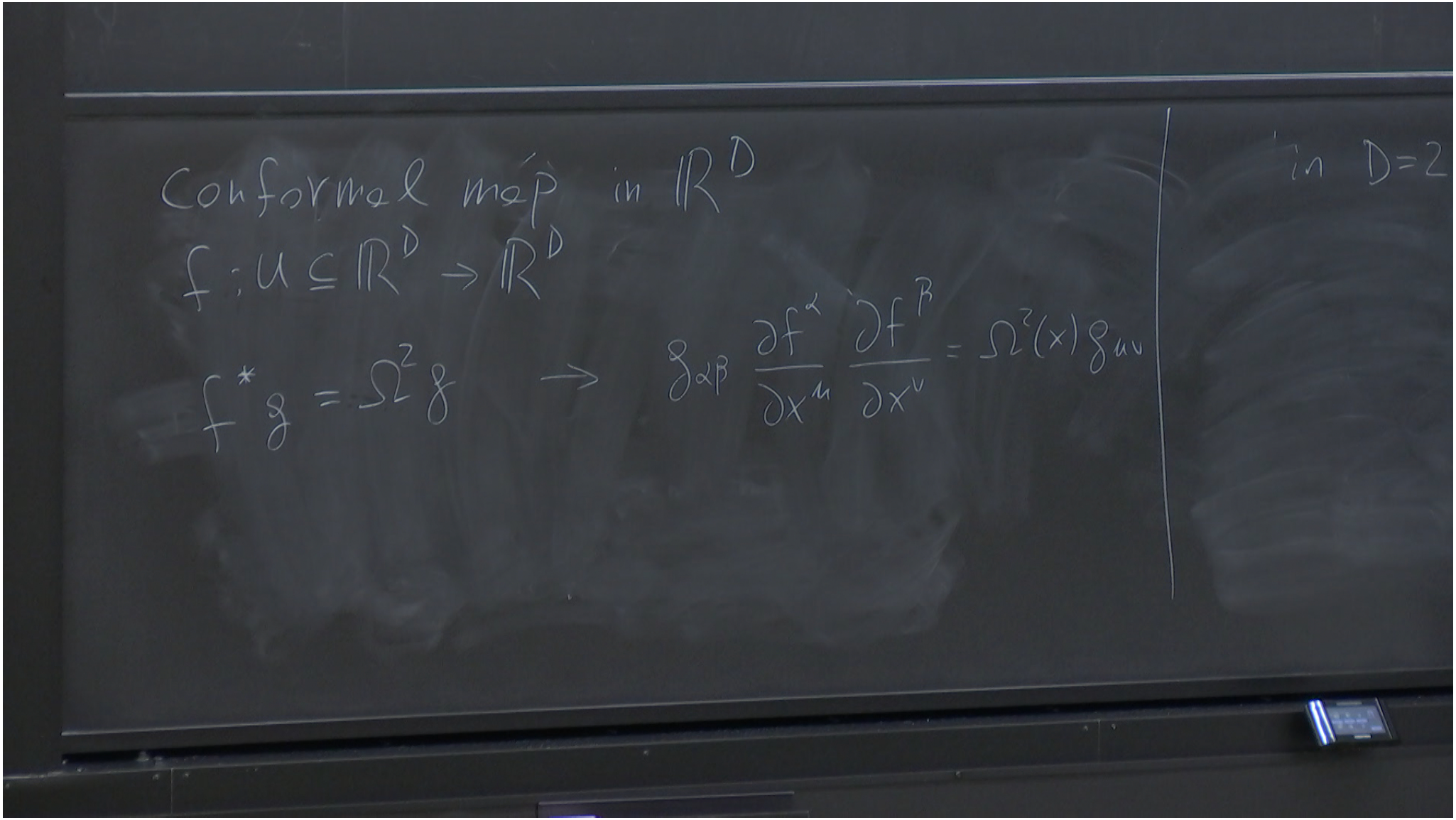
Speakers: Jaume Gomis

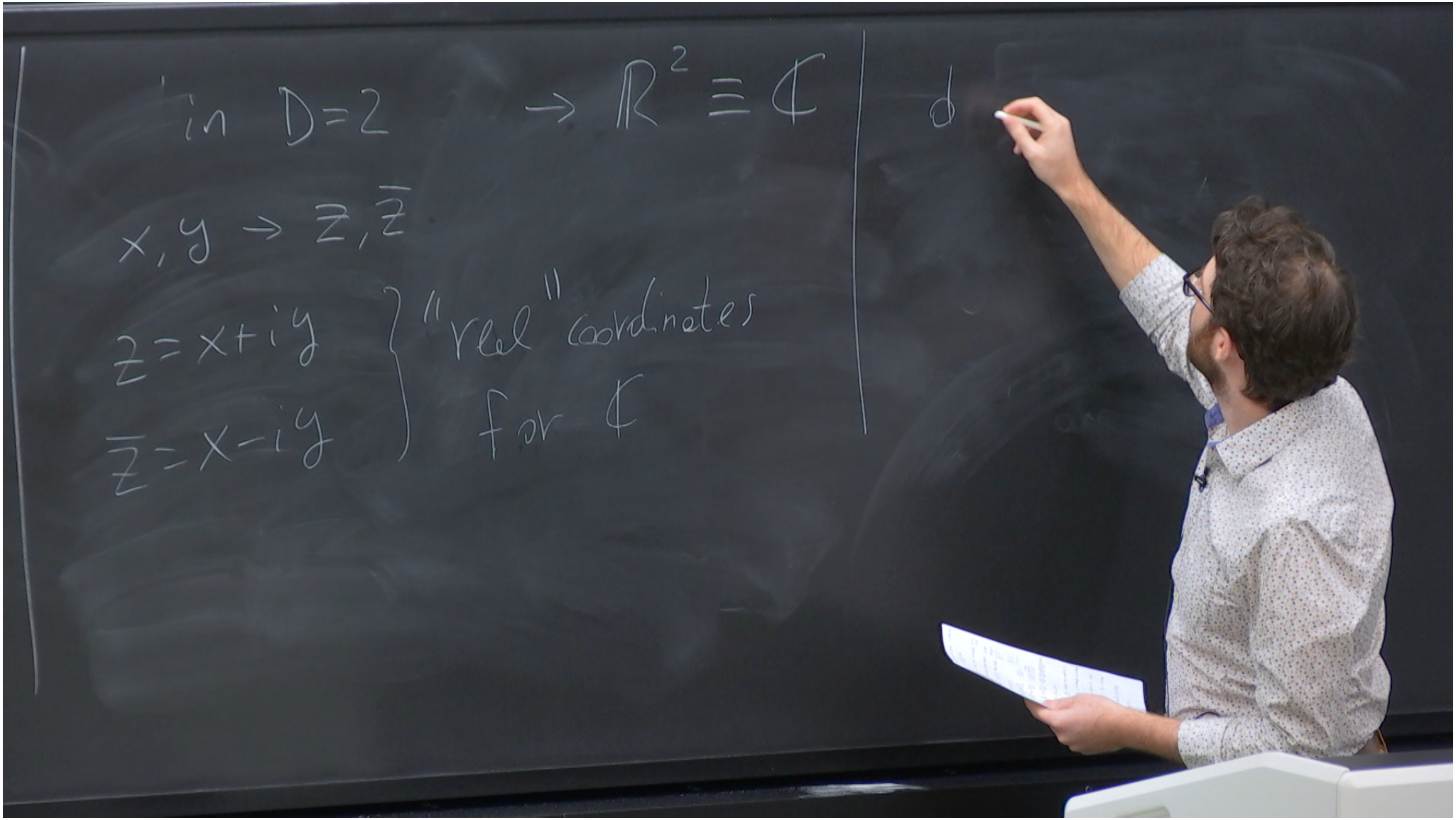
Collection: PSI 2019/2020 - QFT III

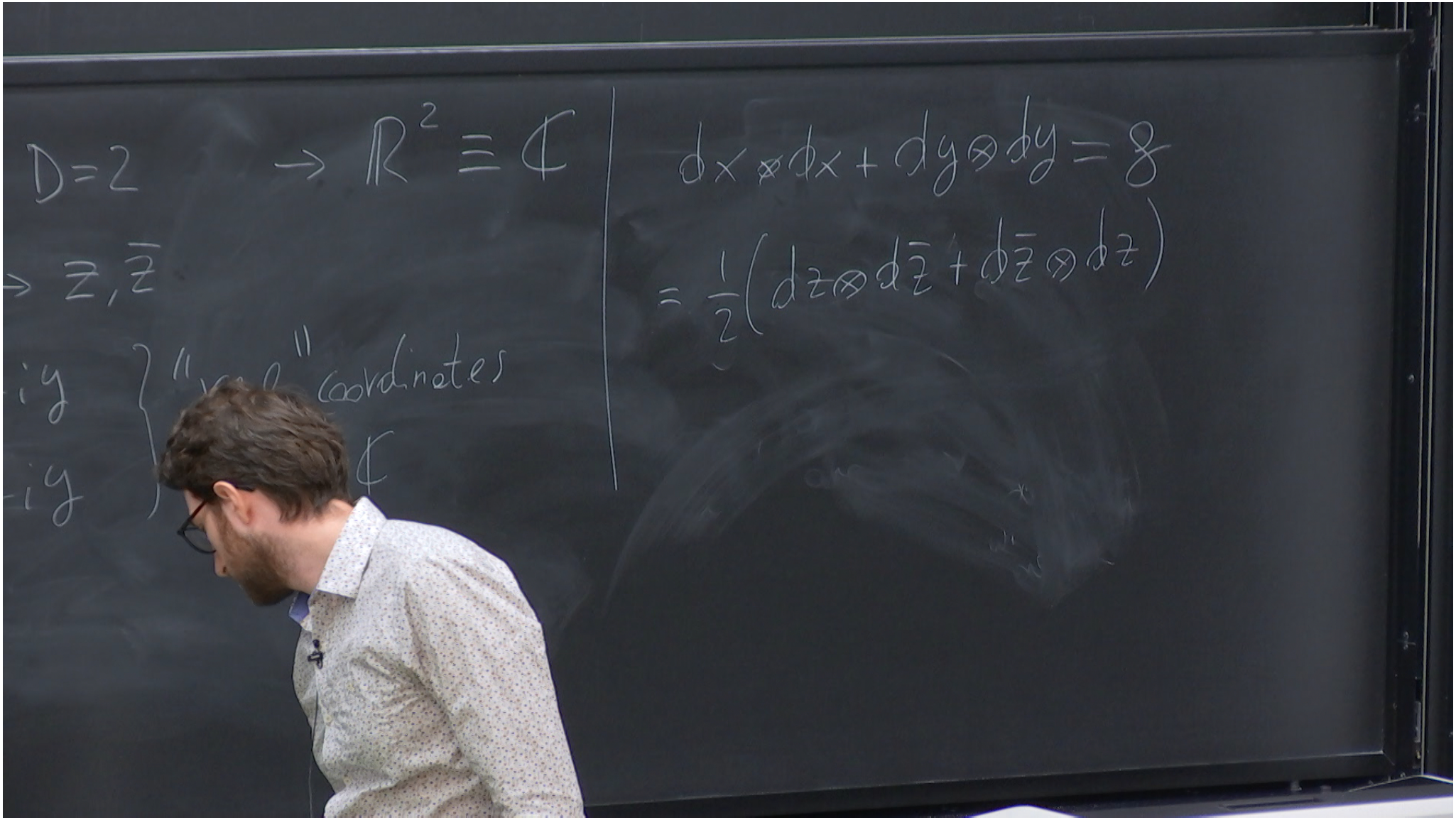
Date: February 28, 2020 - 11:15 AM

URL: http://pirsa.org/20020038

- Conformal maps in $\mathbb{R}^2$ ($D=2$, Euclidean)
- Conformal group of $\mathbb{R}^2$
- Local and global conformal algebra
- Primary and quasi-primary operators


Conformal map in $\mathbb{R}^D$
$f: U \subseteq \mathbb{R}^D \to \mathbb{R}^D$
$f^* g = \Omega^2 g \quad \to \quad g_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu} = \Omega^2(x) g_{\mu\nu}$
in $D=2$


in D=2 → $\mathbb{R}^2 \equiv \mathbb{C}$
x, y → z, $\bar{z}$
z = x + iy
$\bar{z}$ = x − iy
"real" coordinates for $\mathbb{C}$


D=2 → $\mathbb{R}^2 \equiv \mathbb{C}$
→ z, $\bar{z}$
coordinates
$dx \otimes dx + dy \otimes dy = g$
$= \frac{1}{2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$

$D=2 \quad \to \mathbb{R}^2 \equiv \mathbb{C}$

$\to z, \bar{z}$

$-iy$ } "real" coordinates
$-iy$ } for $\mathbb{C}$

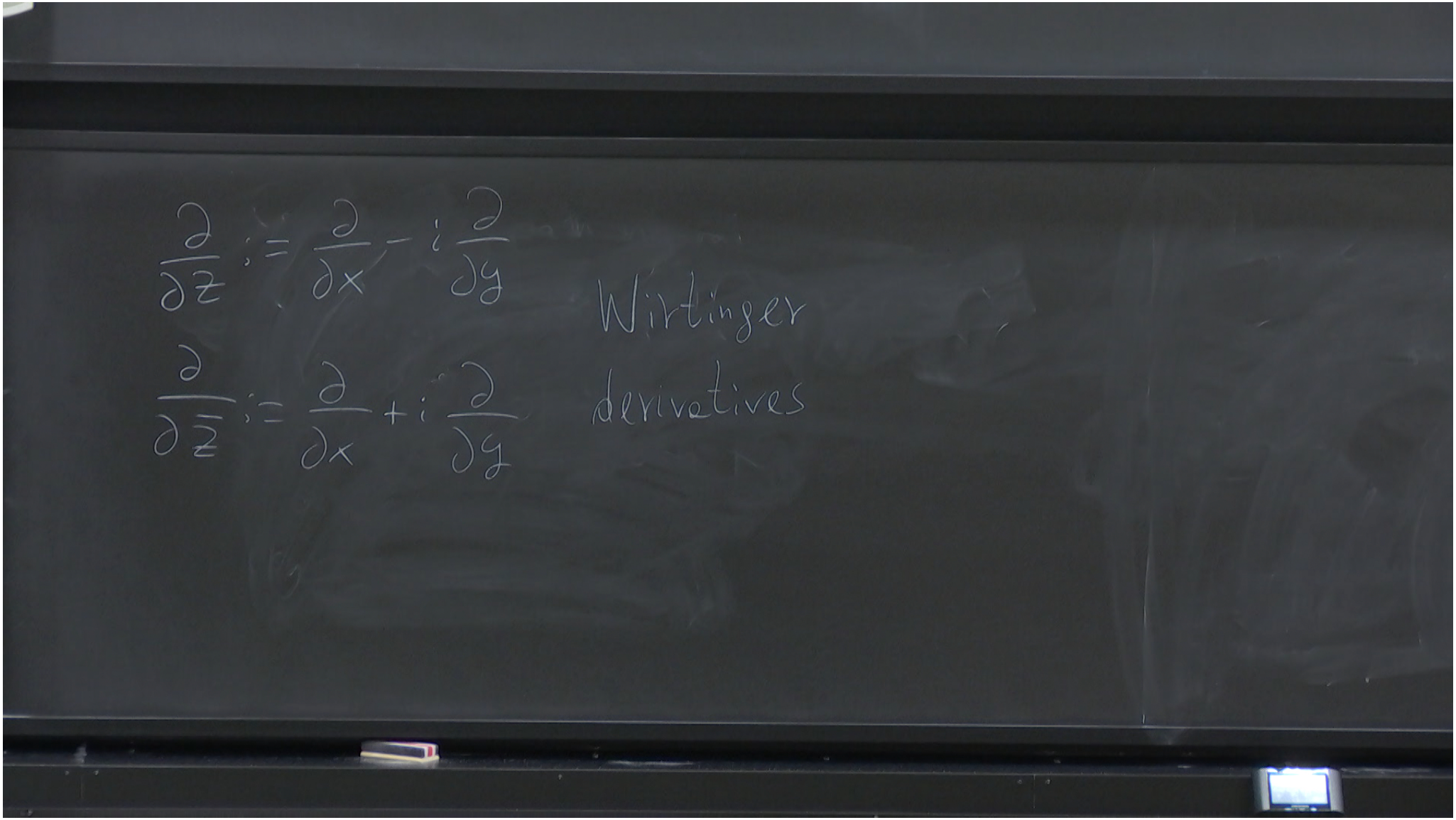
$dx \otimes dx + dy \otimes dy = g$

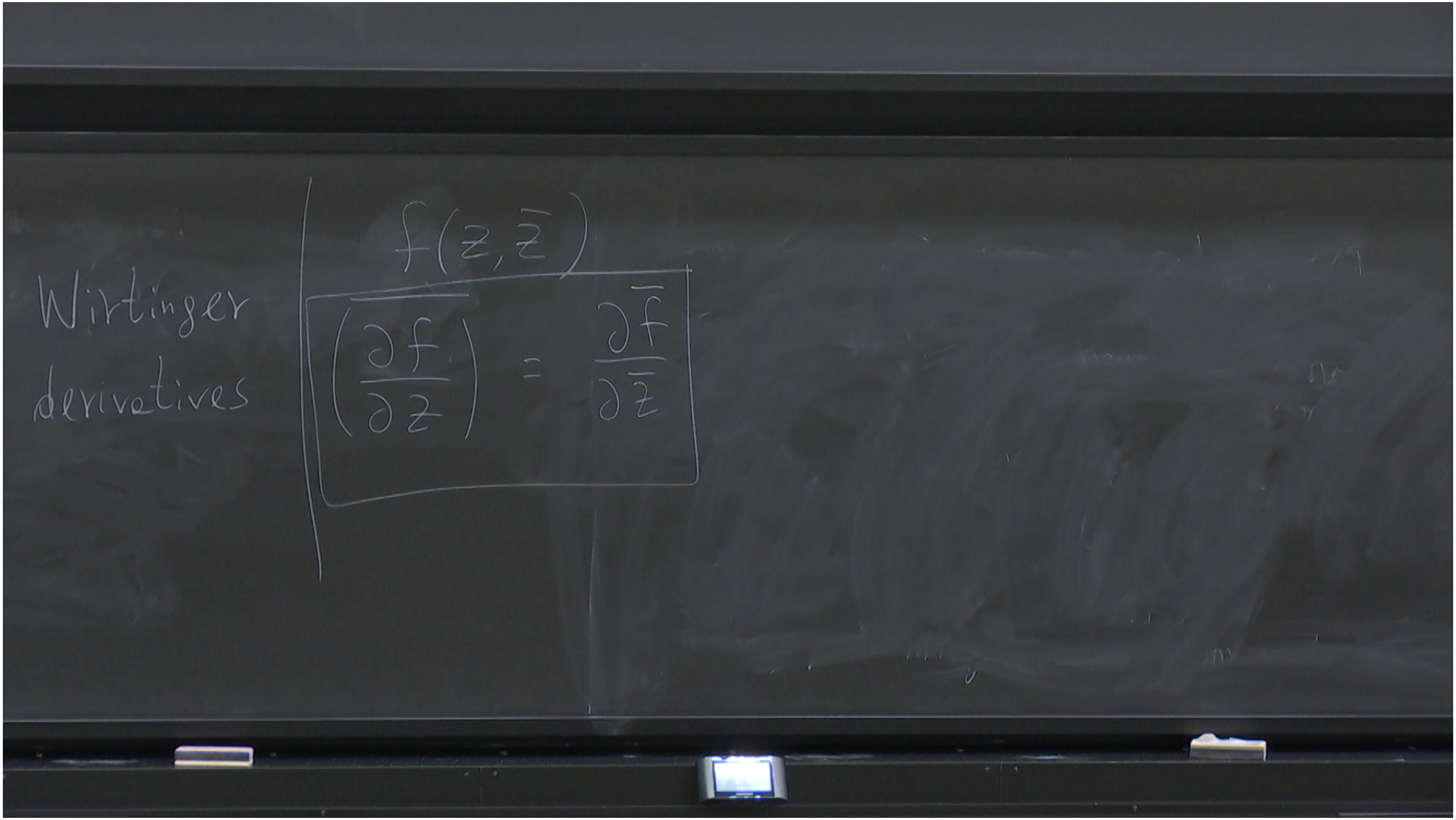
$= \frac{1}{2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$

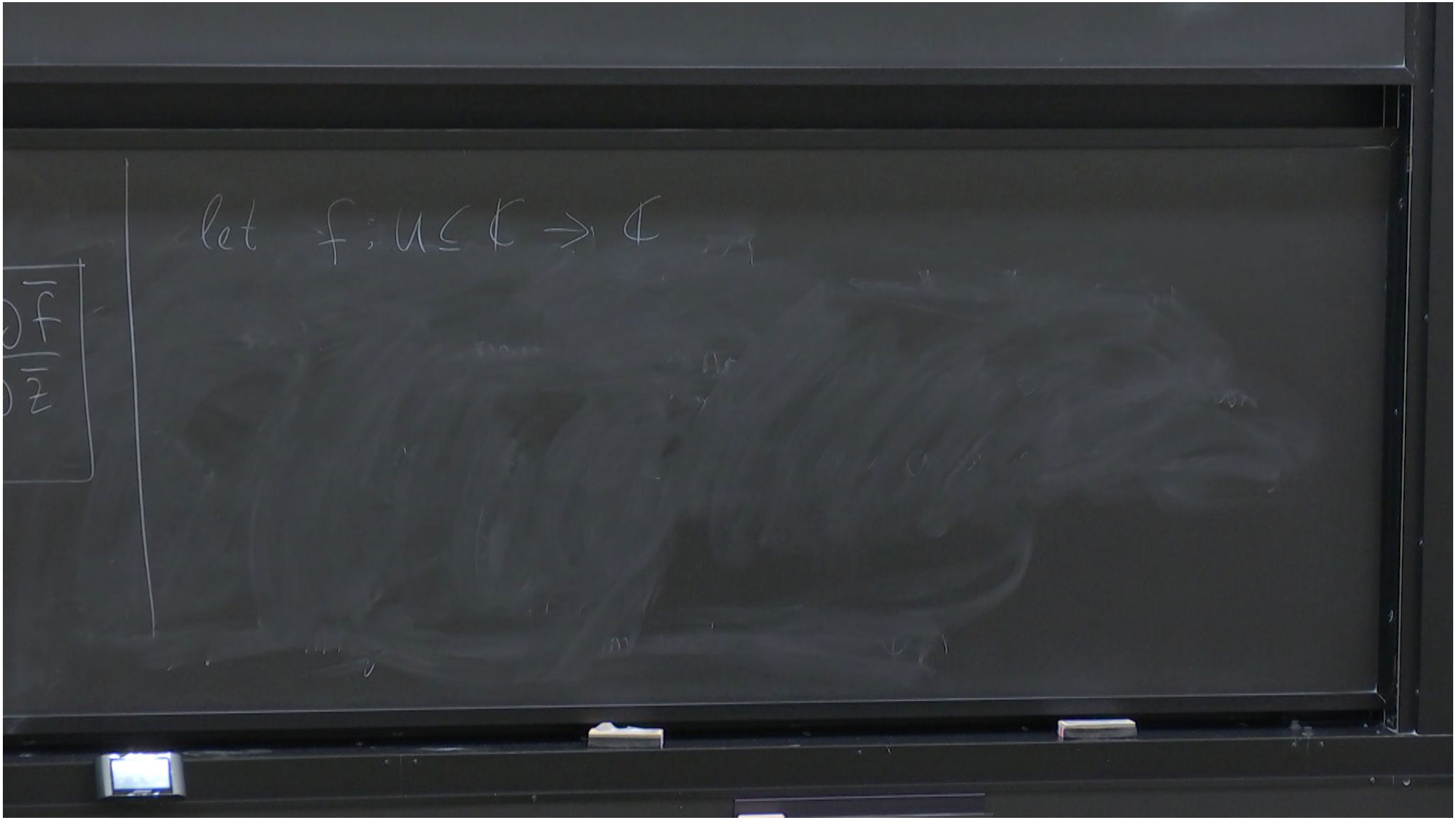
$g_{zz} = 0 = g_{\bar{z}\bar{z}}$

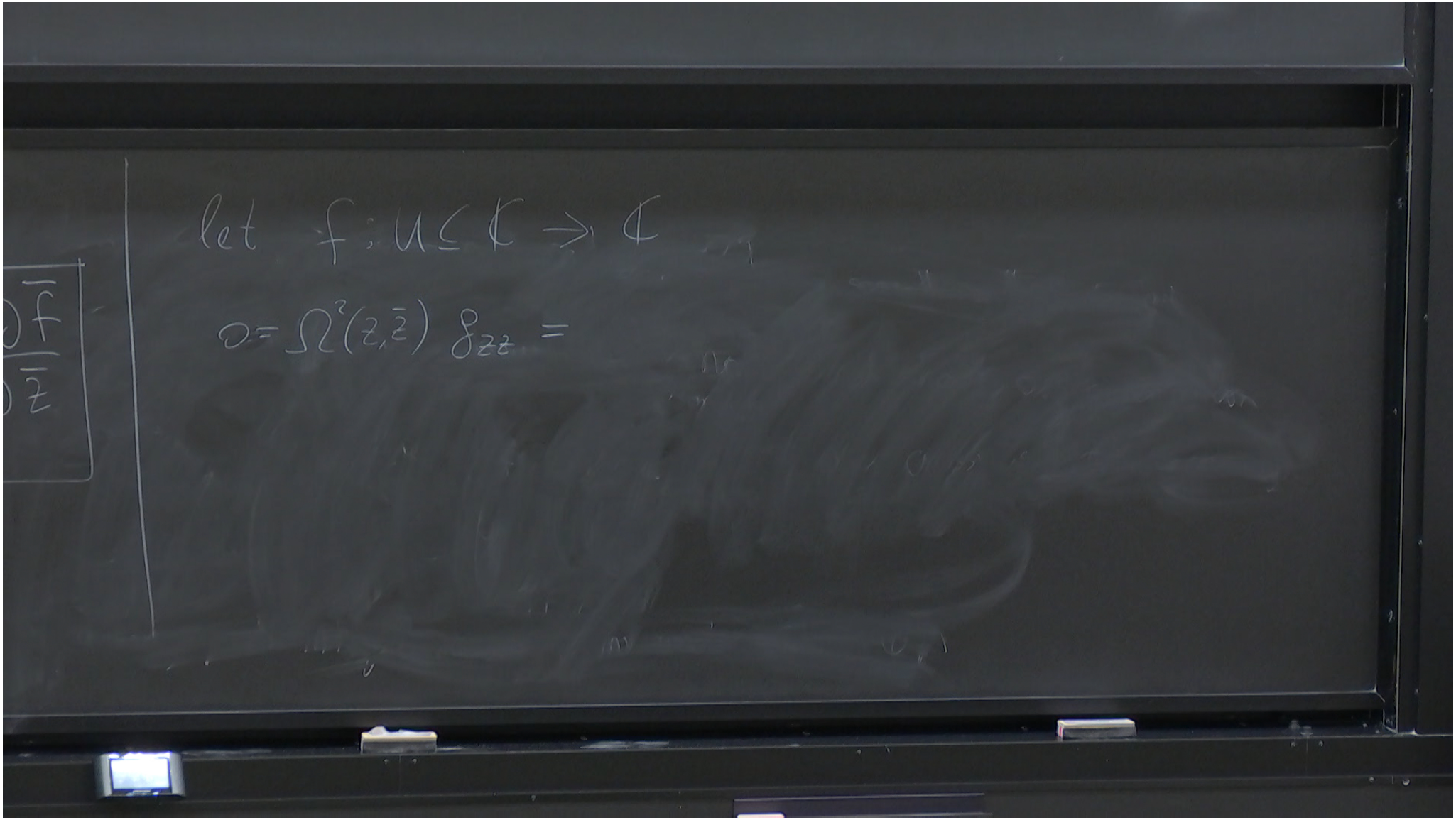
$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$

$\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$


$\frac{\partial}{\partial z} := \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$
$\frac{\partial}{\partial \bar{z}} := \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$
Wirtinger
derivatives


Wirtinger
derivatives
$f(z,\bar{z})$
$\left(\frac{\partial f}{\partial z}\right) = \frac{\partial f}{\partial \bar{z}}$


let f : U ⊆ ℂ → ℂ


let $f: U \subseteq \mathbb{C} \to \mathbb{C}$
$g = \Omega^2(z,\bar{z})\, \delta_{z\bar{z}} =$

let $f : U \subseteq \mathbb{C} \to \mathbb{C}$

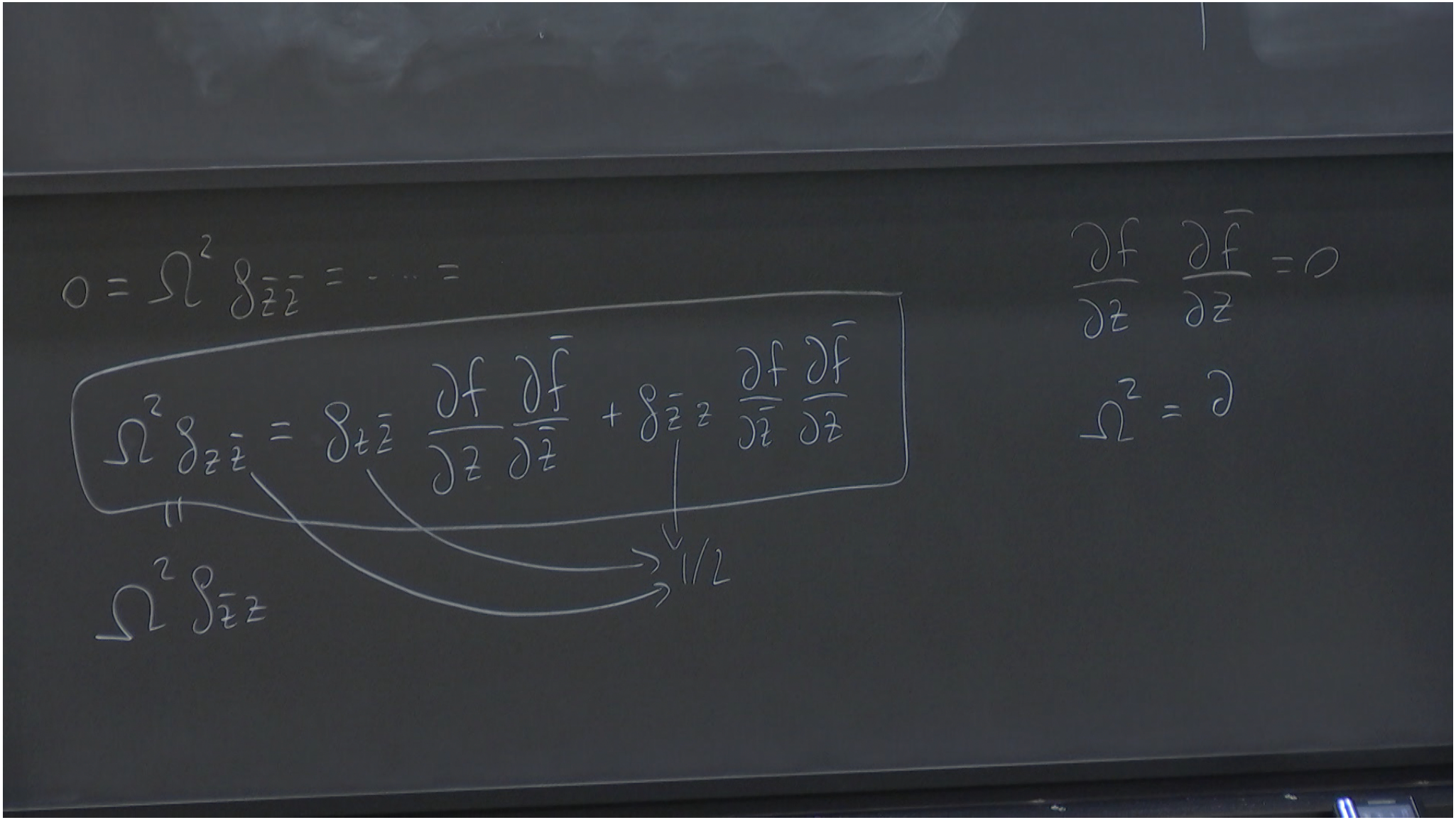
$$0 = \Omega^2(z,\bar{z})\, g_{zz} = g_{z\bar{z}} \frac{\partial f}{\partial z} \frac{\partial \bar{f}}{\partial z} + g_{\bar{z}z} \frac{\partial \bar{f}}{\partial z} \frac{\partial f}{\partial z}$$

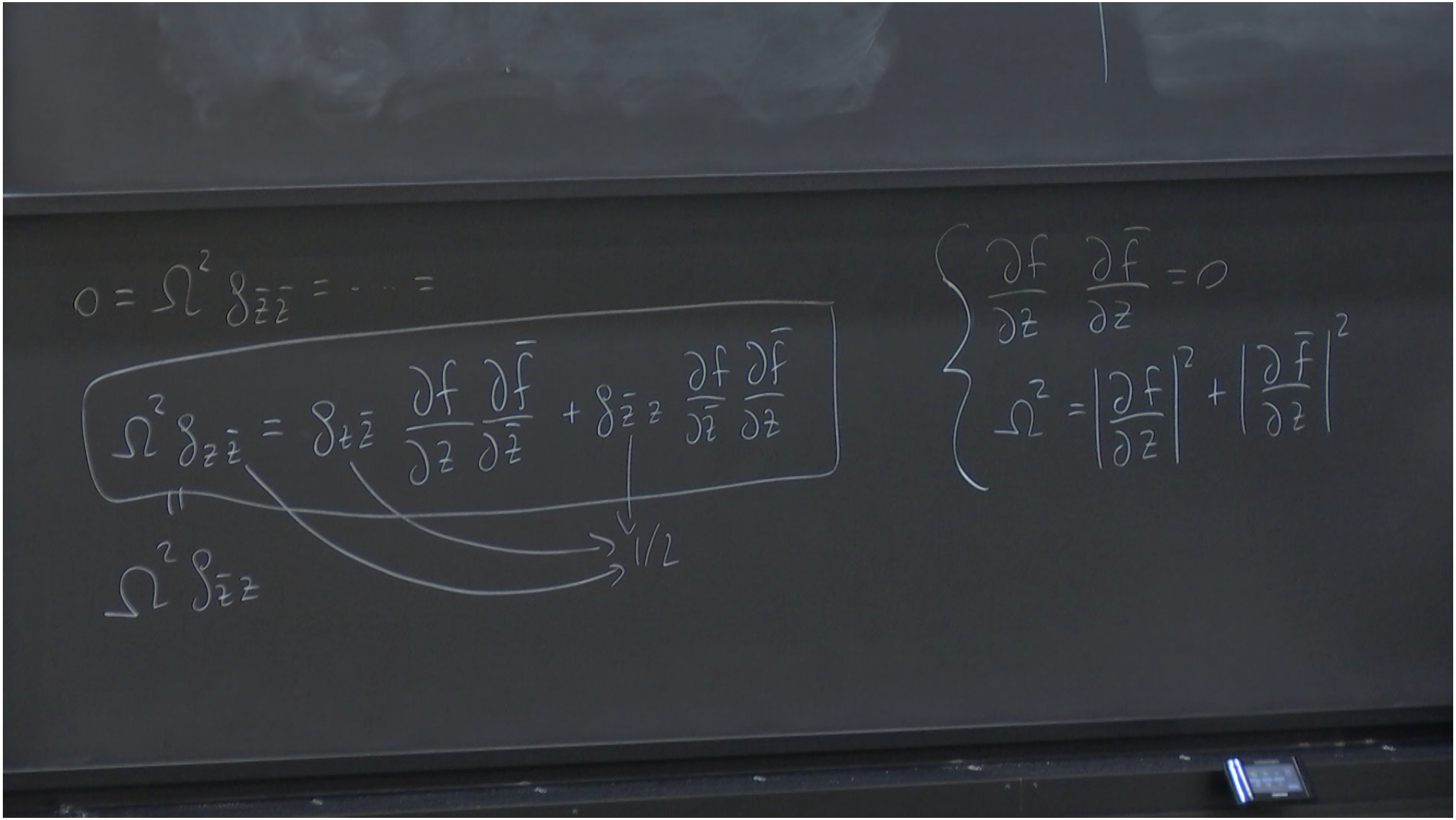
$$\rightarrow \boxed{\frac{\partial f}{\partial z} \frac{\partial \bar{f}}{\partial z} = 0}$$

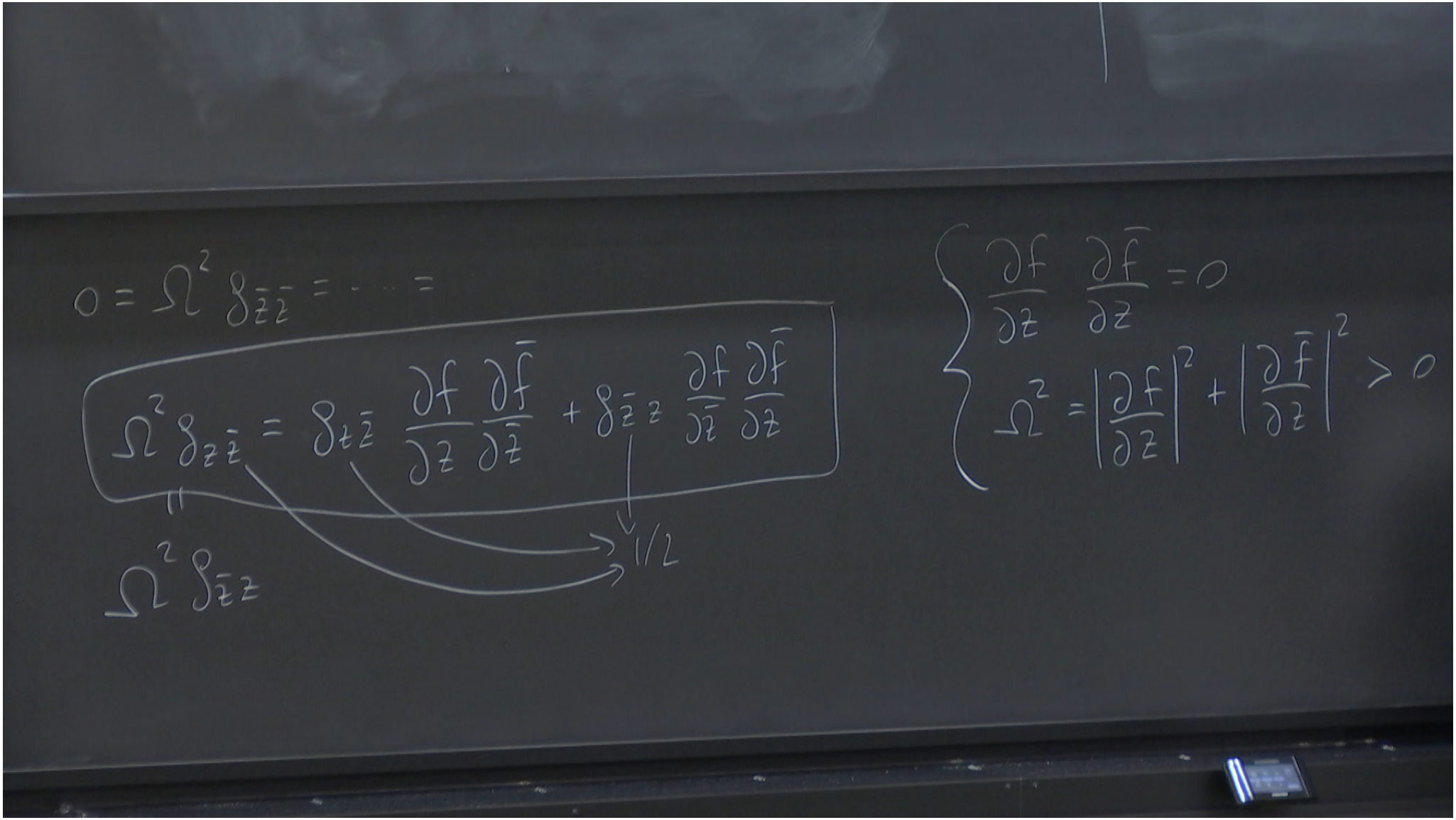
$$0 = \Omega^2 g_{\bar{z}\bar{z}} = \dots =$$

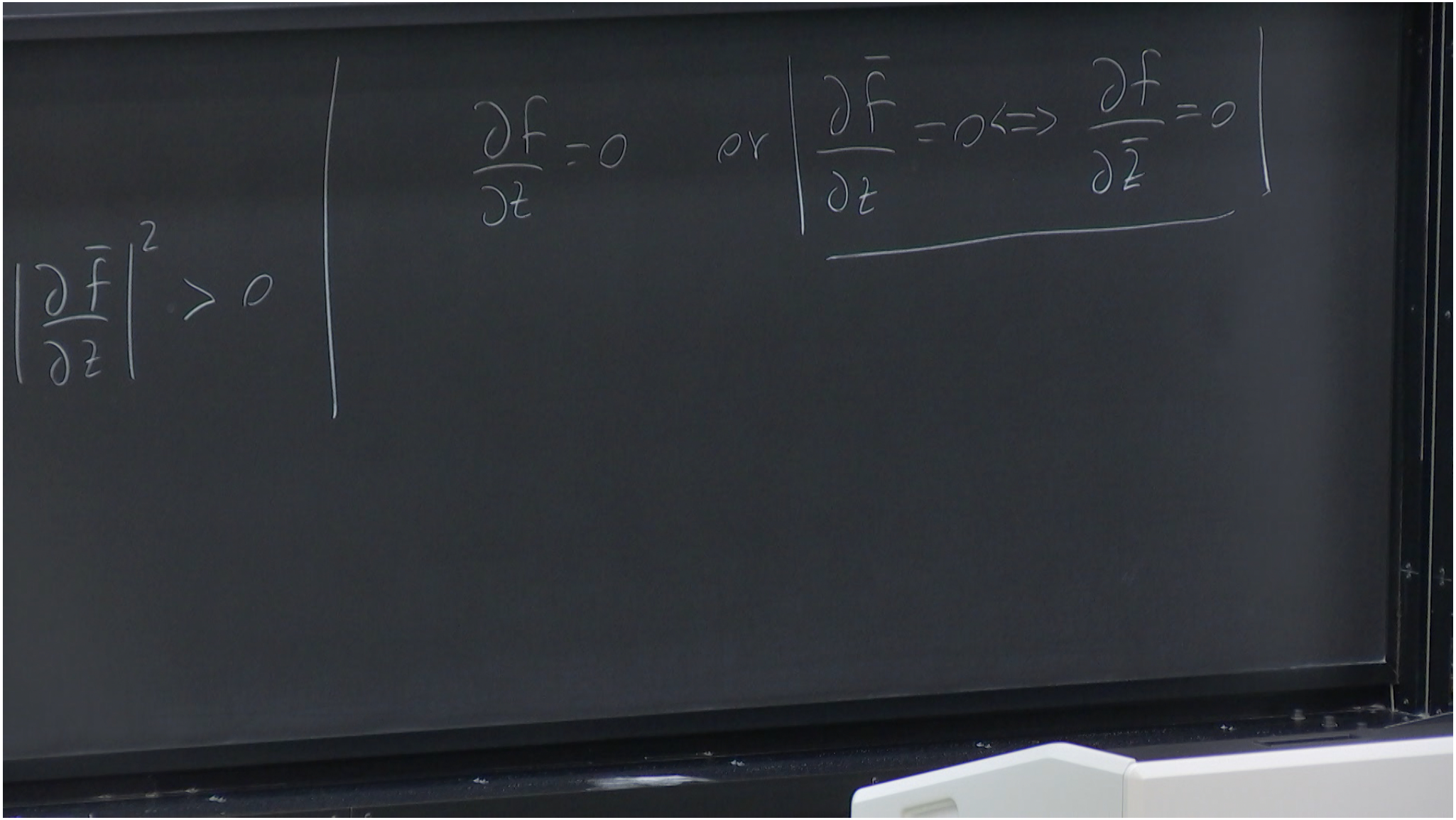
$$\Omega^2 g_{z\bar{z}} = g_{z\bar{z}} \frac{\partial f}{\partial z} \frac{\partial \bar{f}}{\partial \bar{z}} + g_{\bar{z}z} \frac{\partial f}{\partial \bar{z}} \frac{\partial \bar{f}}{\partial z}$$

$$\Omega^2 g_{\bar{z}z}$$


$0 = \Omega^2 g_{\bar{z}\bar{z}} = \dots =$
$\Omega^2 g_{z\bar{z}} = g_{z\bar{z}} \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} + g_{\bar{z}z}\frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z}$
$\Omega^2 g_{\bar{z}z}$
1/2
$\frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial z} = 0$
$\Omega^2 = \partial$


$0 = \Omega^2 g_{\bar{z}\bar{z}} = \dots =$
$\Omega^2 g_{z\bar{z}} = g_{z\bar{z}} \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} + g_{\bar{z}z}\frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z}$
$\Omega^2 g_{\bar{z}z}$
1/2
$\begin{cases} \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial z} = 0 \\ \Omega^2 = \left|\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\partial \bar{f}}{\partial z}\right|^2 \end{cases}$


$0 = \Omega^2 g_{\bar{z}\bar{z}} = \dots =$
$\Omega^2 g_{z\bar{z}} = g_{z\bar{z}} \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} + g_{\bar{z}z}\frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z}$
$\Omega^2 g_{\bar{z}z}$
1/2
$\frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial z} = 0$
$\Omega^2 = \left|\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\partial \bar{f}}{\partial z}\right|^2 > 0$


$\left|\frac{\partial \bar{f}}{\partial z}\right|^2 > 0$
$\frac{\partial f}{\partial z} = 0$ or $\frac{\partial \bar{f}}{\partial z} = 0 \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0$

$\frac{\partial \bar{f}}{\partial z} = 0$

$\left|\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\partial \bar{f}}{\partial z}\right|^2 > 0$

$\frac{\partial f}{\partial \bar{z}} = 0$ or $\frac{\partial \bar{f}}{\partial z} = 0 \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0$

$\downarrow$ anti-holomorphic

$\downarrow$ holomorphic

$\rightarrow$ f is locally invertible



$$\begin{cases} \dfrac{\partial f}{\partial z}\dfrac{\partial \bar{f}}{\partial z} = 0 \\ \Omega^2 = \left|\dfrac{\partial f}{\partial z}\right|^2 + \left|\dfrac{\partial \bar{f}}{\partial z}\right|^2 > 0 \end{cases}$$

$\Omega^2 > 0$ → f is locally invertible

$\dfrac{\partial f}{\partial z} = 0$ → anti-holomorphic

or

$\dfrac{\partial \bar{f}}{\partial z} = 0 \Leftrightarrow \dfrac{\partial f}{\partial \bar{z}} = 0$ → holomorphic

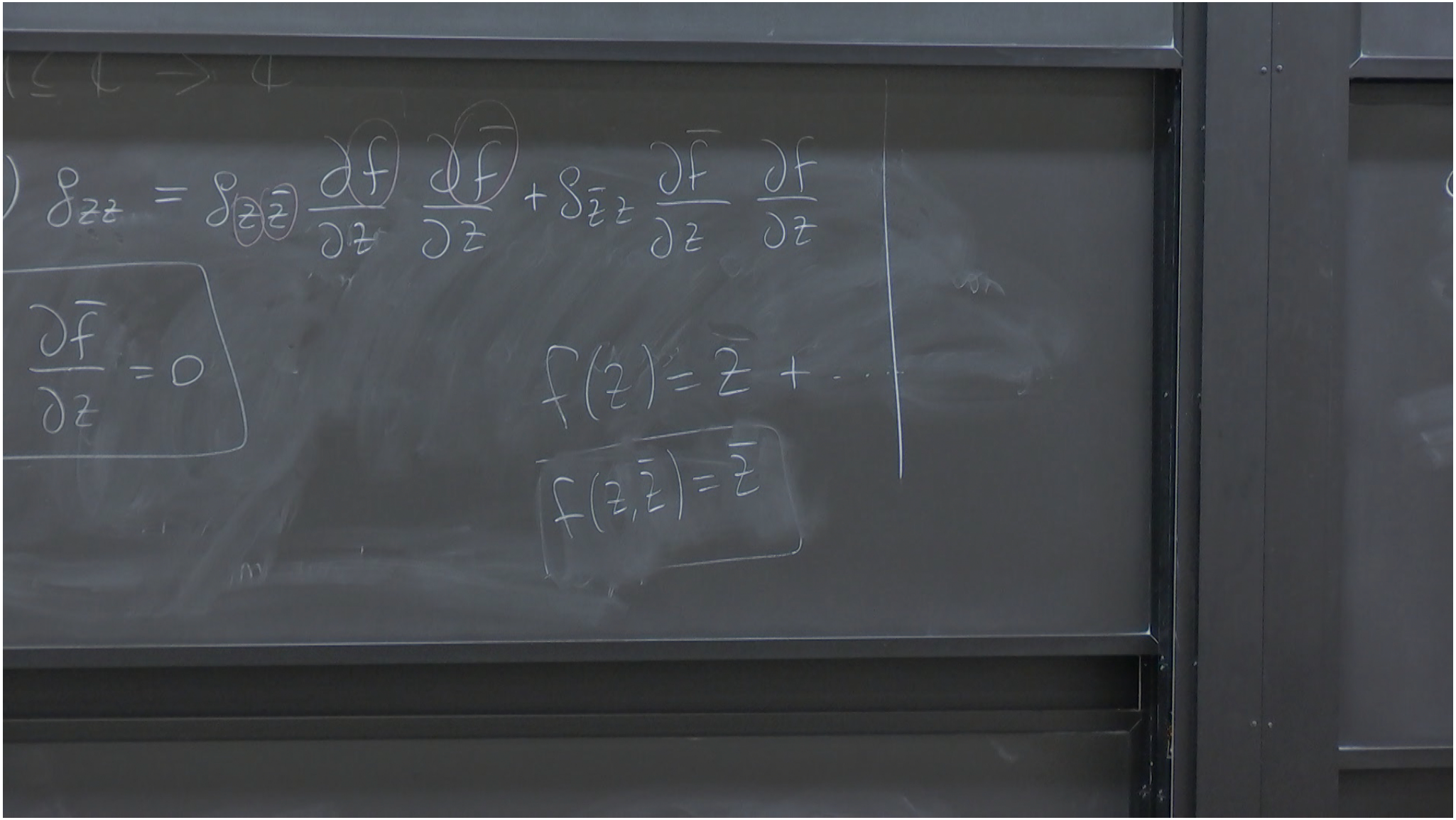
let $f : U \subseteq \mathbb{C} \to \mathbb{C}$

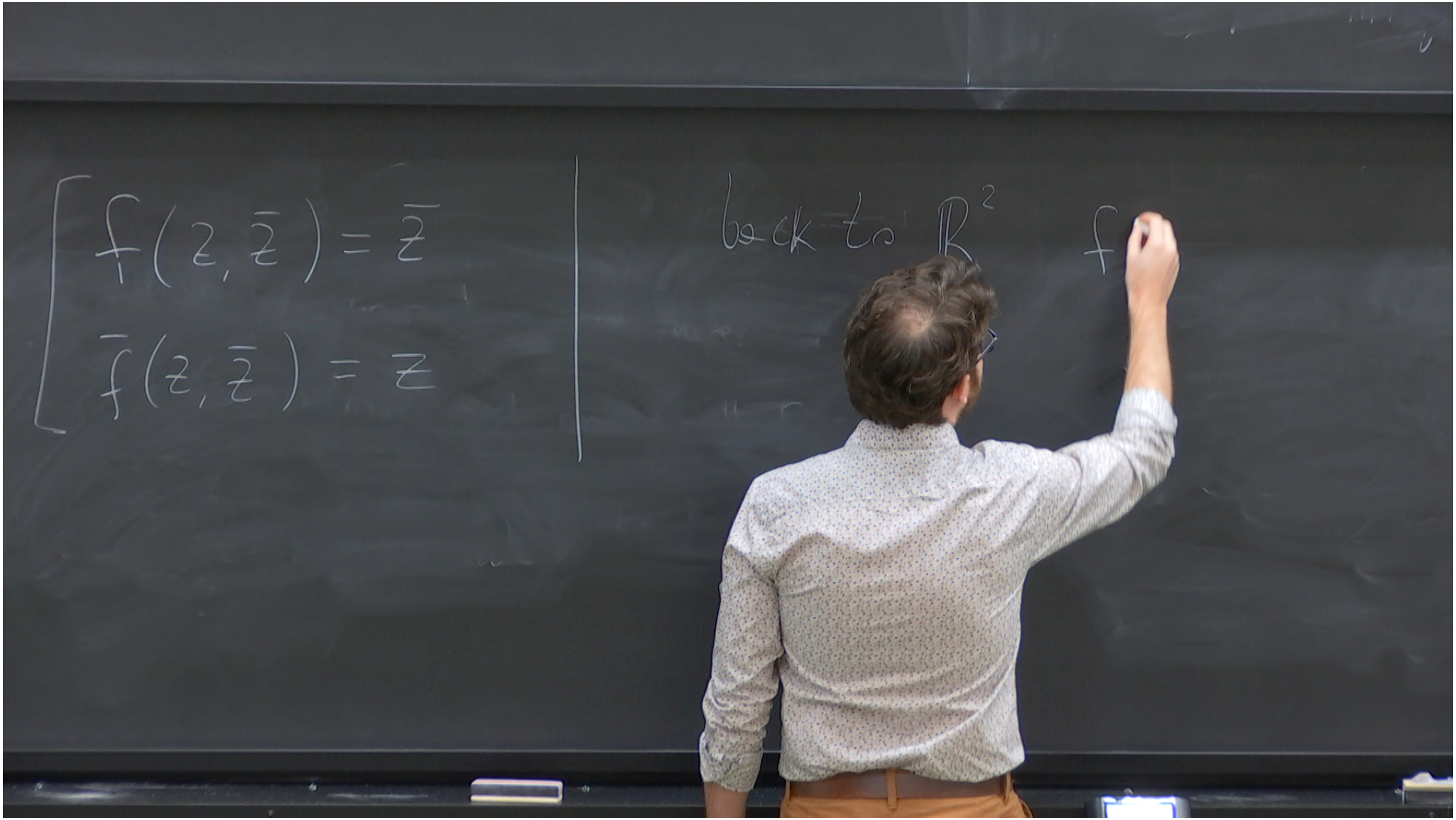
$$0 = \Omega^2(z,\bar{z})\, g_{z\bar{z}} = g_{z\bar{z}} \frac{\partial f}{\partial z} \frac{\partial \bar{f}}{\partial z} + g_{\bar{z}z} \frac{\partial \bar{f}}{\partial z} \frac{\partial f}{\partial z}$$

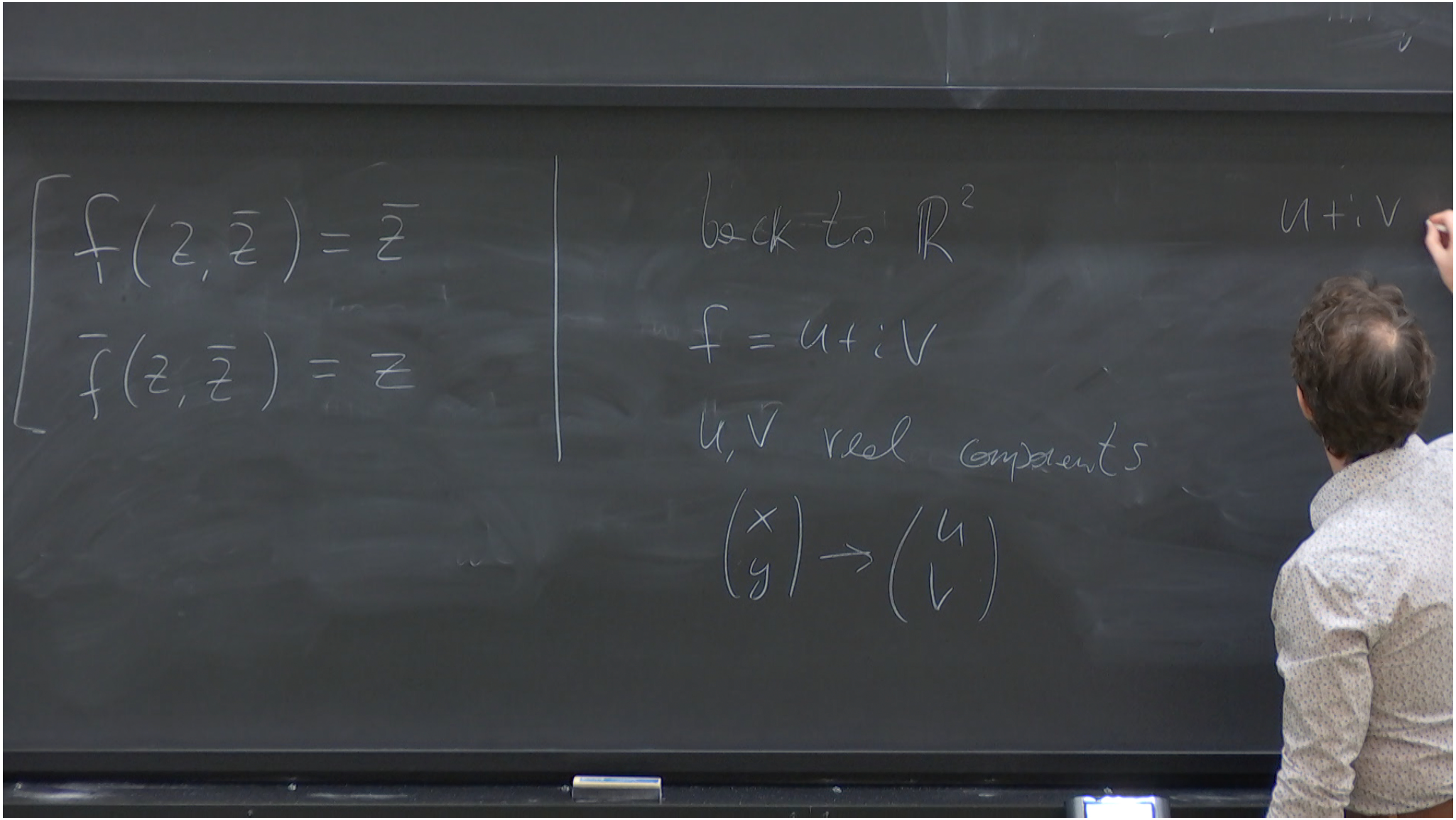
$$\to \boxed{\frac{\partial f}{\partial z} \frac{\partial \bar{f}}{\partial z} = 0}$$

$$f(z) = \bar{z} + \dots$$

$$\boxed{f(z) = \bar{z}}$$


$g_{zz} = g_{(z\bar{z})} \frac{\partial f}{\partial z} \frac{\partial \bar{f}}{\partial z} + g_{\bar{z}z} \frac{\partial \bar{f}}{\partial z} \frac{\partial f}{\partial z}$
$\frac{\partial \bar{f}}{\partial z} = 0$
$f(z) = \bar{z} + \dots$
$f(z, \bar{z}) = \bar{z}$


$f(z,\bar{z}) = \bar{z}$
$\bar{f}(z,\bar{z}) = z$
back to $\mathbb{R}^2$


$f(z,\bar{z}) = \bar{z}$
$\bar{f}(z,\bar{z}) = z$
back to $\mathbb{R}^2$
$f = u + iv$
u, v real components
$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} u \\ v \end{pmatrix}$
$u + iv$

back to $\mathbb{R}^2$

$f = u + iv$

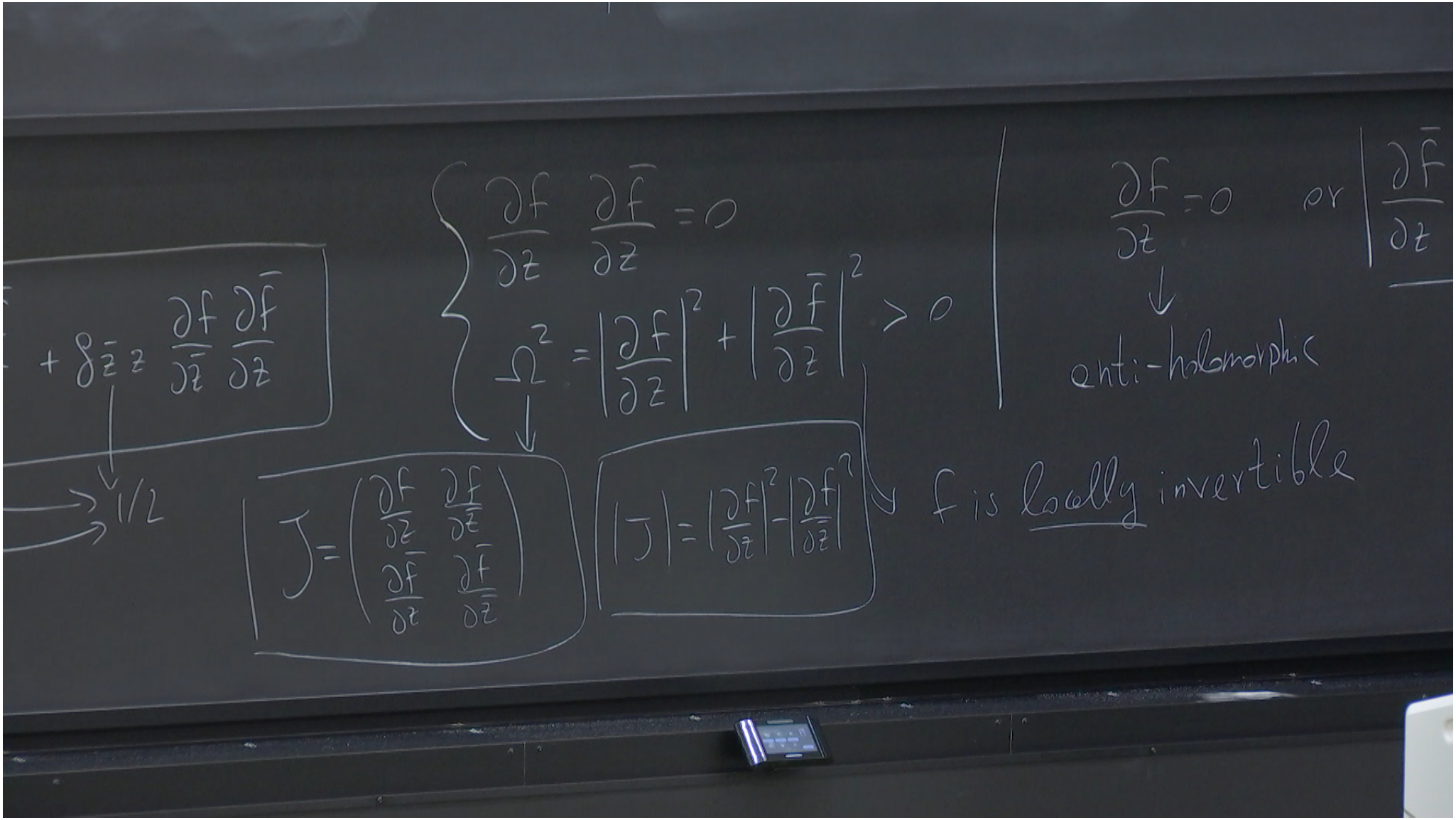
u, v real components

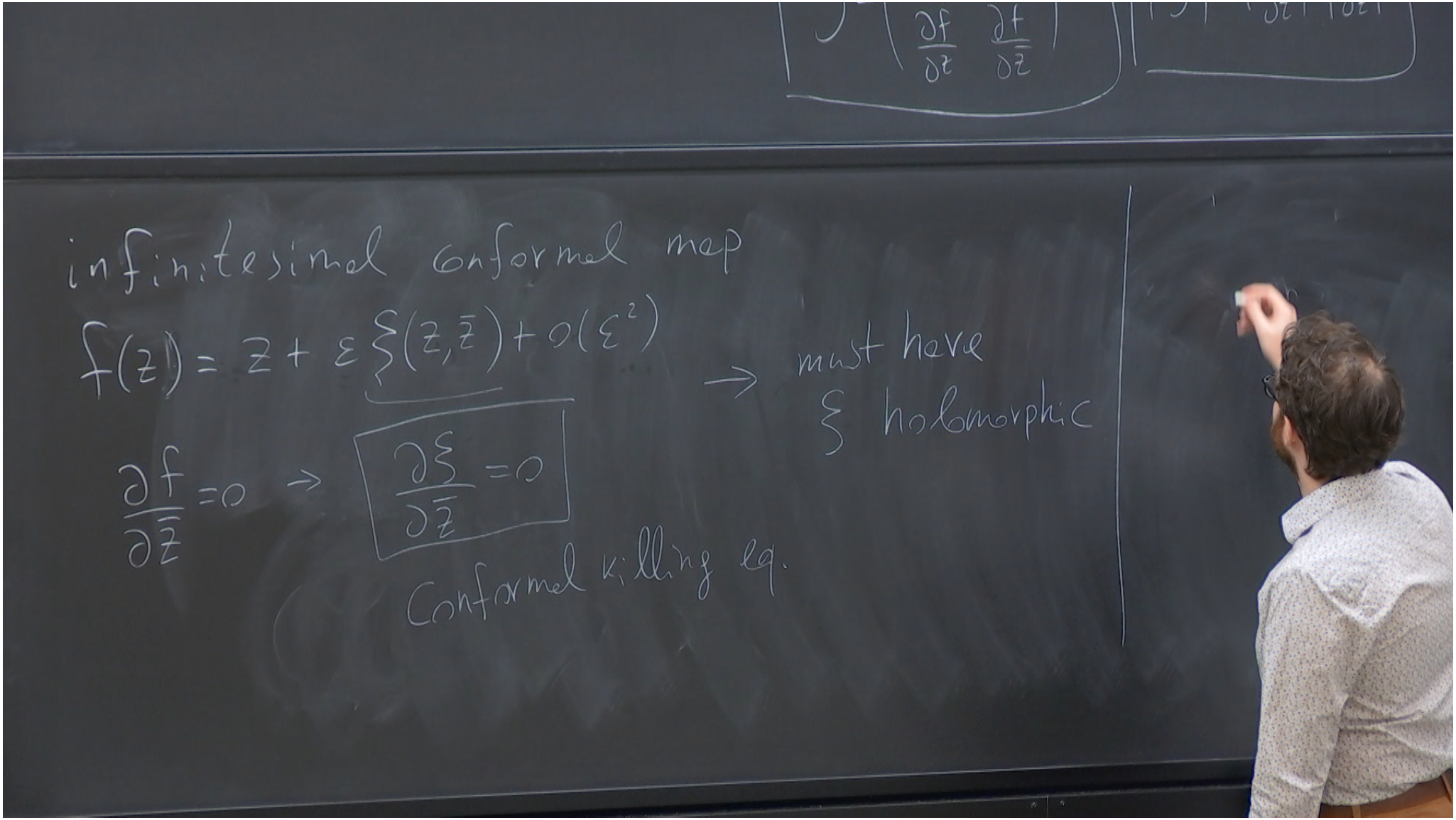
$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

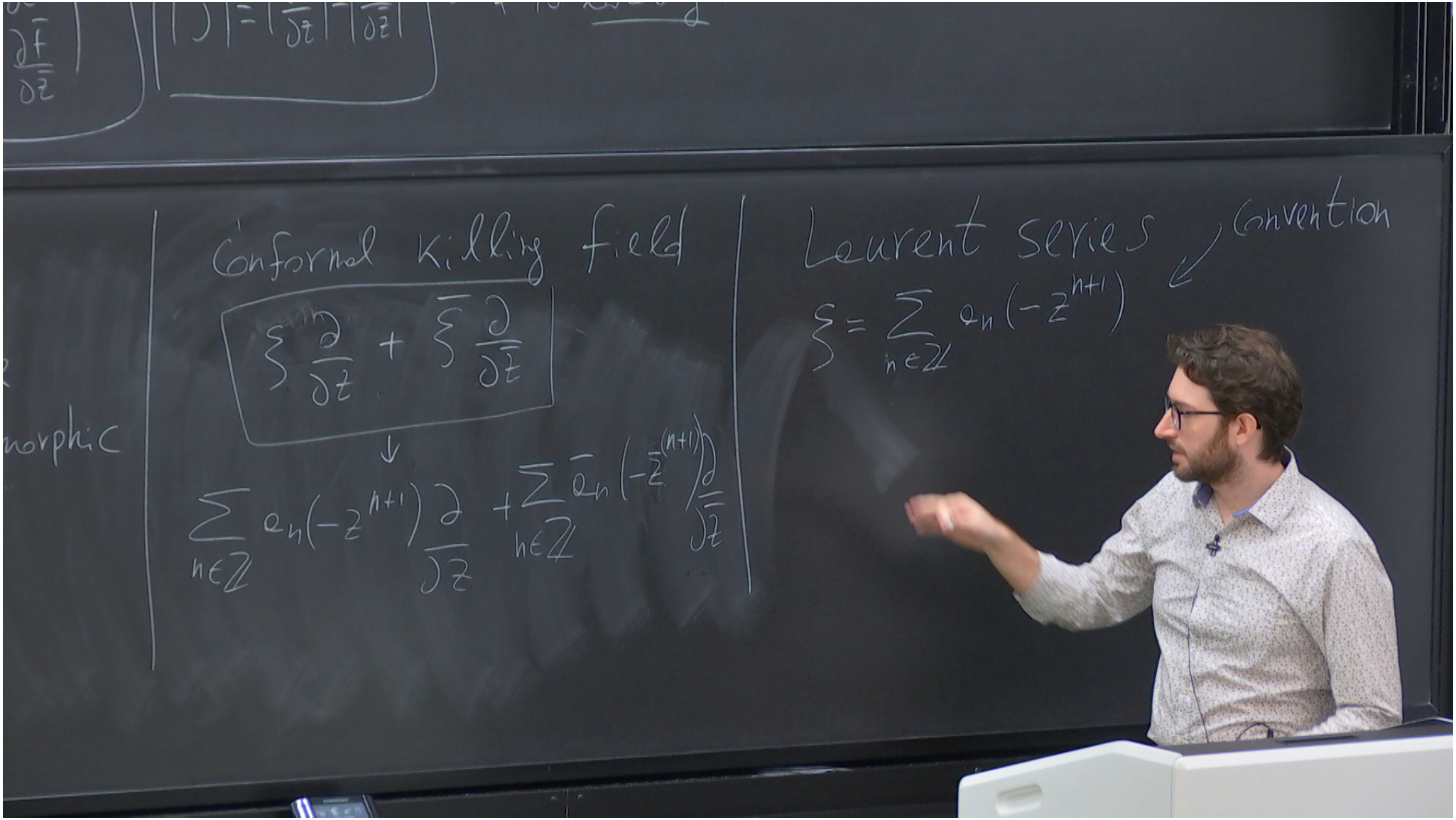
$u + iv = x - iy$

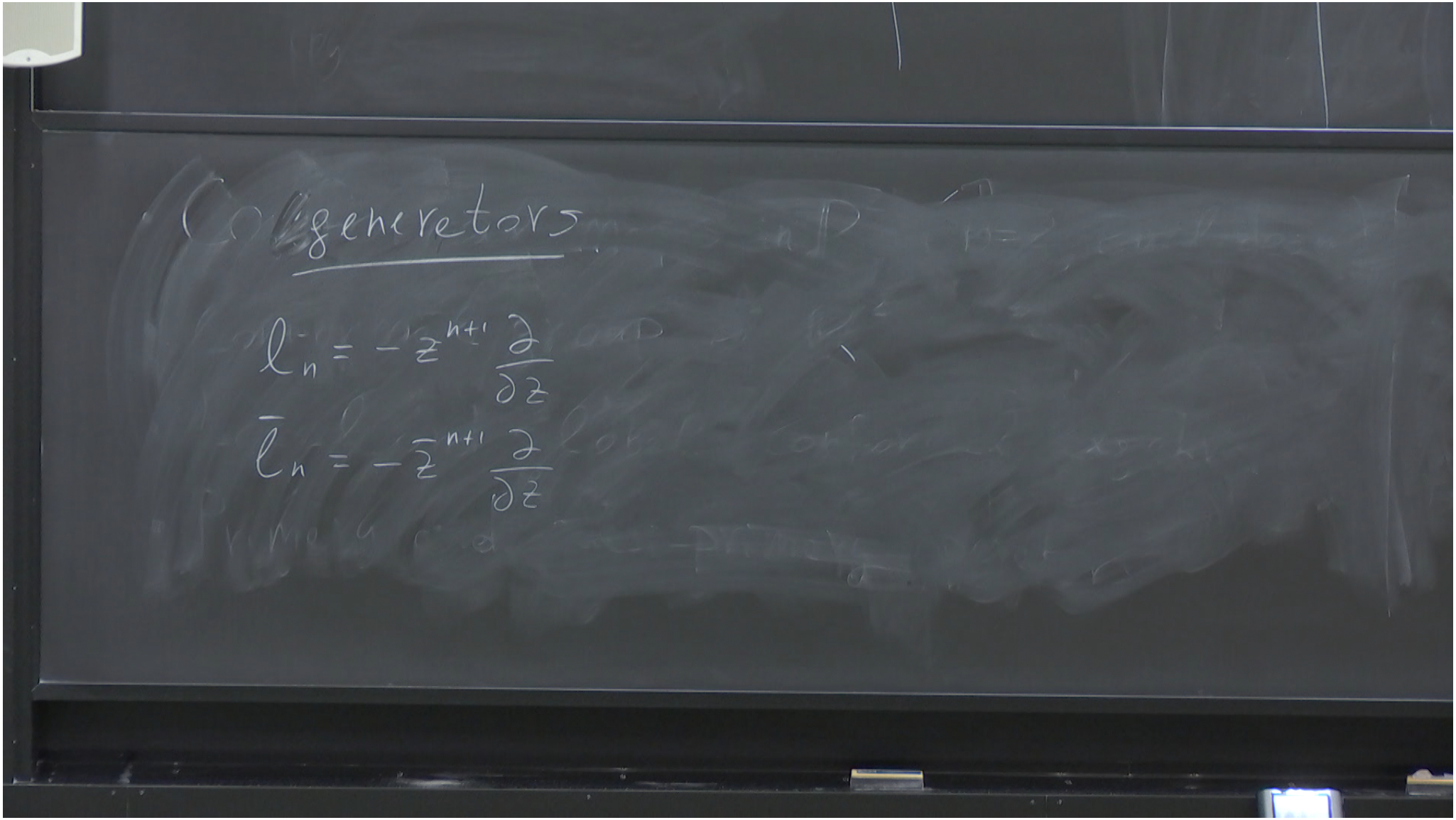
$u = x$

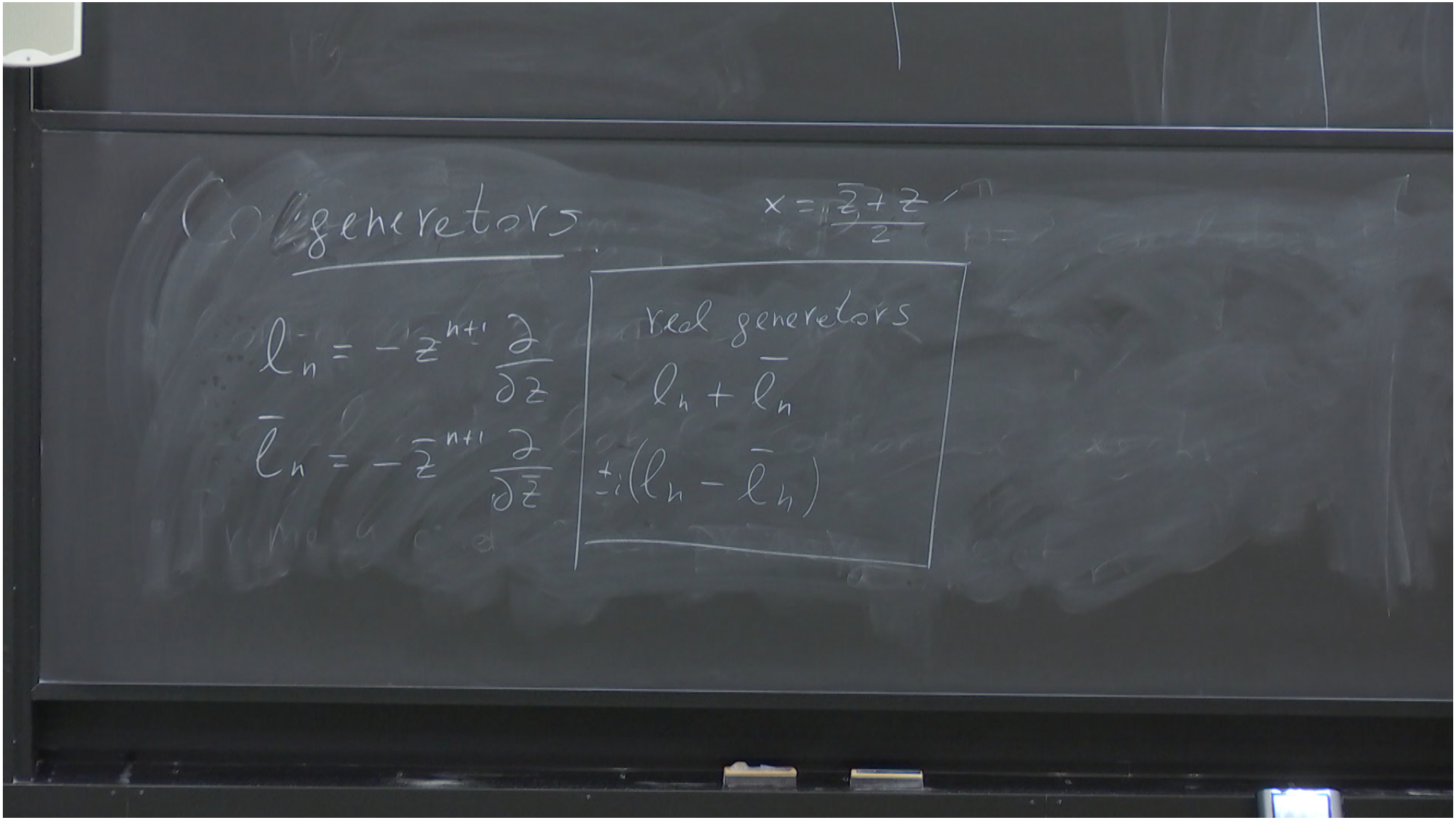
$v = -y$


$+\delta_{\bar{z}z}\,\frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z}$
$\frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial z}=0$
$\Omega^2=\left|\frac{\partial f}{\partial z}\right|^2+\left|\frac{\partial \bar{f}}{\partial z}\right|^2>0$
$\frac{\partial f}{\partial z}=0$ or $\frac{\partial \bar{f}}{\partial z}$
anti-holomorphic
$J=\begin{pmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial \bar{z}}\\ \frac{\partial \bar{f}}{\partial z} & \frac{\partial \bar{f}}{\partial \bar{z}}\end{pmatrix}$
$|J|=\left|\frac{\partial f}{\partial z}\right|^2-\left|\frac{\partial f}{\partial \bar{z}}\right|^2$
f is locally invertible
1/2


infinitesimal conformal map
$f(z) = z + \varepsilon \xi(z,\bar{z}) + o(\varepsilon^2)$
→ must have ξ holomorphic
$\frac{\partial f}{\partial \bar{z}} = 0 \rightarrow \boxed{\frac{\partial \xi}{\partial \bar{z}} = 0}$
Conformal Killing eq.


Conformal Killing field
$\xi \frac{\partial}{\partial z} + \bar{\xi} \frac{\partial}{\partial \bar{z}}$
$\sum_{n \in \mathbb{Z}} a_n (-z^{n+1}) \frac{\partial}{\partial z} + \sum_{n \in \mathbb{Z}} \bar{a}_n (-\bar{z}^{(n+1)}) \frac{\partial}{\partial \bar{z}}$
Laurent series
convention
$\xi = \sum_{n \in \mathbb{Z}} a_n (-z^{n+1})$

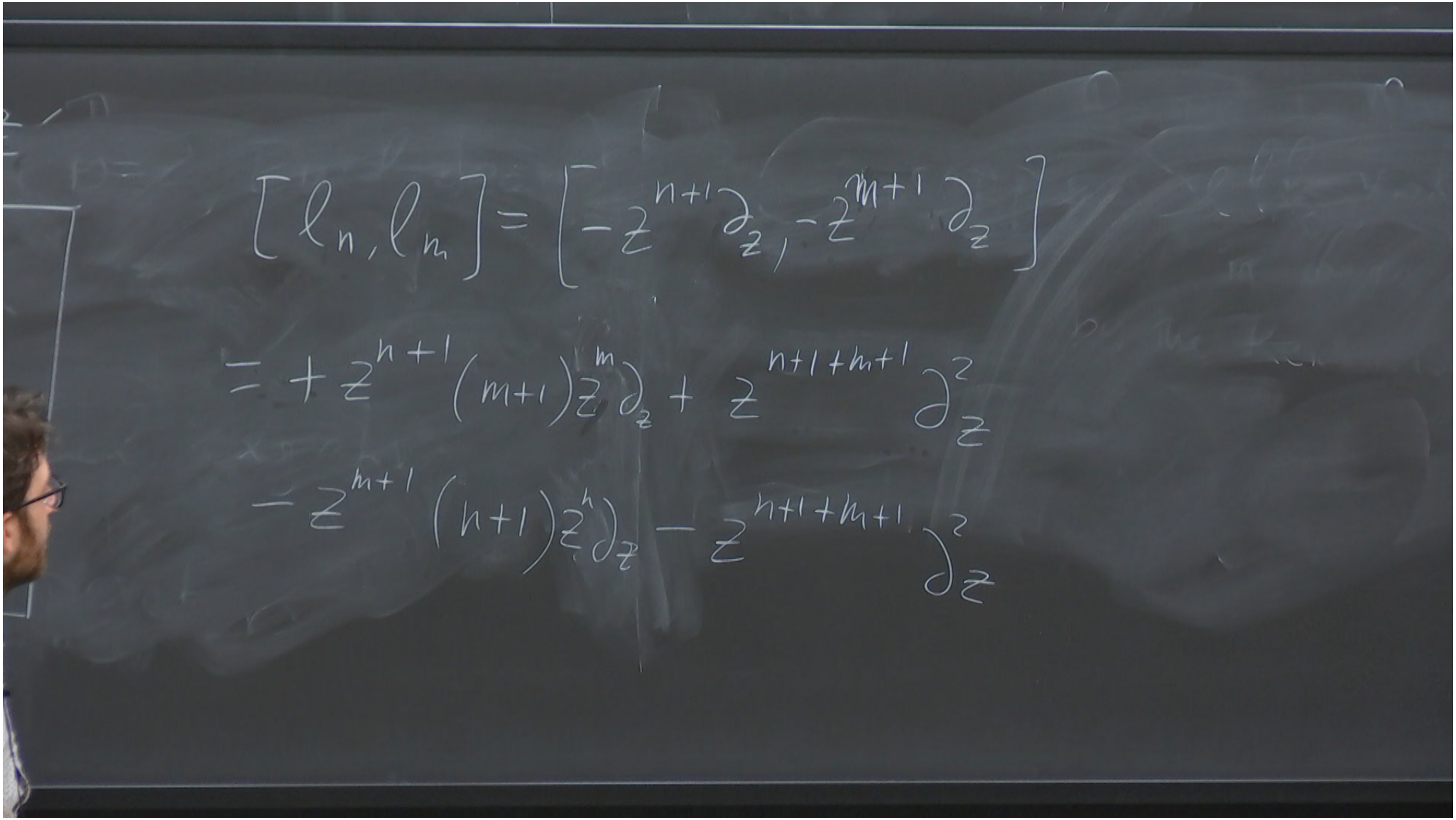

Conformal generators
$\ell_n = -z^{n+1}\frac{\partial}{\partial z}$
$\bar{\ell}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}$

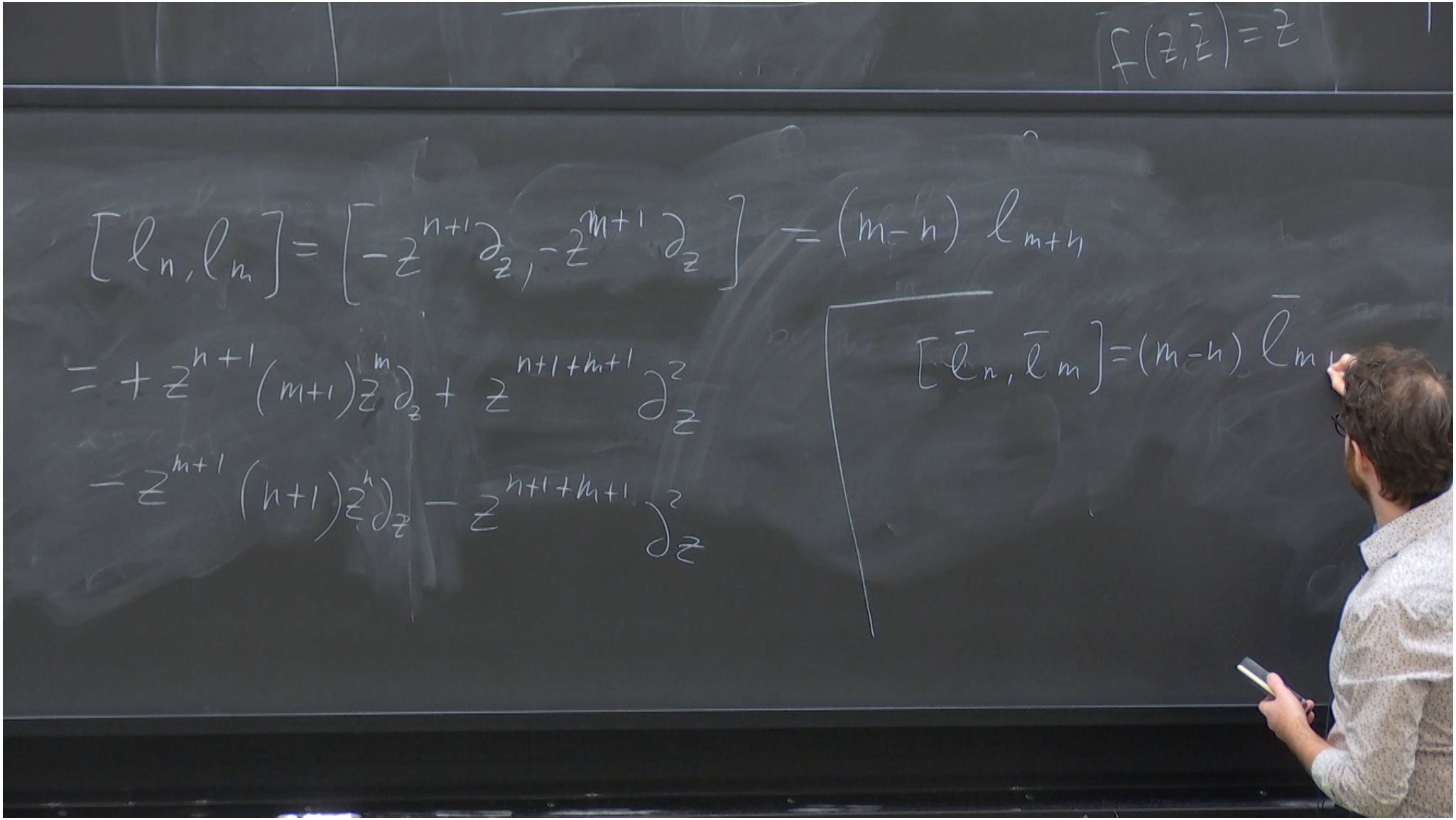

generators
$x = \frac{\bar{z}+z}{2}$
$\ell_n = -z^{n+1}\frac{\partial}{\partial z}$
$\bar{\ell}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}$
real generators
$\ell_n + \bar{\ell}_n$
$\pm i(\ell_n - \bar{\ell}_n)$

$$[\ell_n, \ell_m] = \left[-z^{n+1}\partial_z, -z^{m+1}\partial_z\right]$$

$$= +z^{n+1}(m+1)\partial_z + z^{n+1+m+1}\partial_z^2$$

$$-z^{m+1}(n+1)\partial_z - z^{n+1+m+1}\partial_z^2$$


$[\ell_n, \ell_m] = [-z^{n+1}\partial_z, -z^{m+1}\partial_z]$
$= +z^{n+1}(m+1)z^m\partial_z + z^{n+1+m+1}\partial_z^2$
$-z^{m+1}(n+1)z^n\partial_z - z^{n+1+m+1}\partial_z^2$


$f(z,\bar{z}) = z$
$[\ell_n, \ell_m] = [-z^{n+1}\partial_z, -z^{m+1}\partial_z] = (m-n)\,\ell_{m+n}$
$= +z^{n+1}(m+1)z^m\partial_z + z^{n+1+m+1}\partial_z^2$
$-z^{m+1}(n+1)z^n\partial_z - z^{n+1+m+1}\partial_z^2$
$[\bar{\ell}_n, \bar{\ell}_m] = (m-n)\,\bar{\ell}_{m+n}$

$$] = (m-n)\, \ell_{m+n}$$

$$[\bar{\ell}_n, \bar{\ell}_m] = (m-n)\, \bar{\ell}_{m+n}$$

$$[\ell_n, \bar{\ell}_m] = 0$$



$$] = (m-n)\, \ell_{m+n}$$

$$[\bar{\ell}_n, \bar{\ell}_m] = (m-n)\, \bar{\ell}_{m+n}$$

$$[\ell_n, \bar{\ell}_m] = 0$$

2 copies of an infinite-dim Lie algebra

$$] = (m-n)\, \ell_{m+n}$$

$$[\bar{\ell}_n, \bar{\ell}_m] = (m-n)\, \bar{\ell}_{m+n}$$

$$[\ell_n, \bar{\ell}_m] = 0$$

2 copies of the infinite-dim Lie algebra

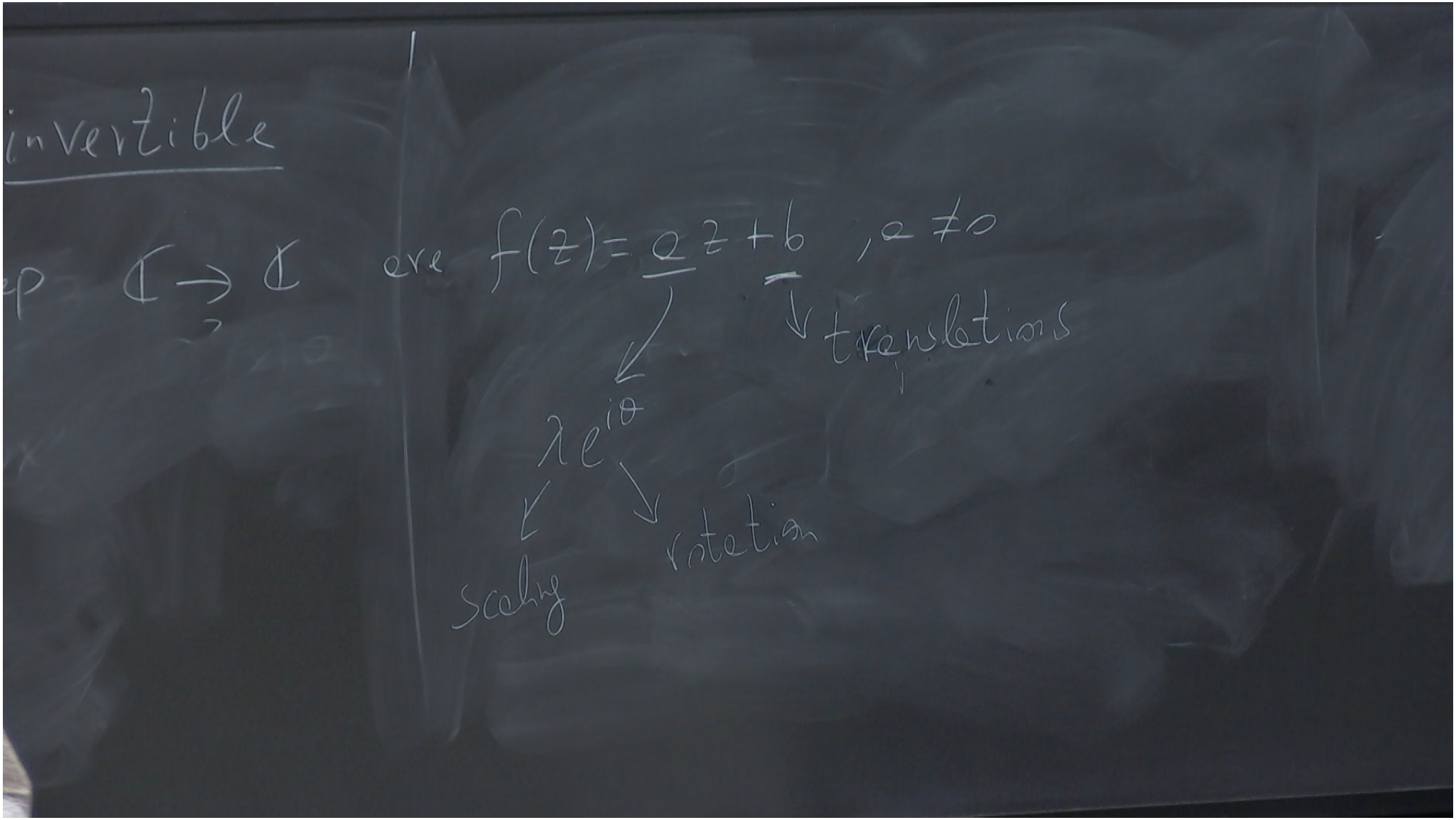
Witt algebra

Conformal group → $f(z,\bar{z})$ invertible

- the only holomorphic invertible map $\mathbb{C} \to \mathbb{C}$ are $f(z) = az + b$, $a \neq 0$

Conformal group → $f(z,\bar{z})$ invertible

- the only holomorphic invertible map $\mathbb{C} \to \mathbb{C}$ are $f(z) = az + b$, $a \neq 0$
  - b: translations
  - az: $\lambda e^{i\theta}$
    - λ: scaling
    - $e^{i\theta}$: rotation

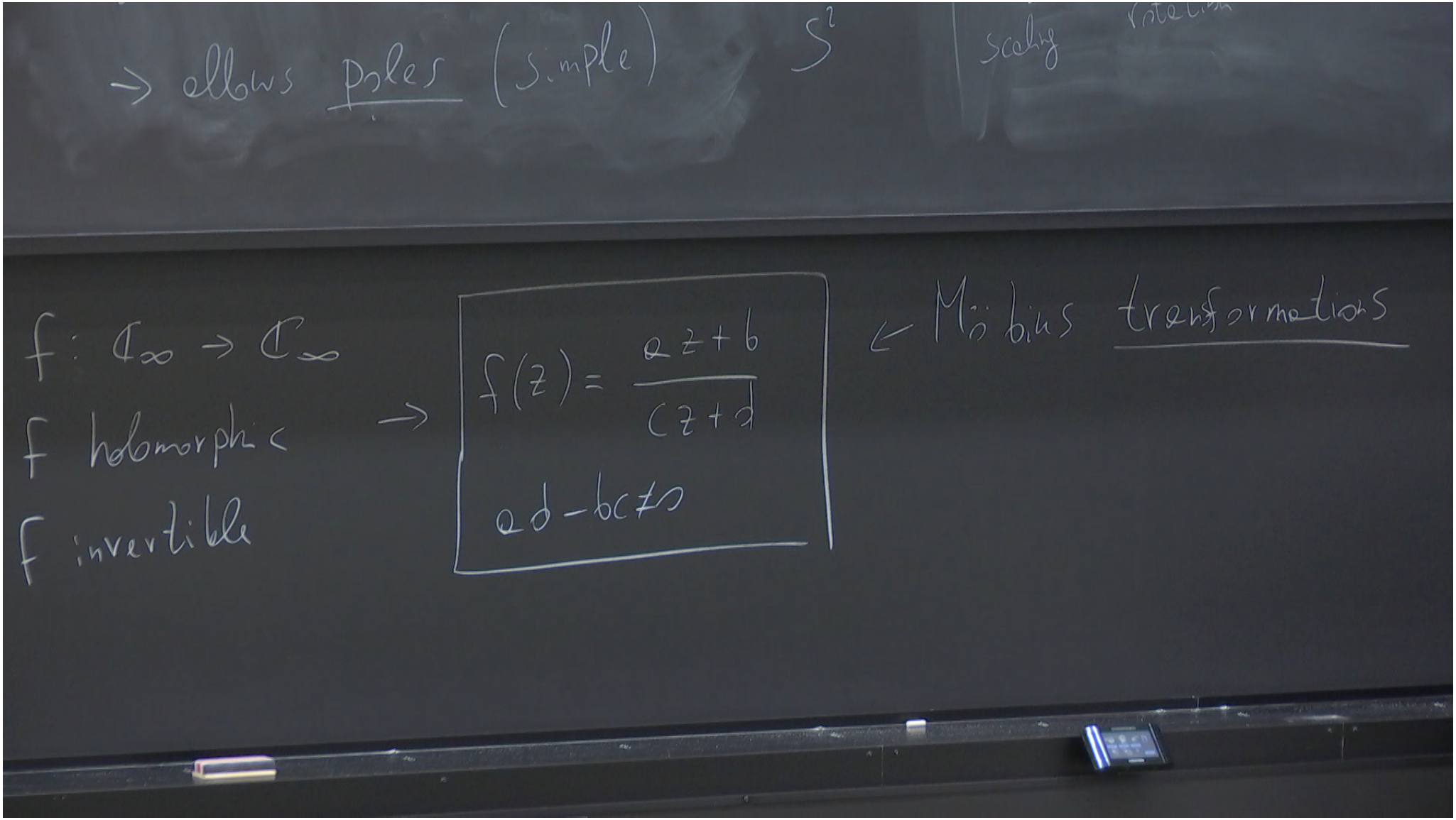

invertible
$f(z) = az + b$, $a \neq 0$
translations
$\lambda e^{i\theta}$
scaling
rotation

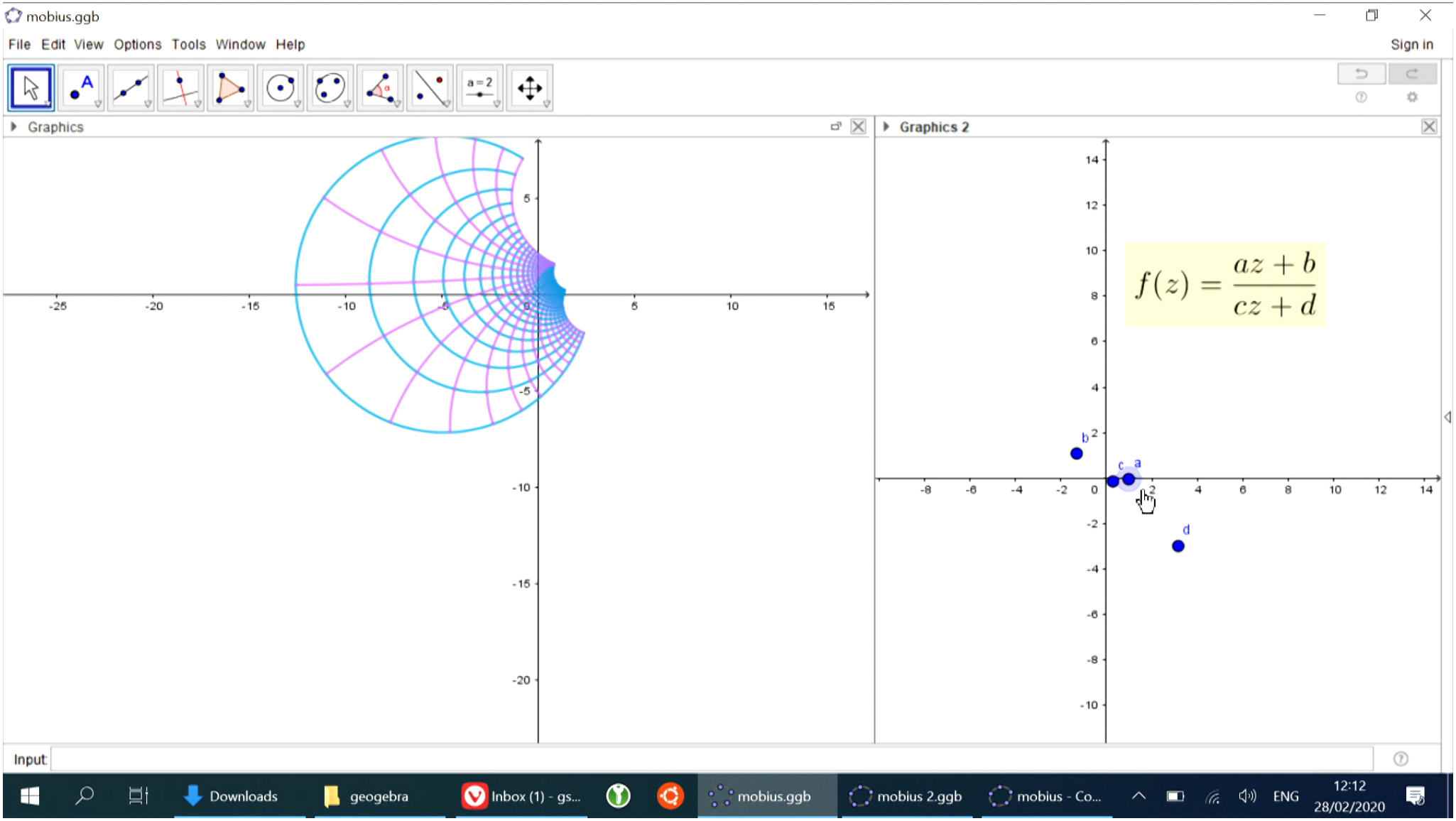
$\frac{\partial f}{\partial z}$ $\frac{\partial f}{\partial \bar{z}}$

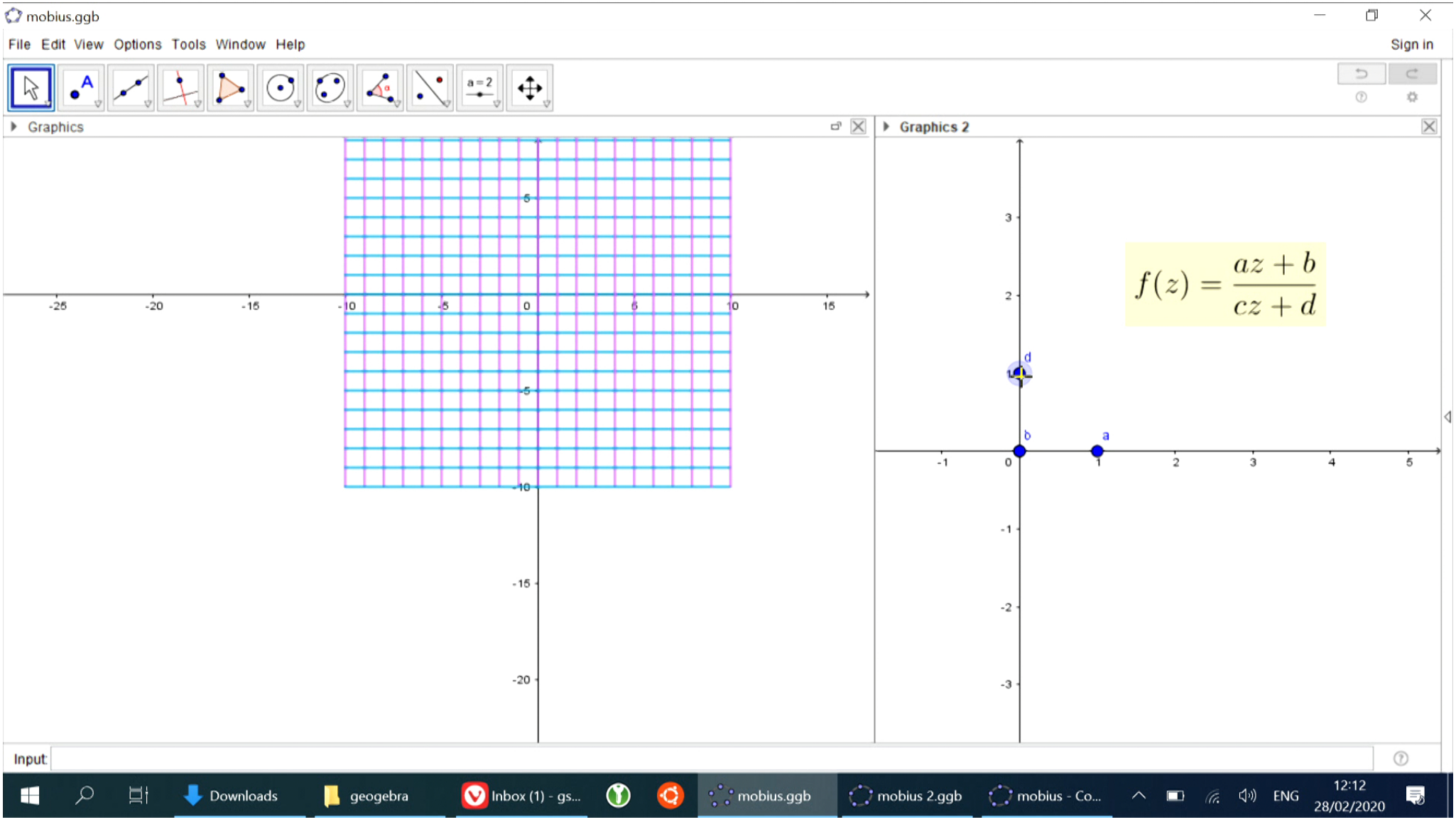
Conformal group → $f(z, \bar{z})$ invertible

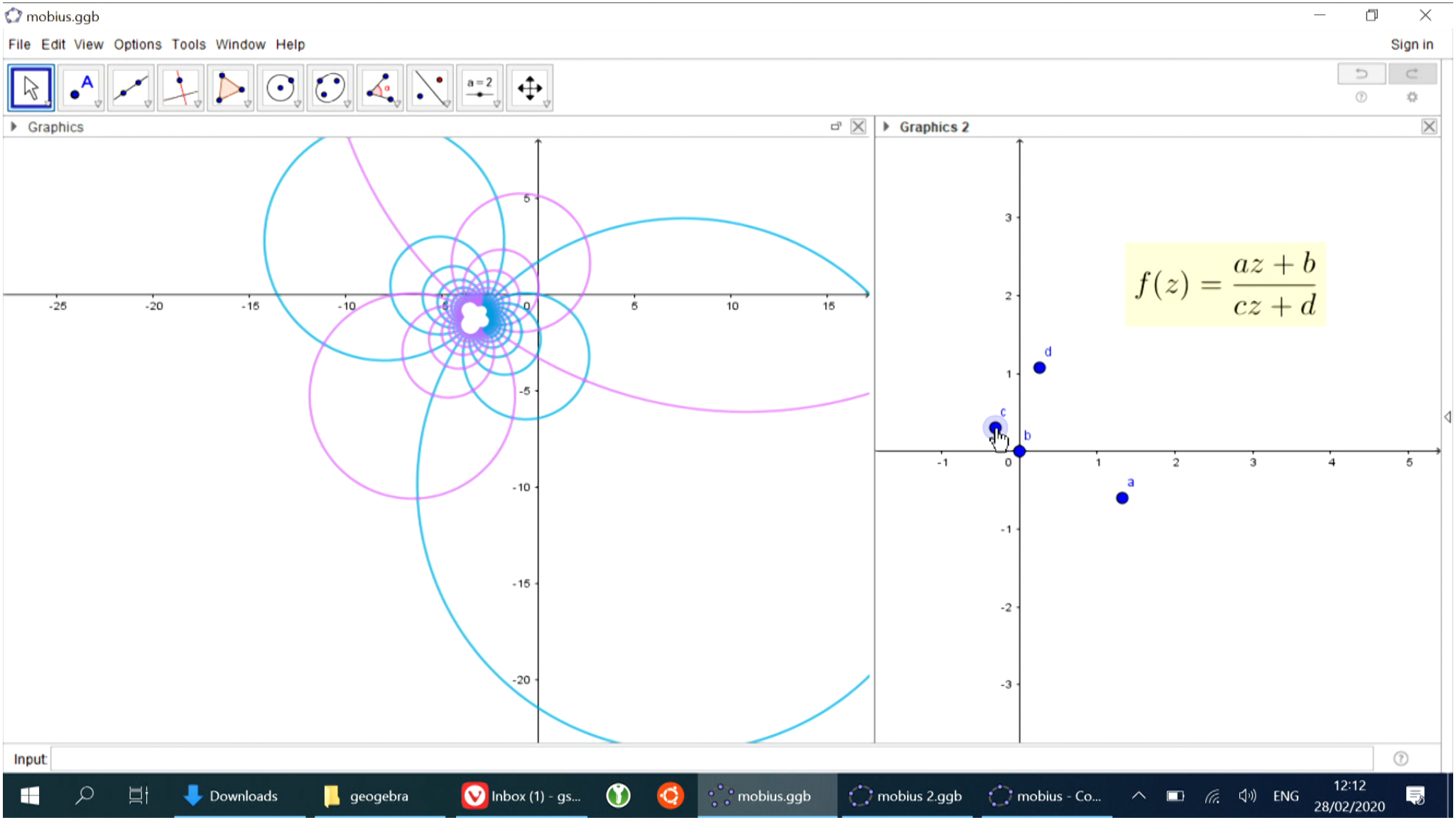
- the only holomorphic invertible map $\mathbb{C} \to \mathbb{C}$ are $f(z) = az + b$, a ↓ tra
- to get more, we extend $\mathbb{C}$ to $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ ≅ S^2
  → allows poles (simple)

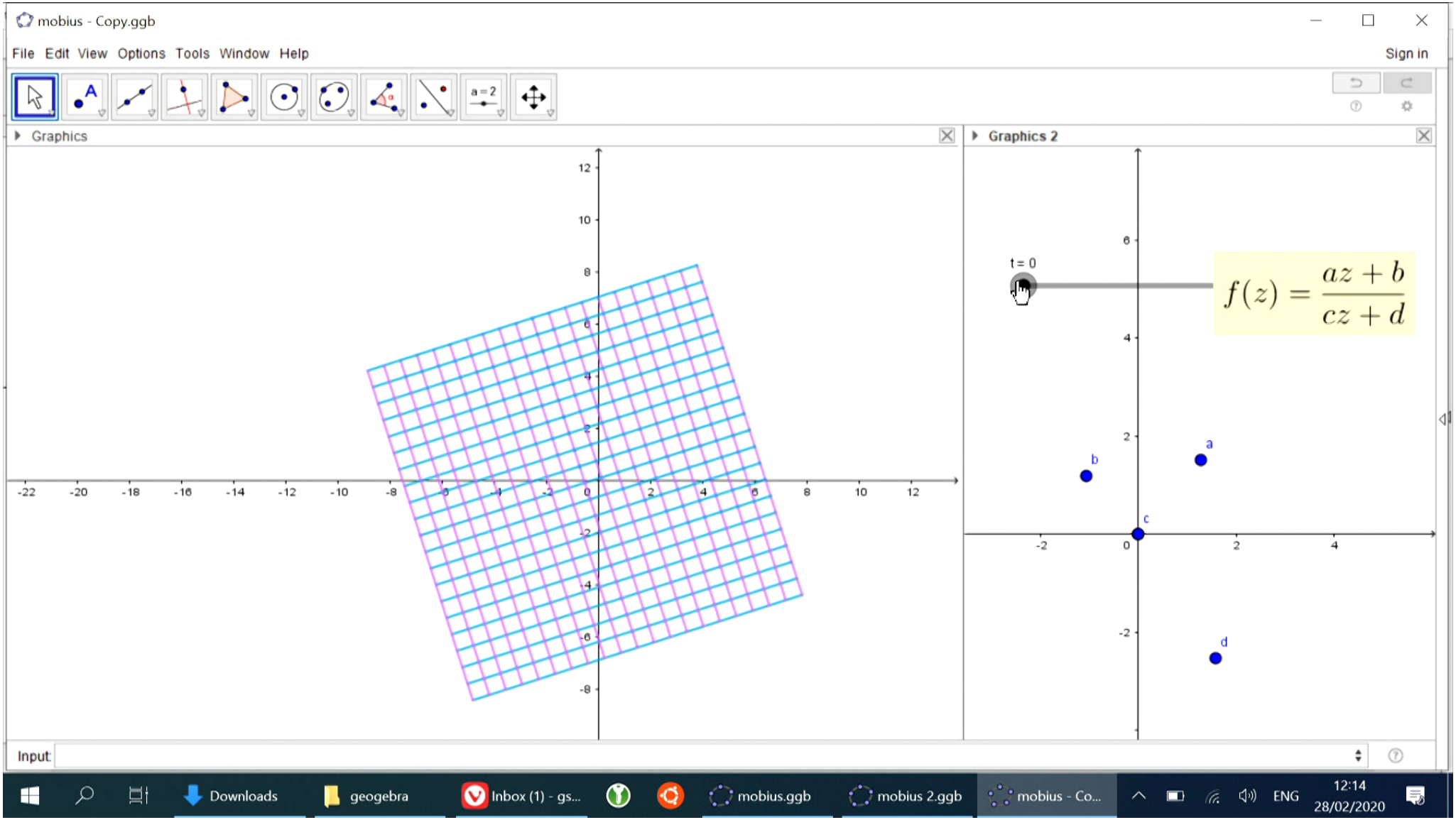
$\lambda e^{i\theta}$ ↙ scaling ↘ rotation

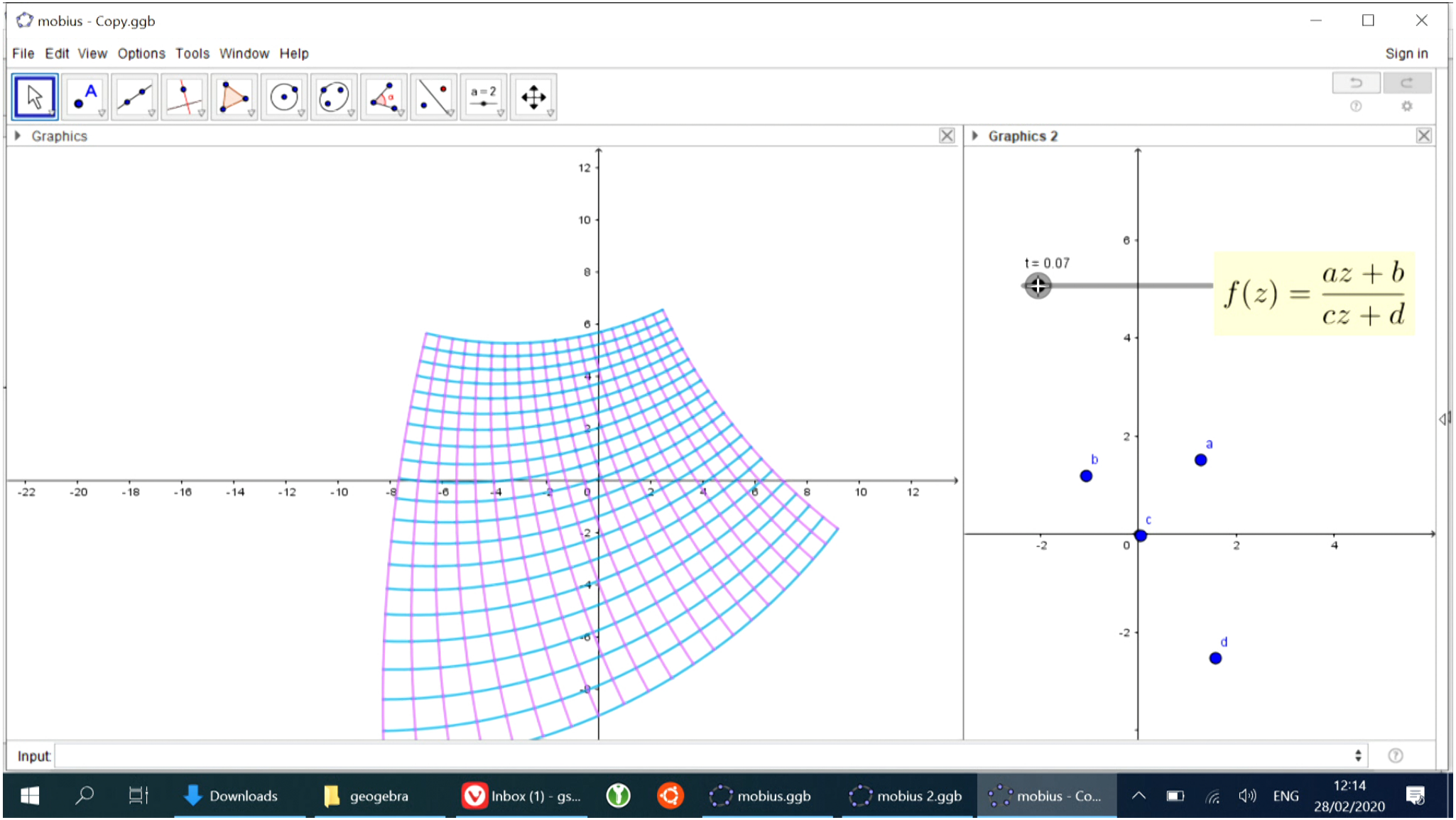

→ allows poles (simple)
S^2
scaling
$f: \mathbb{C}_\infty \to \mathbb{C}_\infty$
f holomorphic
f invertible
→
$f(z) = \frac{az+b}{cz+d}$
$ad - bc \neq 0$
← Möbius transformations

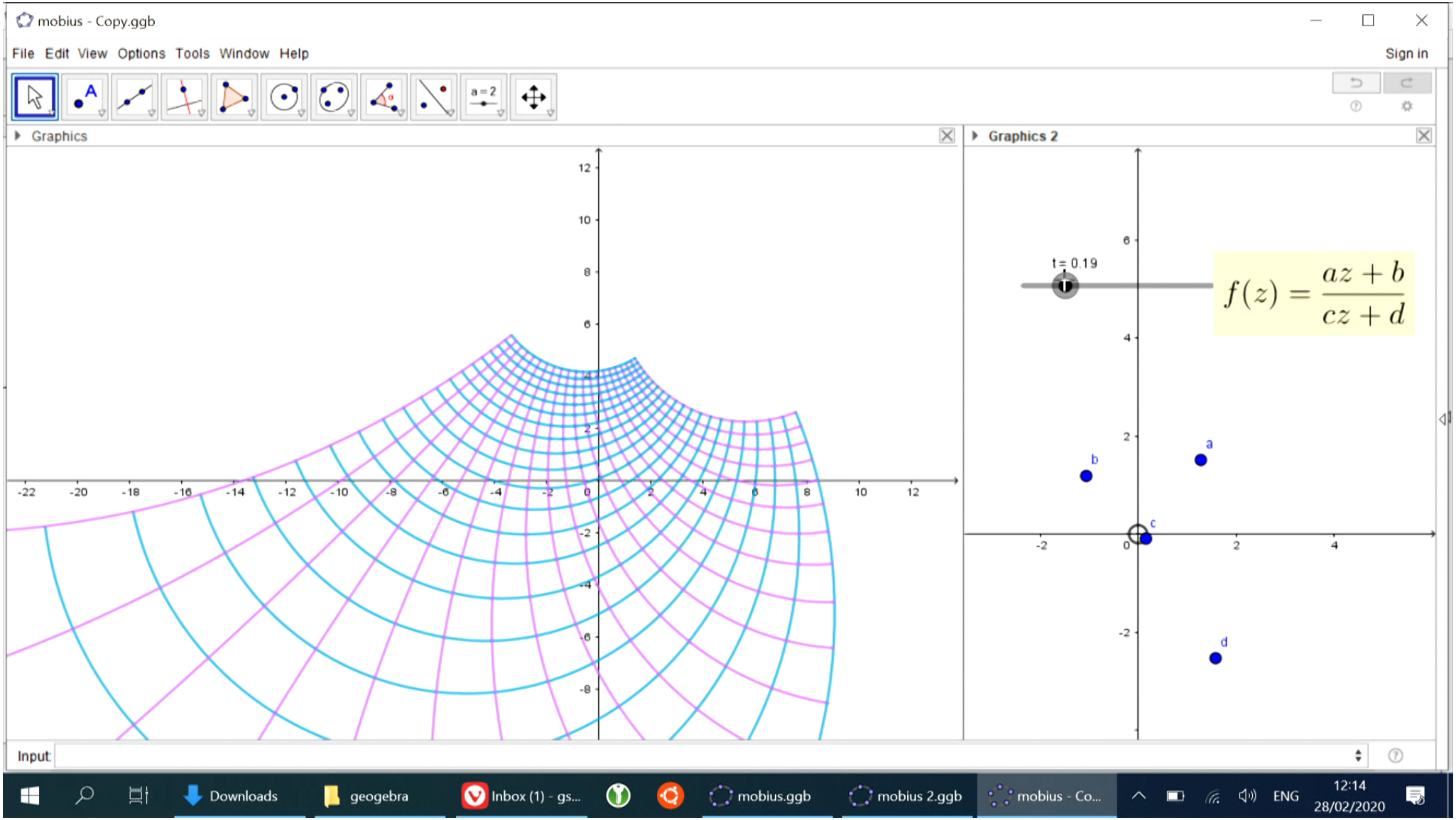

mobius.ggb
File Edit View Options Tools Window Help
Sign in
Graphics
Graphics 2
$f(z) = \frac{az + b}{cz + d}$
b
c
a
d
Input:
Downloads
geogebra
Inbox (1) - gs...
mobius.ggb
mobius 2.ggb
mobius - Co...
ENG
12:12
28/02/2020

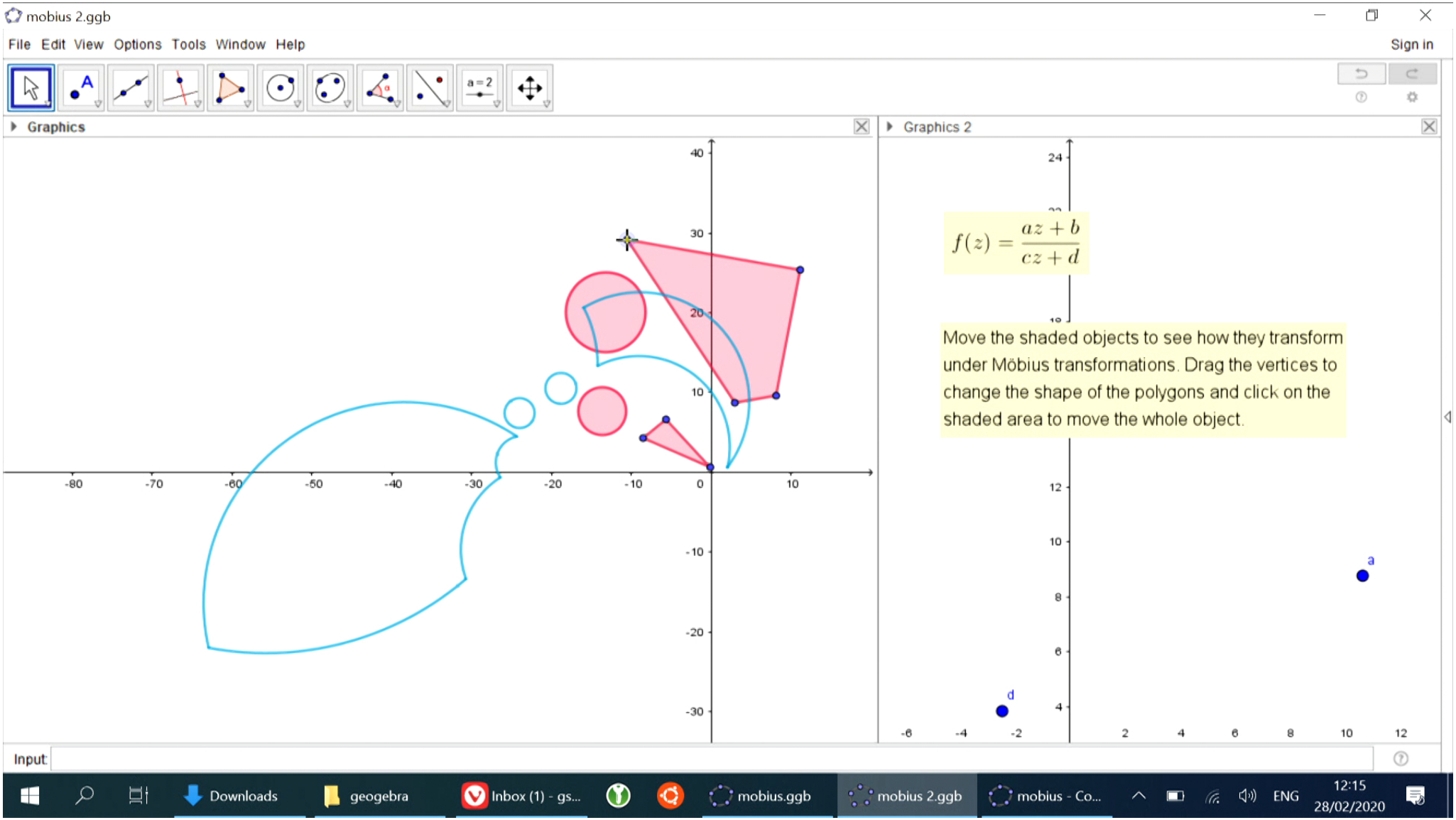

mobius.ggb
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Graphics
Graphics 2
$f(z) = \frac{az + b}{cz + d}$
d
b
a
Input:
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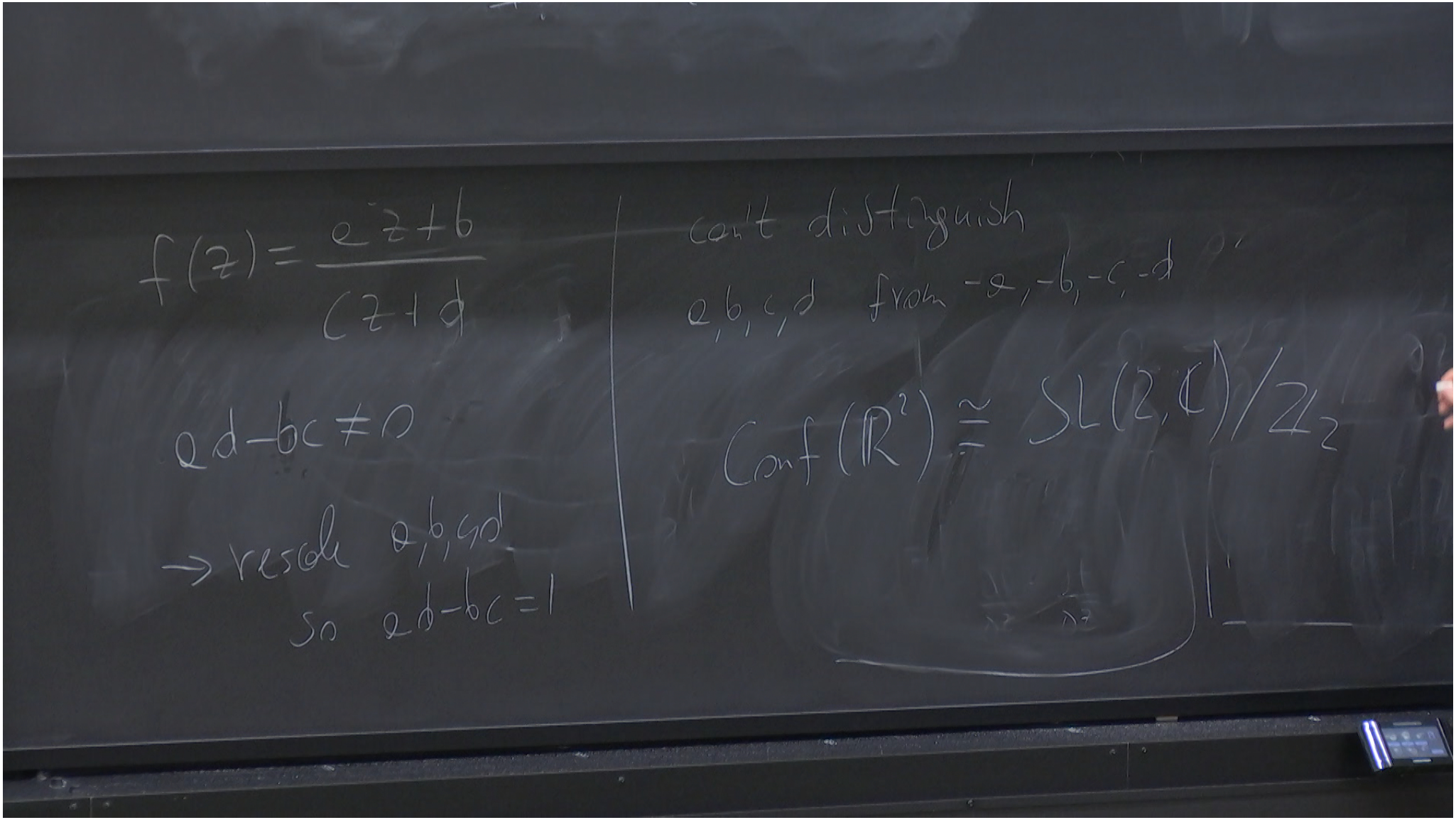

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Graphics
Graphics 2
$f(z) = \frac{az + b}{cz + d}$
a
b
c
d
Input:
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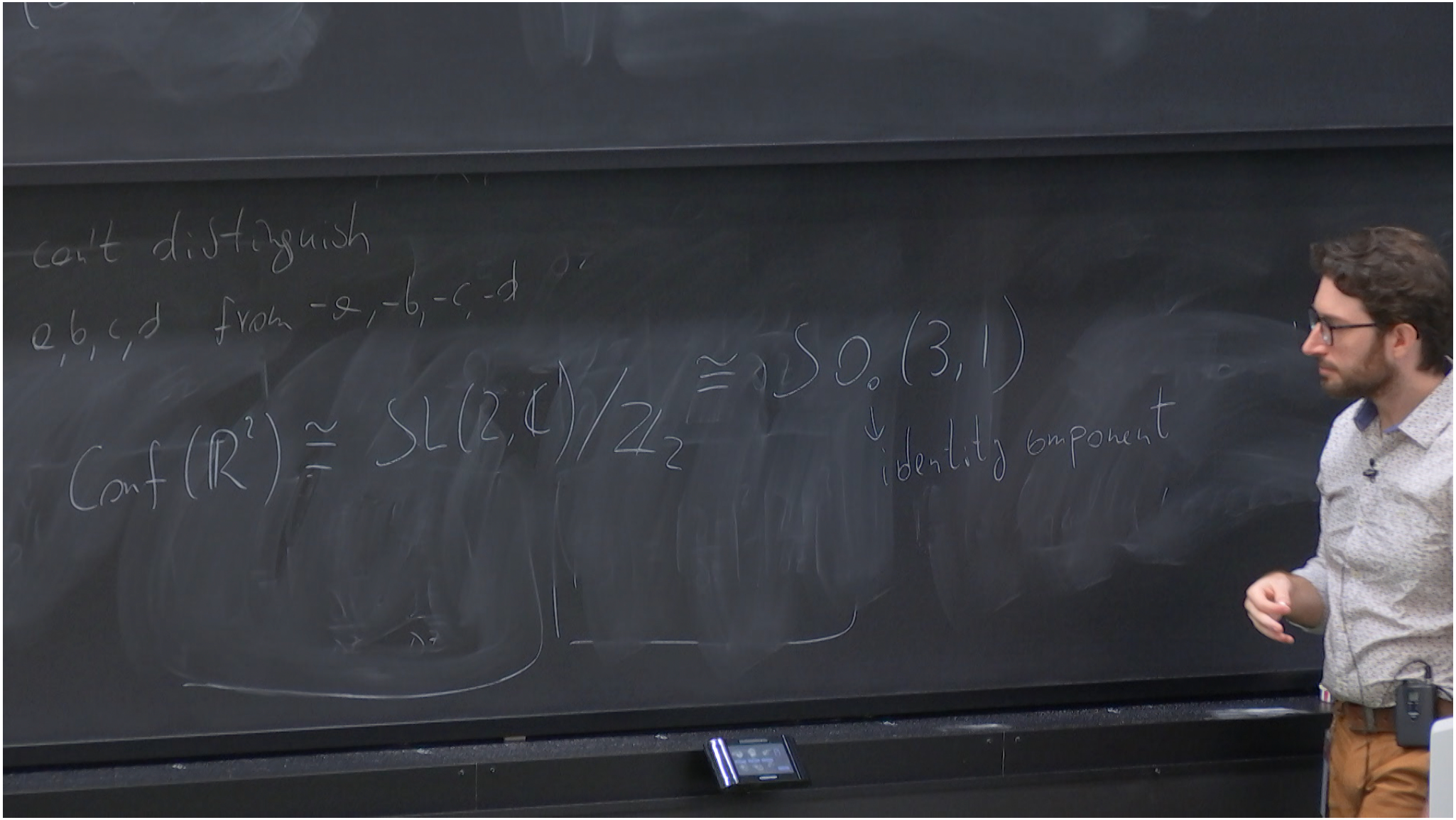

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Graphics
Graphics 2
t = 0
$f(z) = \frac{az + b}{cz + d}$
a
b
c
d
Input:
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Graphics
Graphics 2
t = 0.07
$f(z) = \frac{az + b}{cz + d}$
a
b
c
d
Input:
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Graphics
Graphics 2
t = 0.19
$f(z) = \frac{az + b}{cz + d}$
a
b
c
d
Input:
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mobius - Co...
ENG
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mobius 2.ggb
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Sign in
Graphics
Graphics 2
$f(z) = \frac{az + b}{cz + d}$
Move the shaded objects to see how they transform under Möbius transformations. Drag the vertices to change the shape of the polygons and click on the shaded area to move the whole object.
a
d
Input:
Downloads
geogebra
Inbox (1) - gs...
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$f(z) = \frac{az+b}{cz+d}$
$ad-bc \neq 0$
→ rescale a,b,c,d
so $ad-bc=1$
can't distinguish
a,b,c,d from $-a,-b,-c,-d$
$Conf(\mathbb{R}^2) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$


can't distinguish
a, b, c, d from -a, -b, -c, -d
$\mathrm{Conf}(\mathbb{R}^2) \cong SL(2,\mathbb{C})/\mathbb{Z}_2 \cong SO_0(3,1)$
identity component

$l_{\pm 1}, l_0, \bar{l}_{\pm 1}, \bar{l}_0$ generate the Lie algebra

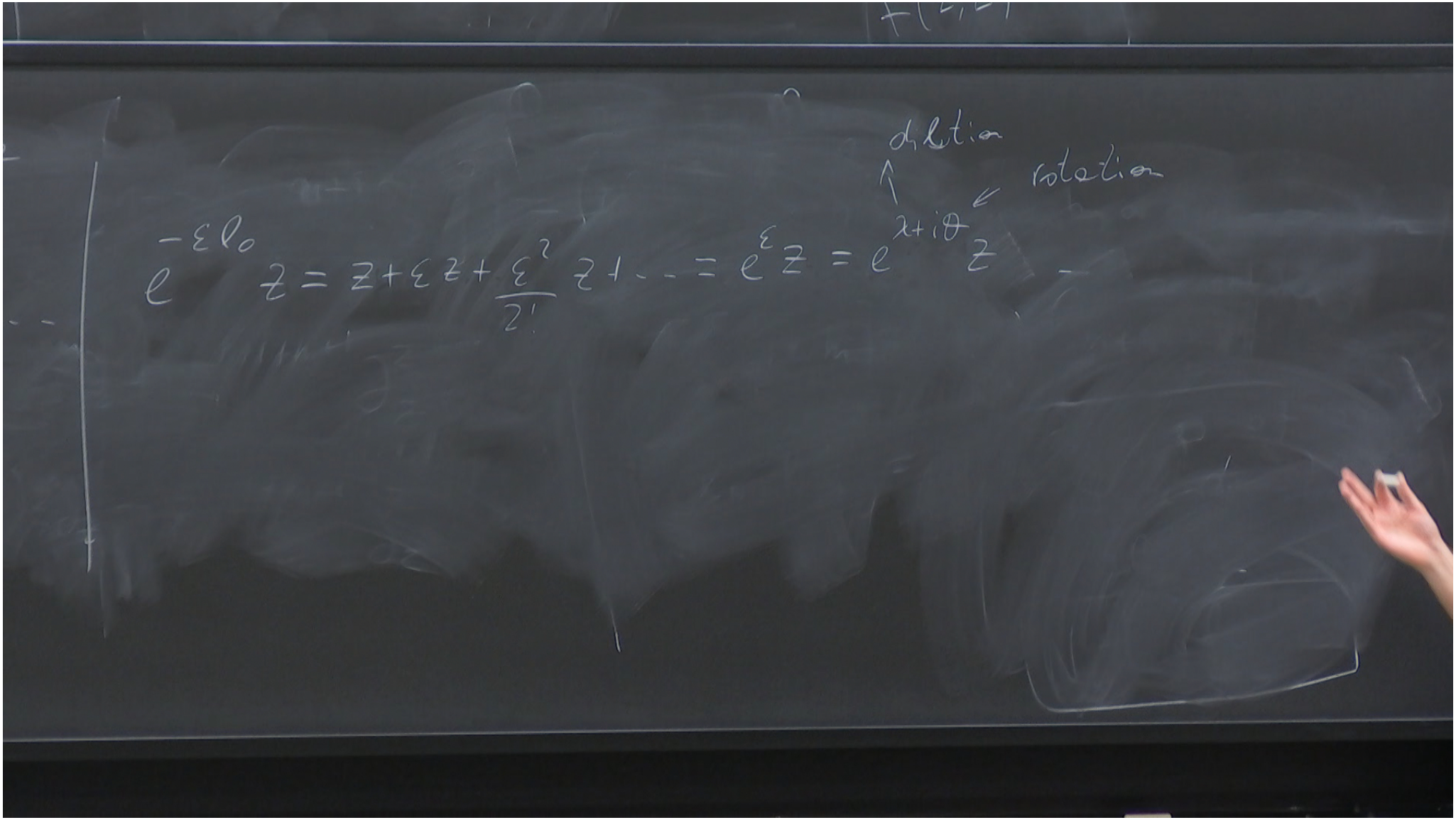
$l_{-1} = -\frac{\partial}{\partial z}$

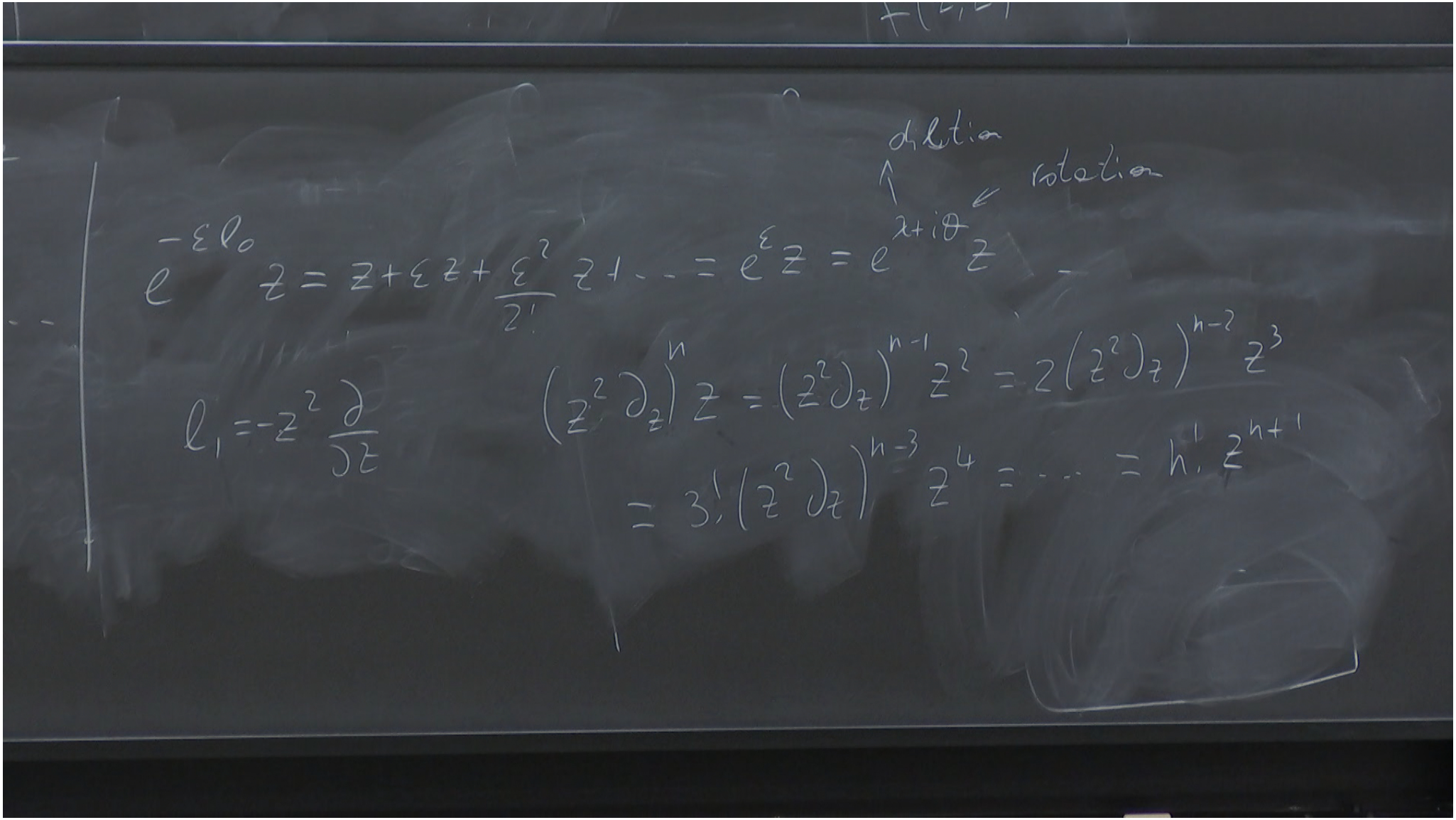
$l_0 = -z\frac{\partial}{\partial z}$

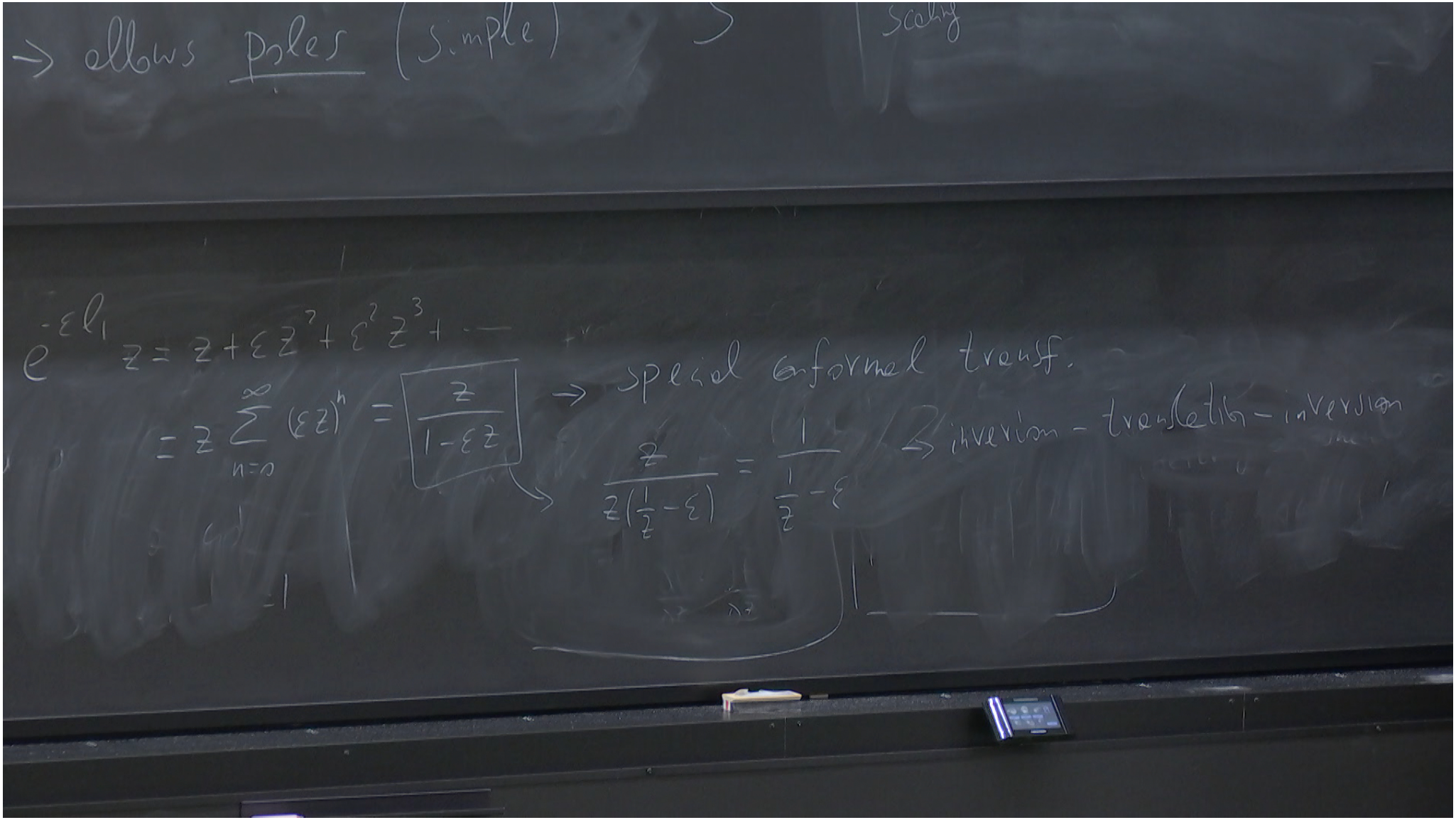
$e^{-\varepsilon l_{-1}} z = z + \varepsilon \partial_z z + \varepsilon^2 \partial_z^2 z + \dots$

$= z + \varepsilon$

$(z\partial_z)^n z = (z\partial_z)^{n-1} z = \dots = z$


dilation
rotation


dilation
rotation
$e^{-\varepsilon l_0} z = z + \varepsilon z + \frac{\varepsilon^2}{2!} z + \dots = e^{\varepsilon} z = e^{\lambda + i\theta} z$
$l_1 = -z^2 \frac{\partial}{\partial z}$
$(z^2 \partial_z)^n z = (z^2 \partial_z)^{n-1} z^2 = 2 (z^2 \partial_z)^{n-2} z^3$
$= 3! (z^2 \partial_z)^{n-3} z^4 = \dots = n! z^{n+1}$


allows poles (simple)
scaling
$e^{-\varepsilon l_1} z = z + \varepsilon z^2 + \varepsilon^2 z^3 + \cdots$
$= z \sum_{n=0}^{\infty} (\varepsilon z)^n = \frac{z}{1-\varepsilon z}$
special conformal transf.
$\frac{z}{z(\frac{1}{z} - \varepsilon)} = \frac{1}{\frac{1}{z} - \varepsilon}$
inversion - translation - inversion

real generators

$$D = -\frac{1}{2}(\ell_0 + \bar{\ell}_0) = x\partial_x + y\partial_y$$

$$R = \frac{1}{2i}(\ell_0 - \bar{\ell}_0) = x\partial_y - y\partial_x$$

$$P_x = -\frac{1}{2}(\ell_{-1} + \bar{\ell}_{-1}) = \partial_x$$

$$P_y = \frac{1}{2i}(\ell_{-1} - \bar{\ell}_{-1}) = \partial_y$$

$$P_y = \frac{1}{2i}(\ell_{-1} - \bar{\ell}_{-1}) = \partial_y$$

$$K_x = \frac{-1}{2}(\ell_1 + \bar{\ell}_1) = -(x^2+y^2)\partial_x + 2x(x\partial_x + y\partial_y)$$

$$K_y = \frac{-1}{2i}(\ell_1 - \bar{\ell}_1) = -(x^2+y^2)\partial_y + 2y(x\partial_x + y\partial_y)$$