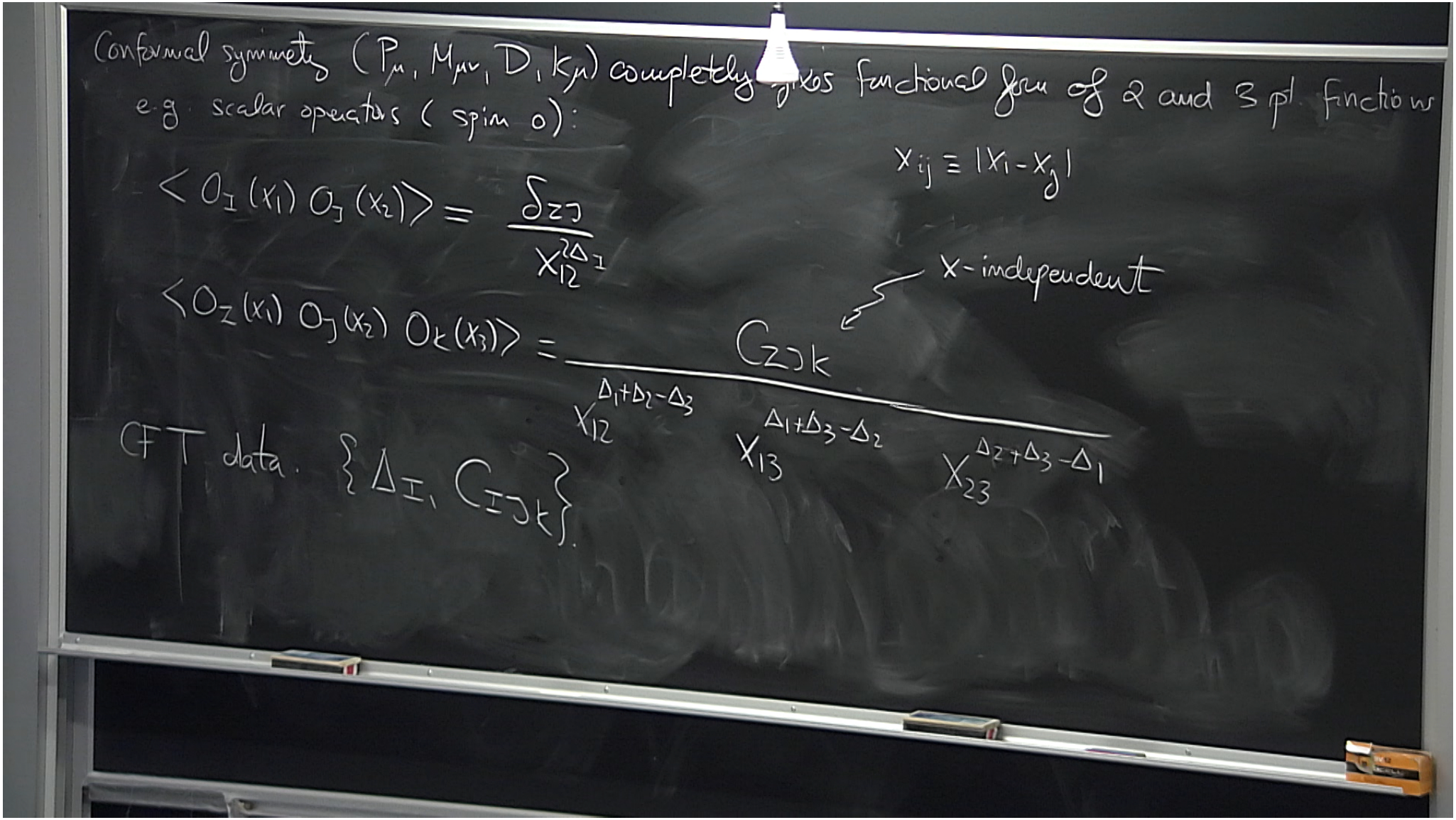
Title: PSI 2019/2020 - QFT III - Lecture 10

Speakers: Jaume Gomis

Collection: PSI 2019/2020 - QFT III

Date: February 21, 2020 - 11:15 AM

URL: http://pirsa.org/20020036


Conformal symmetry ($P_\mu, M_{\mu\nu}, D, K_\mu$) completely fixes functional form of 2 and 3 pt. functions
e.g. scalar operators (spin 0):
$x_{ij} \equiv |x_i - x_j|$
$\langle O_I(x_1) O_J(x_2) \rangle = \frac{\delta_{IJ}}{x_{12}^{2\Delta_1}}$
$\langle O_I(x_1) O_J(x_2) O_K(x_3) \rangle = \frac{C_{IJK}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{\Delta_1+\Delta_3-\Delta_2} x_{23}^{\Delta_2+\Delta_3-\Delta_1}}$
x-independent
CFT data. $\{\Delta_I, C_{IJK}\}$.

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e.g. scalar operators (spin 0): ← primary

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CFT data: $\{\Delta_I, C_{IJK}\}$

(special tensor: $I_{\mu\nu} = \frac{x^2\delta_{\mu\nu} - 2x_\mu x_\nu}{x^2}$)

$\langle O_I O_J O_K \rangle$

Conformal symmetry ($P_\mu, M_{\mu\nu}, D, K_\mu$) completely fixes functional form of 2 and 3 pt. functions

e.g. scalar operators (spin 0): primary

$$\langle O_I(x_1)\, O_J(x_2)\rangle = \frac{\delta_{IJ}}{x_{12}^{2\Delta_I}}$$

$x_{ij} \equiv |x_i - x_j|$

$$\langle O_I(x_1)\, O_J(x_2)\, O_K(x_3)\rangle = \frac{C_{IJK}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\, x_{13}^{\Delta_1+\Delta_3-\Delta_2}\, x_{23}^{\Delta_2+\Delta_3-\Delta_1}}$$

C_{IJK} — x-independent

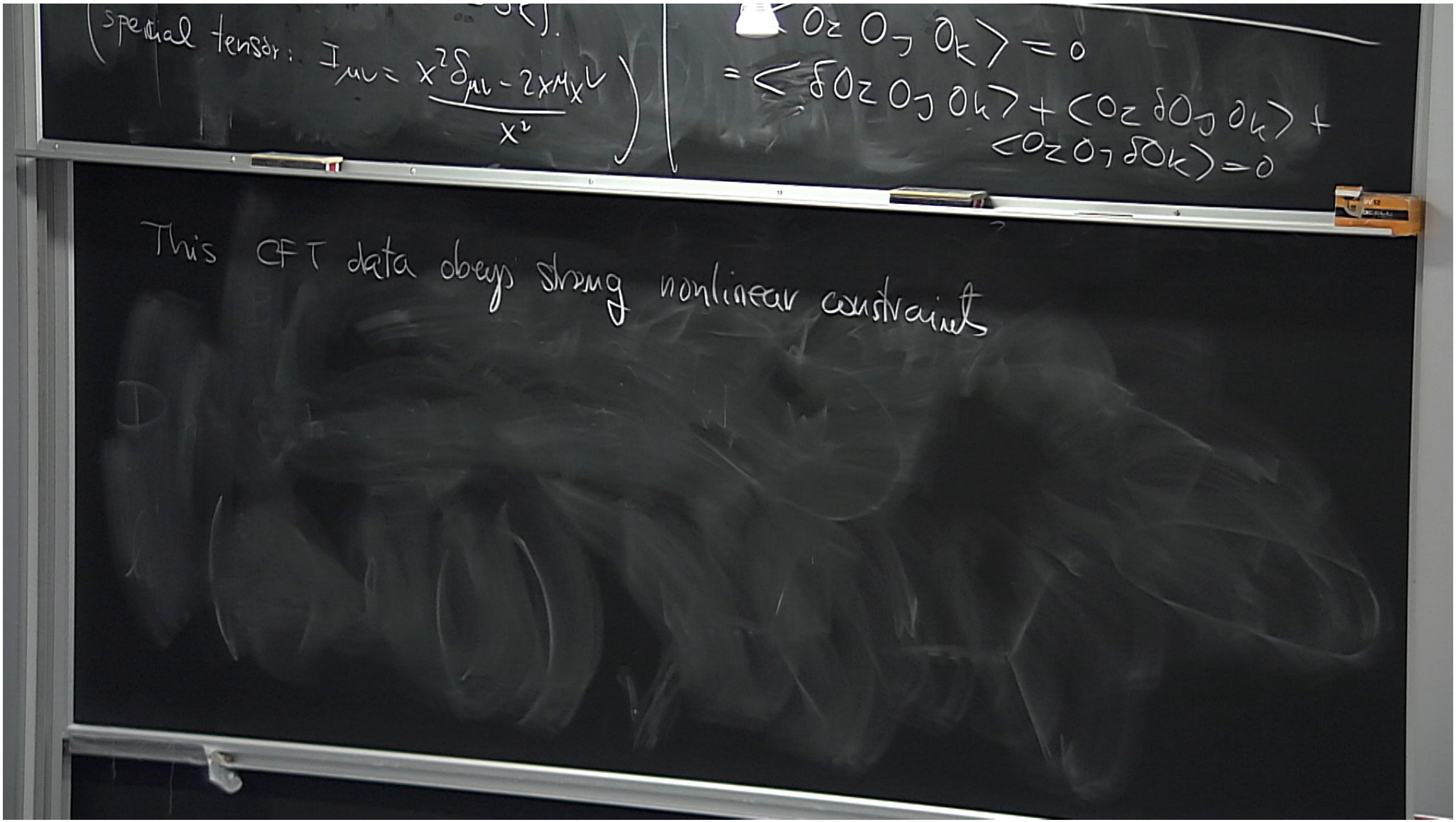
scaling dimensions

CFT data: $\{\Delta_I, C_{IJK}\}$

(special tensor: $I_{\mu\nu} = \frac{x^2\delta_{\mu\nu} - 2x_\mu x_\nu}{x^2}$)

$$\delta\langle O_I O_J O_K\rangle = 0$$

$$= \langle \delta O_I\, O_J\, O_K\rangle + \langle O_I\, \delta O_J\, O_K\rangle + \langle O_I\, O_J\, \delta O_K\rangle = 0$$


special tensor: $I_{\mu\nu} = x^2\delta_{\mu\nu} - 2x_\mu x_\nu / x^2$
$\langle O_i O_j O_k \rangle = 0$
$= \langle \delta O_i O_j O_k \rangle + \langle O_i \delta O_j O_k \rangle + \langle O_i O_j \delta O_k \rangle = 0$
This CFT data obeys strong nonlinear constraints

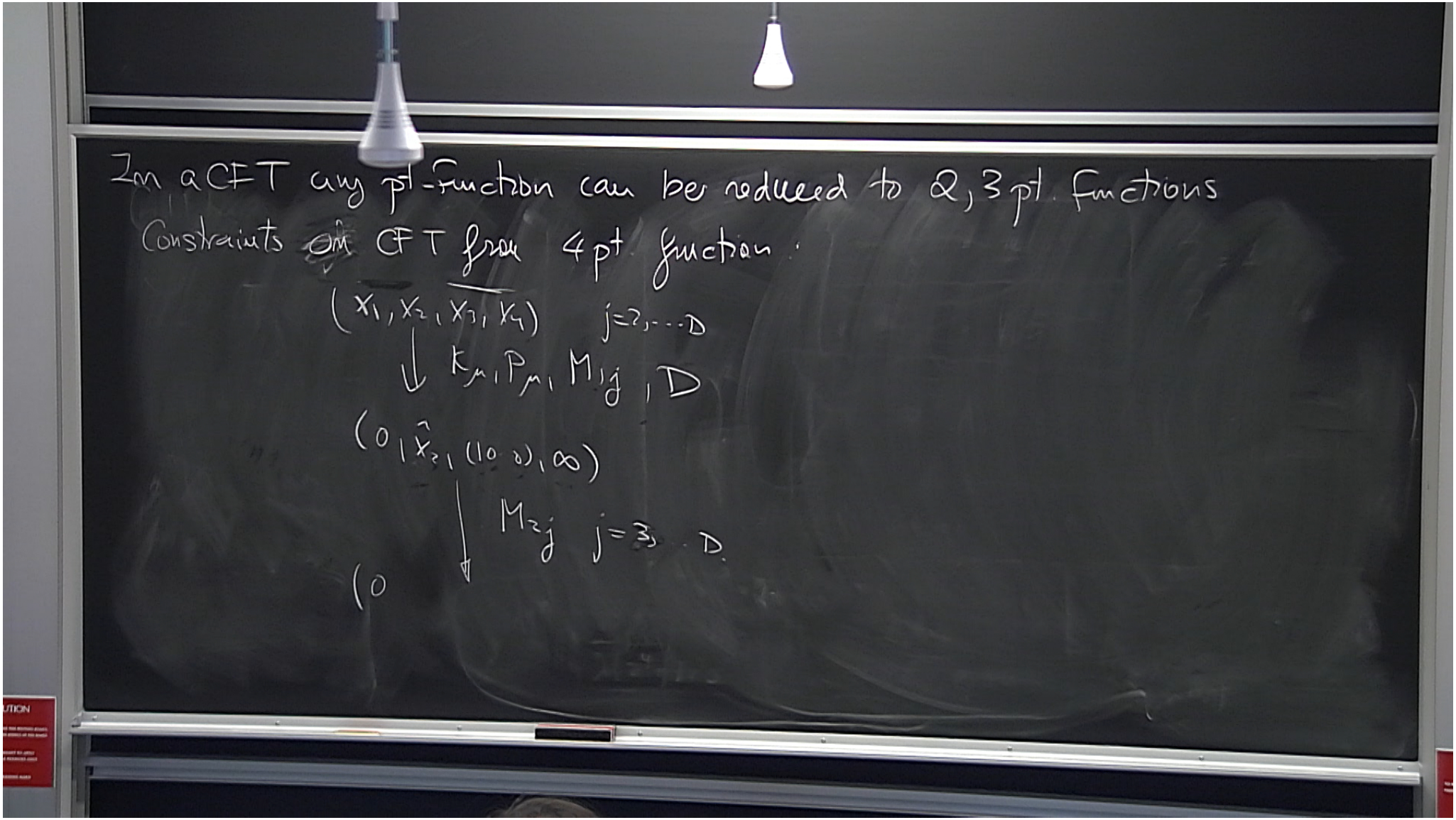
(special tensor: $I_{\mu\nu} = \frac{x^2\delta_{\mu\nu} - 2x_\mu x_\nu}{x^2}$)

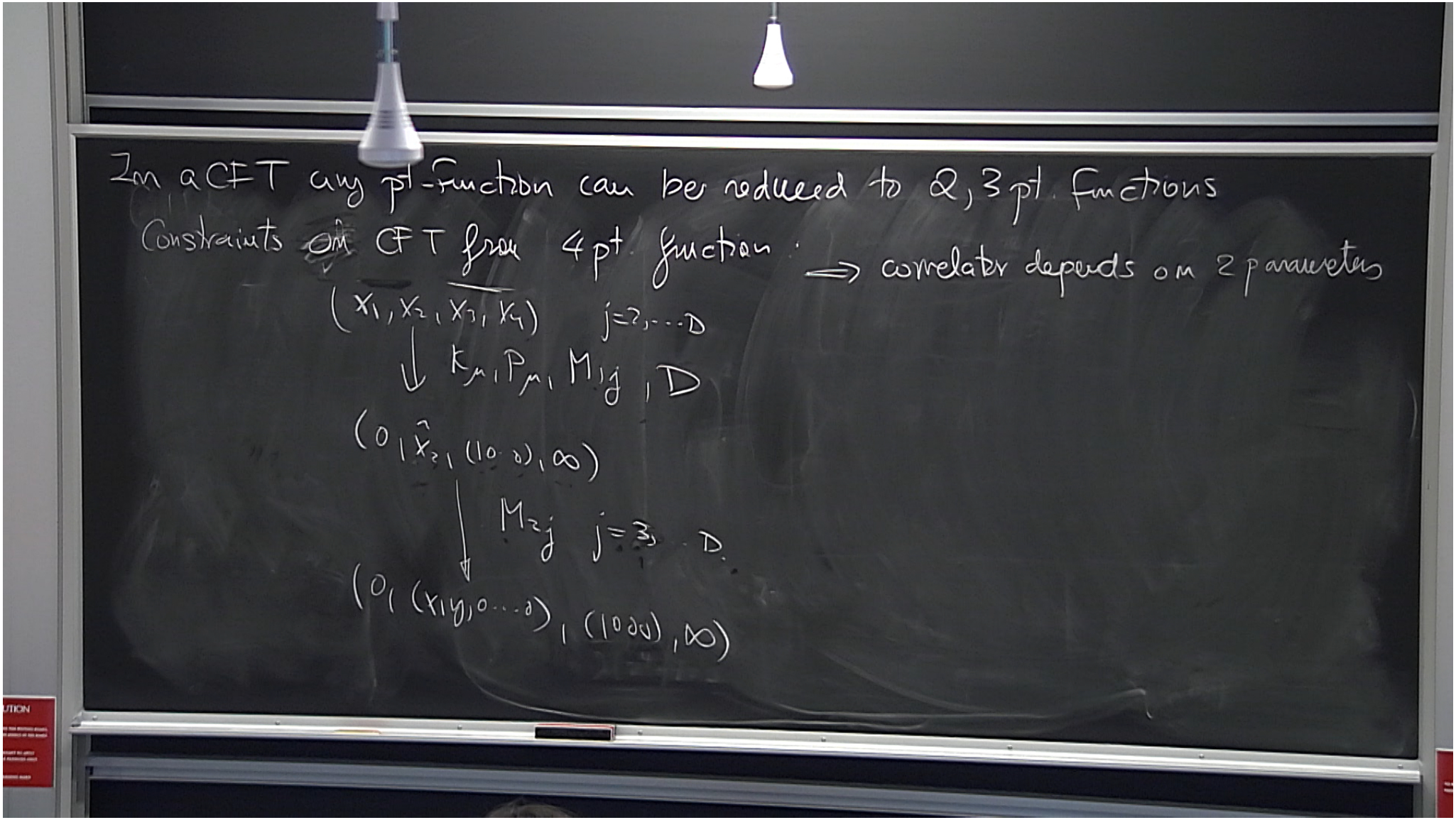
$\langle O_i O_j O_k \rangle = 0$

$= \langle \delta O_i O_j O_k \rangle + \langle O_i \delta O_j O_k \rangle + \langle O_i O_j \delta O_k \rangle = 0$

This CFT data obeys strong nonlinear constraints

$\Rightarrow$ generalization of Jacobi identity in Lie algebra


In a CFT any pt-function can be reduced to 2, 3 pt. functions
Constraints on CFT from 4pt function:
(x_1, x_2, x_3, x_4) $j=2,\dots D$
$\downarrow$ k_μ, P_μ, M_{1j}, D
$(0, 1, \hat{x}_2, (1,0,\dots 0), \infty)$
$\downarrow$ M_{2j} $j=3,\dots D$
$(0$


In a CFT any pt-function can be reduced to 2, 3 pt. functions
Constraints on CFT from 4pt. function: ⟹ correlator depends on 2 parameters
(x_1, x_2, x_3, x_4) $j=2,\dots D$
$\downarrow k_n, P_n, M_{1j}, D$
$(0, 1, \hat{x}_2, (1,0\dots 0), \infty)$
$\downarrow M_{2j}$ $j=3,\dots D$
$(0, (x_1 y, 0\dots 0), (1,0\dots 0), \infty)$

In a CFT any pt-function can be reduced to 2, 3 pt. functions

Constraints on CFT from 4pt. function: $\Longrightarrow$ correlator depends on 2 parameters

(x_1, x_2, x_3, x_4) $j = 2, \ldots D$

$\downarrow$ k_M, P_M, M_{1j}, D

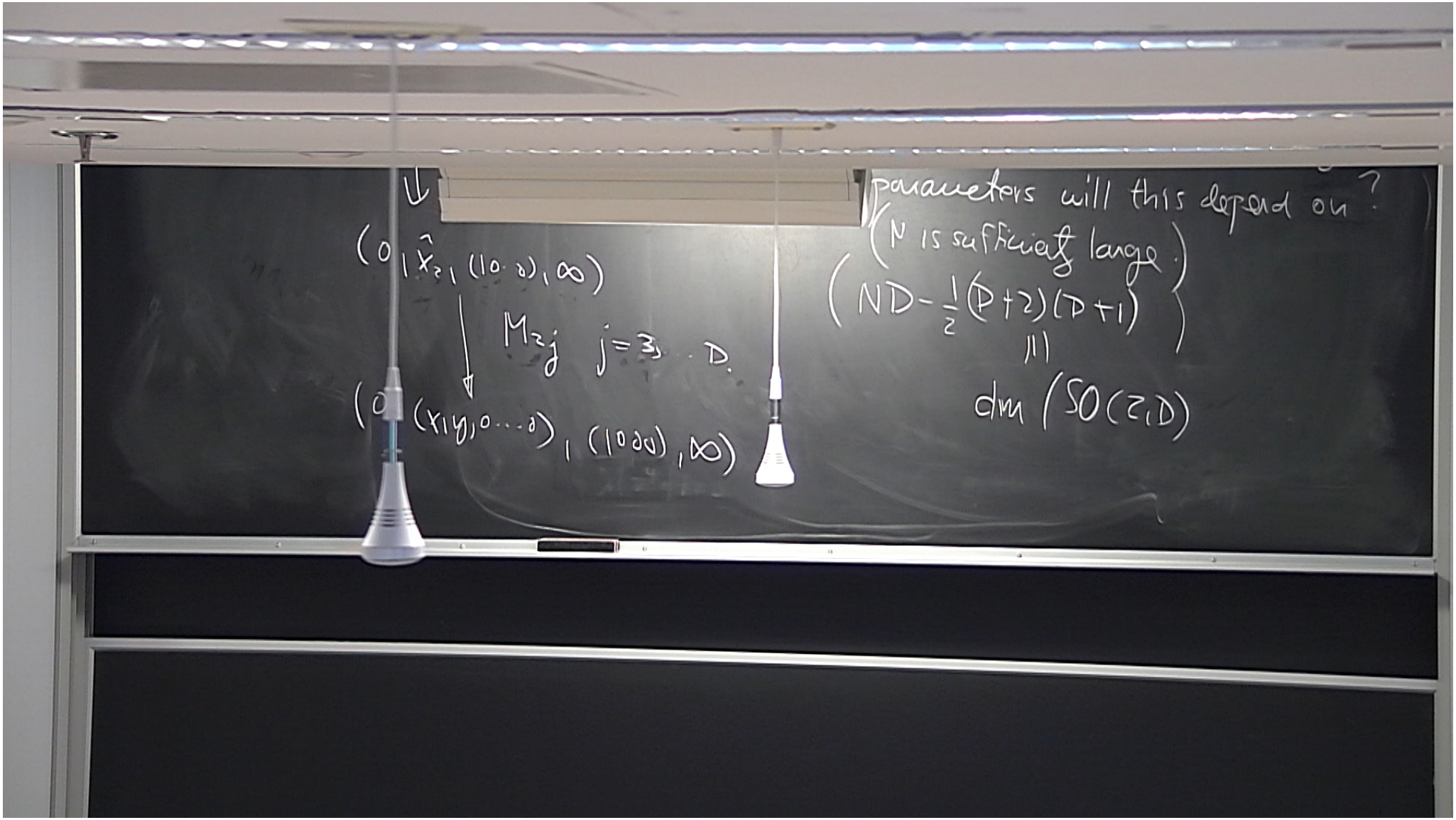
$(0, \hat{x}_2, (1\,0\,\ldots 0), \infty)$

$\downarrow$ M_{2j} $j = 3 \ldots D$

$(0, (x_1 y, 0 \ldots 0), (1\,0\,\ldots 0), \infty)$

N-point function. How many parameters will this depend on? (N is sufficiently large.)

ND


parameters will this depend on?
(N is sufficiently large)
$ND - \frac{1}{2}(D+2)(D+1)$
$\dim SO(2,D)$

$(1000), \infty)$

For 4 pts and higher, 7 conformally invariant cross ratios.

4 pt: $u = \dfrac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2}$ $\qquad v = \dfrac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2}$

2 cross ratios (u, v) $\qquad 1 \leftrightarrow 3$

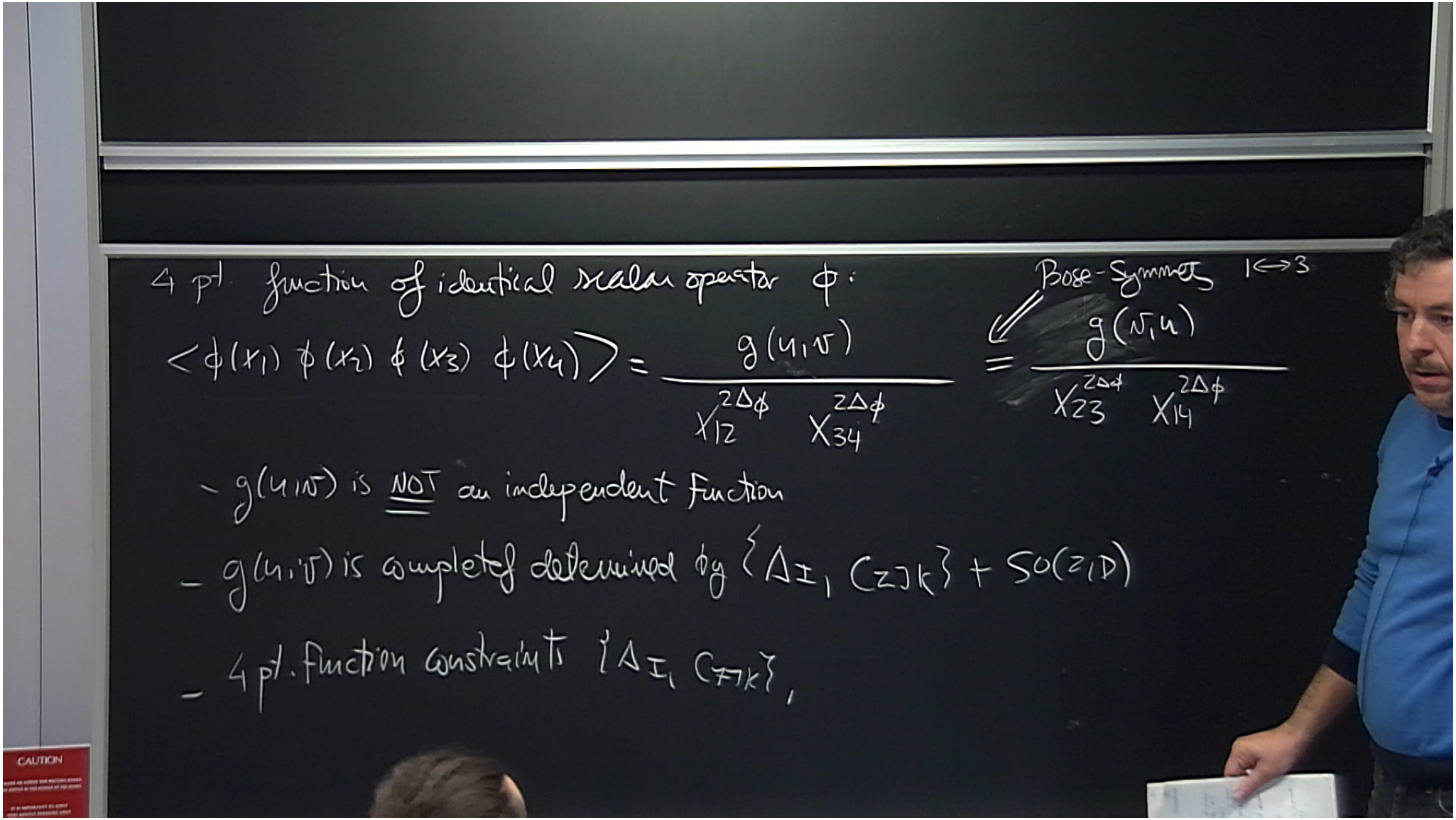
I: $\tilde{x}^\mu = \dfrac{x^\mu}{x^2}$ $\qquad \dfrac{1}{\tilde{x}_{ij}^2} = \dfrac{x_i^2\, x_j^2}{x_{ij}^2}$

4 pt. function of identical scalar operator ϕ:

$$\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle = \frac{g(u,v)}{x_{12}^{2\Delta_\phi}\, x_{34}^{2\Delta_\phi}}$$

- $g(u,v)$ is NOT an independent function
- $g(u,v)$ is completely determined by $\{\Delta_I, C_{ijk}\}$ + $SO(2,D)$
- 4 pt. function constraints $\{\Delta_I, C_{ijk}\}$,

CAUTION


4 pt. function of identical scalar operator ϕ:
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Bose-Symmetry $1 \leftrightarrow 3$
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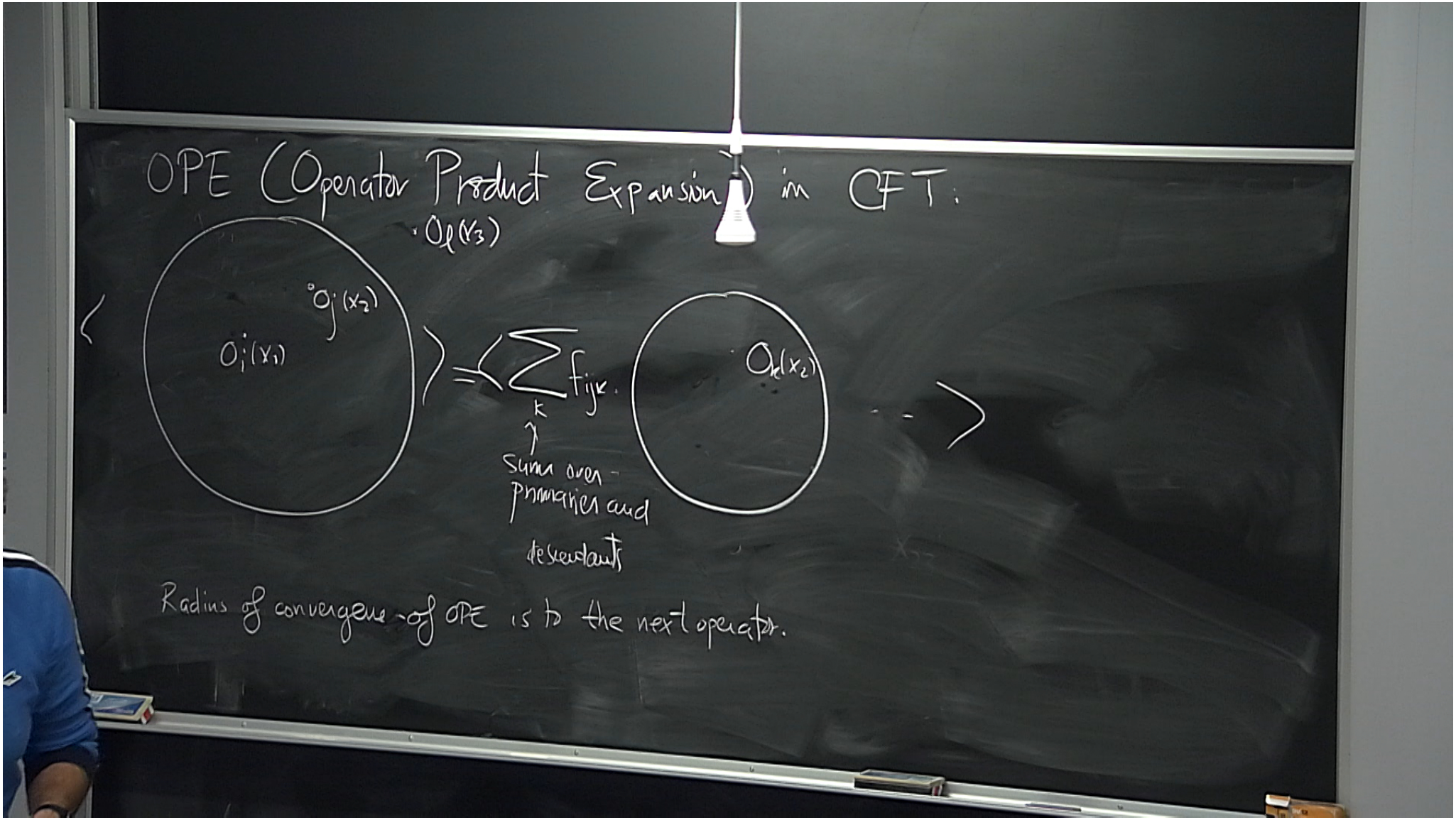
4 pt. function of identical scalar operator ϕ:

$$\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle = \frac{g(u,v)}{x_{12}^{2\Delta_\phi}\, x_{34}^{2\Delta_\phi}} = \frac{g(v,u)}{x_{23}^{2\Delta_\phi}\, x_{14}^{2\Delta_\phi}}$$

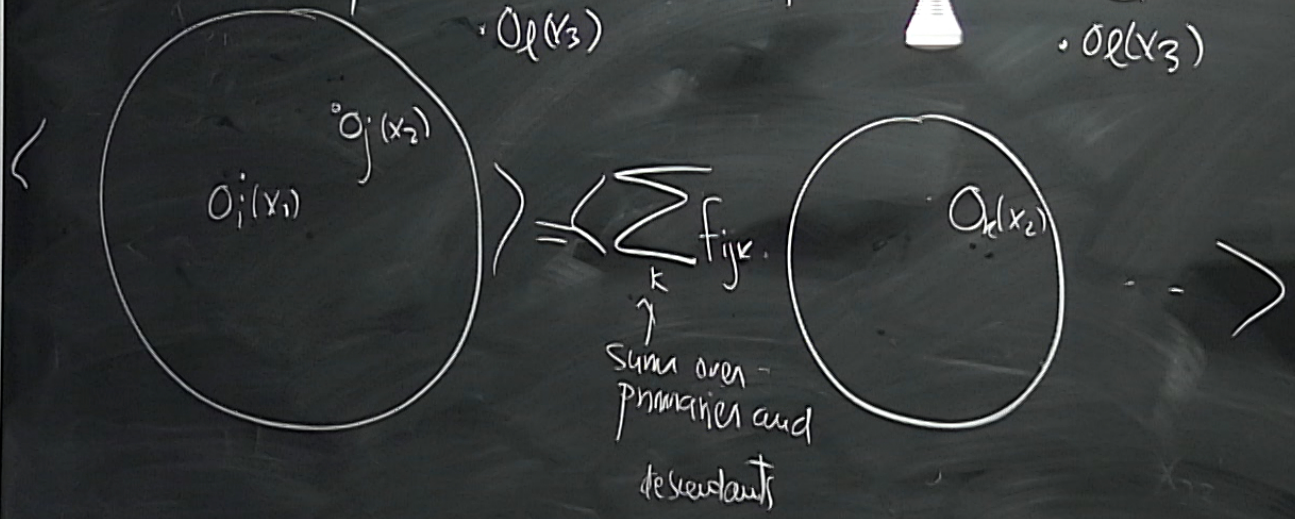
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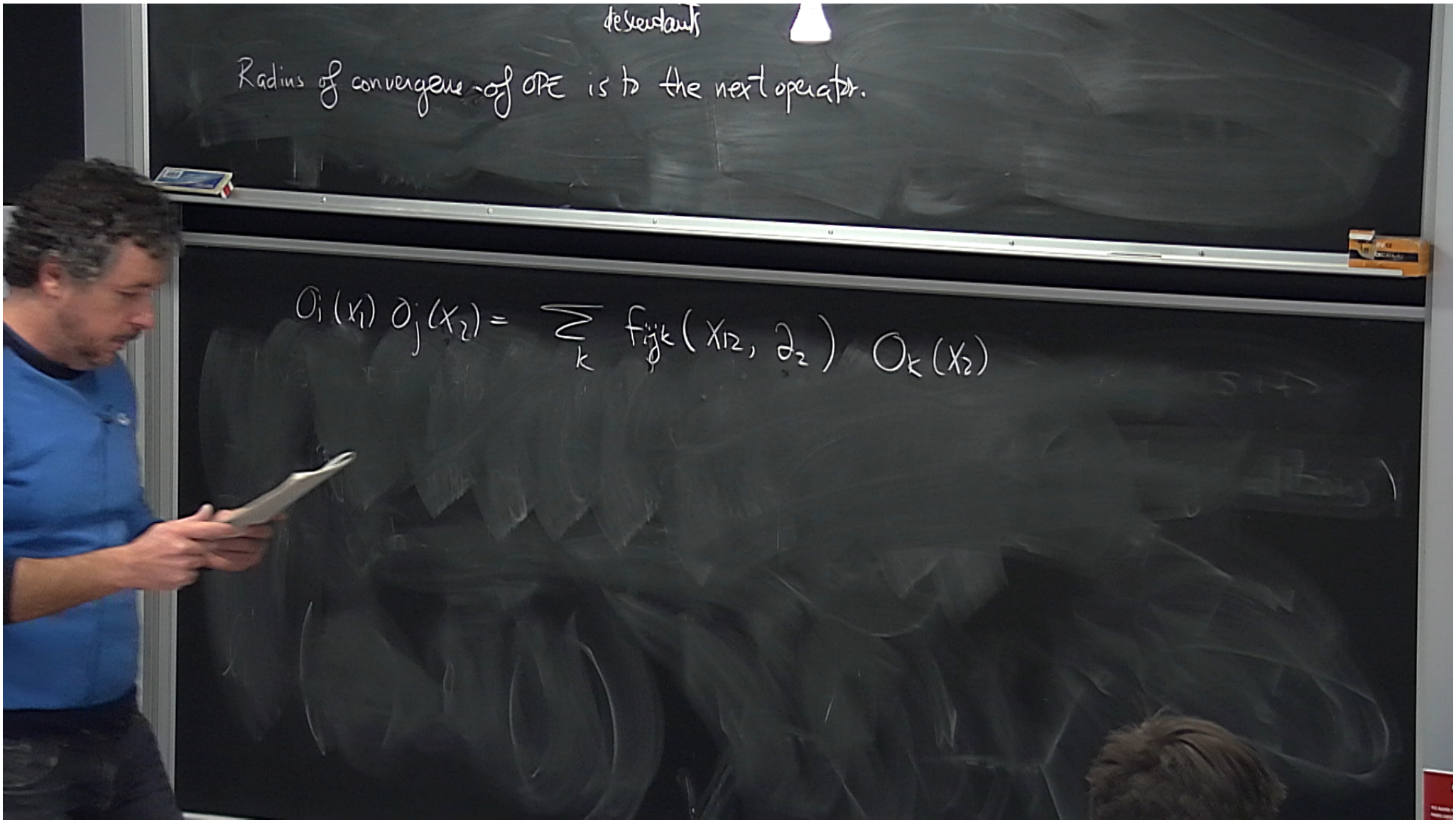
$$v^{\Delta_\phi} g(u,v) = u^{\Delta_\phi} g(v,u)$$

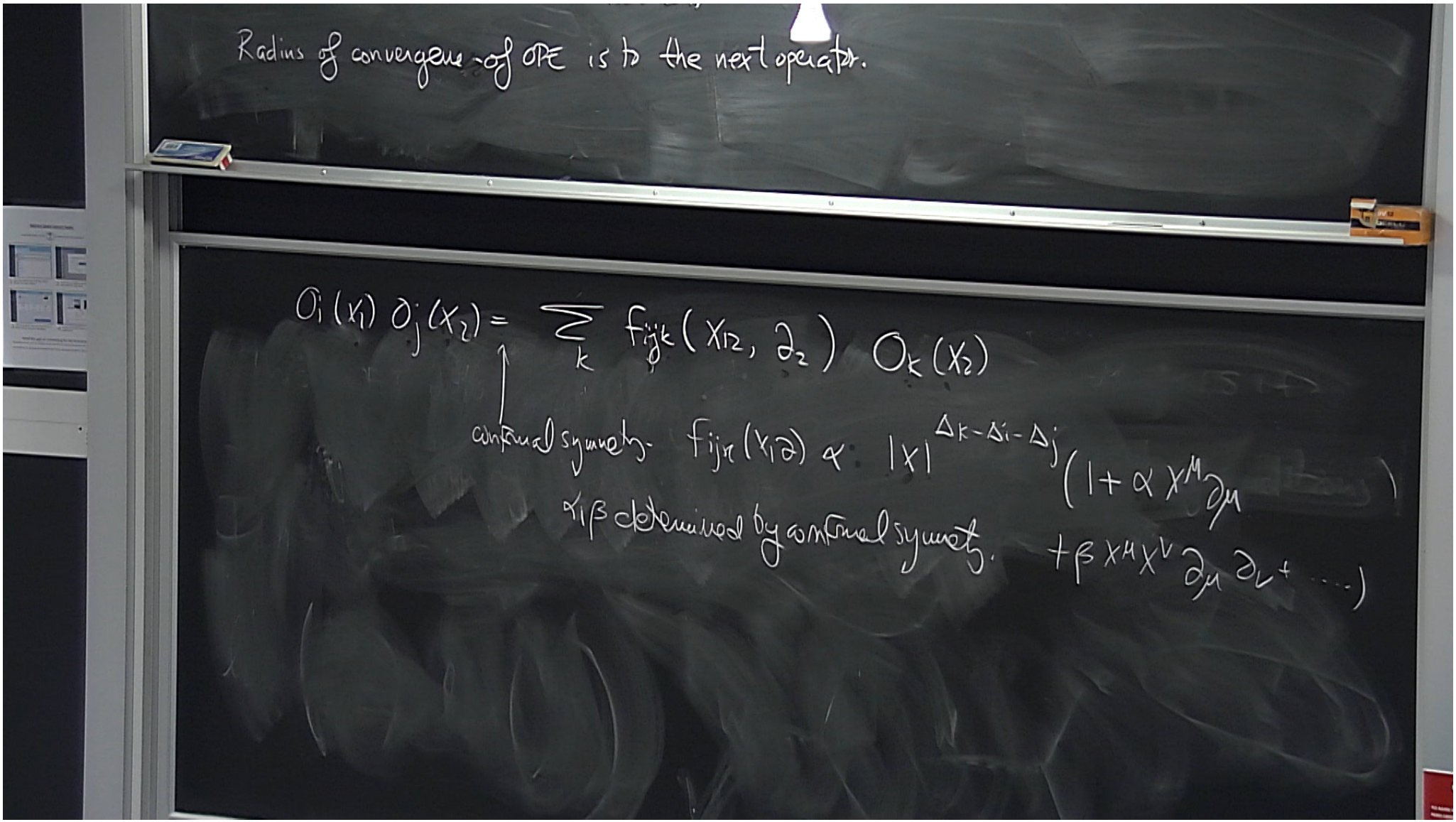

OPE (Operator Product Expansion) in CFT:
$O_l(x_3)$
$O_i(x_1)$
$O_j(x_2)$
$= \sum_k f_{ijk}$
$O_k(x_2)$
sum over primaries and descendants
Radius of convergence of OPE is to the next operator.

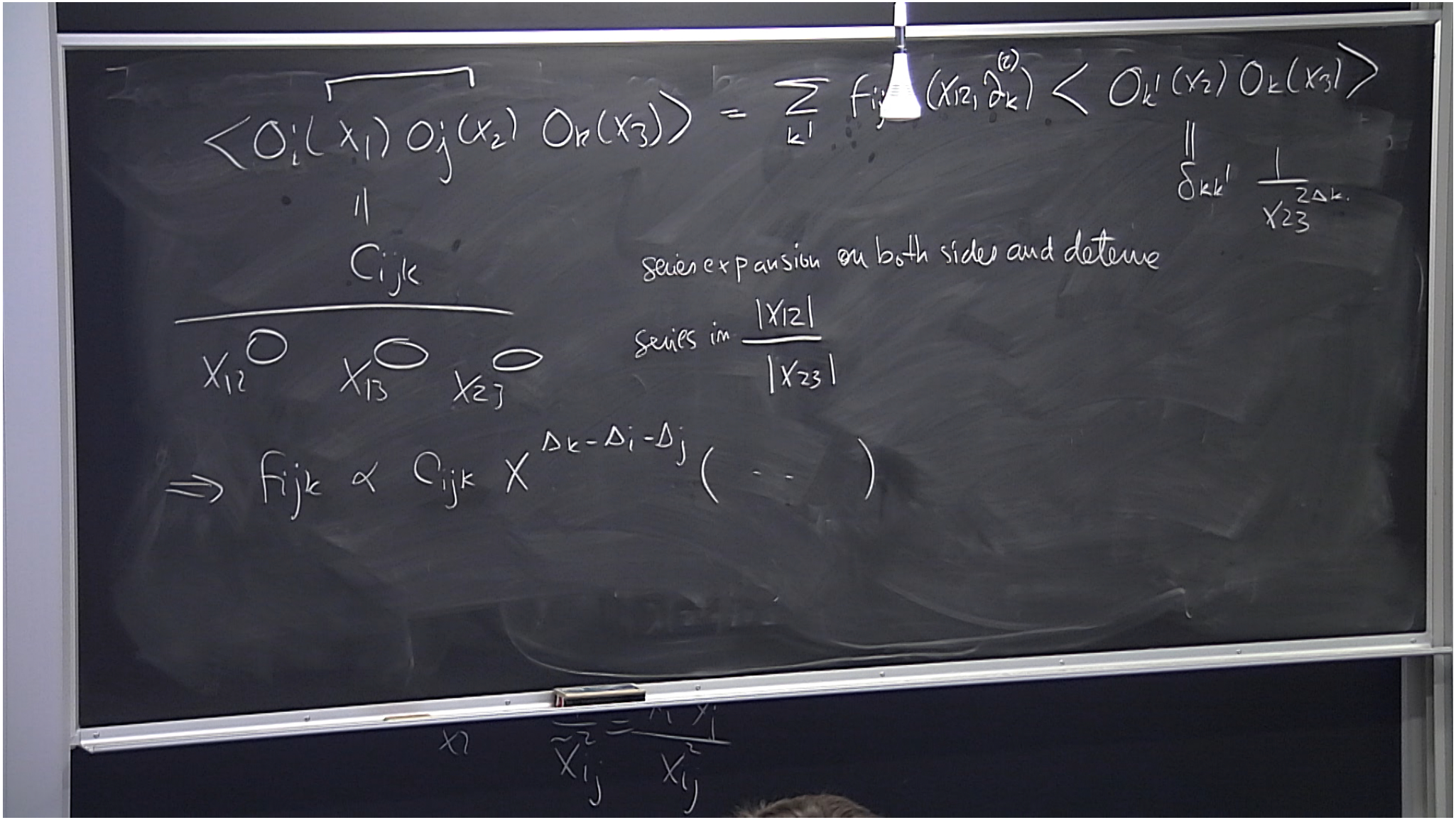
OPE (Operator Product Expansion) in CFT:



Radius of convergence of OPE is to the next operator.


descendants
Radius of convergence of OPE is to the next operator.
$O_i(x_1)\, O_j(x_2) = \sum_k f_{ij}^k(x_{12}, \partial_2)\, O_k(x_2)$


Radius of convergence of OPE is to the next operator.
$O_i(x_1)\, O_j(x_2) = \sum_k f_{ijk}(x_{12}, \partial_2)\, O_k(x_2)$
conformal symmetry. $f_{ijk}(x_{12}) \propto |x|^{\Delta_k - \Delta_i - \Delta_j}$
$(1 + \alpha x^\mu \partial_\mu$
α, β determined by conformal symmetry. $+ \beta x^\mu x^\nu \partial_\mu \partial_\nu + \ldots)$


$\langle O_i(x_1) O_j(x_2) O_k(x_3) \rangle = \sum_{k'} f_{ij}^{k'}(x_{12}, \partial_k^{(2)}) \langle O_{k'}(x_2) O_k(x_3) \rangle$
$= \delta_{kk'} \frac{1}{x_{23}^{2\Delta_k}}$
$= C_{ijk}$
x_{12} x_{13} x_{23}
series expansion on both sides and determine
series in $\frac{|x_{12}|}{|x_{23}|}$
$\Rightarrow f_{ijk} \propto C_{ijk} X^{\Delta_k - \Delta_i - \Delta_j} (\cdots)$

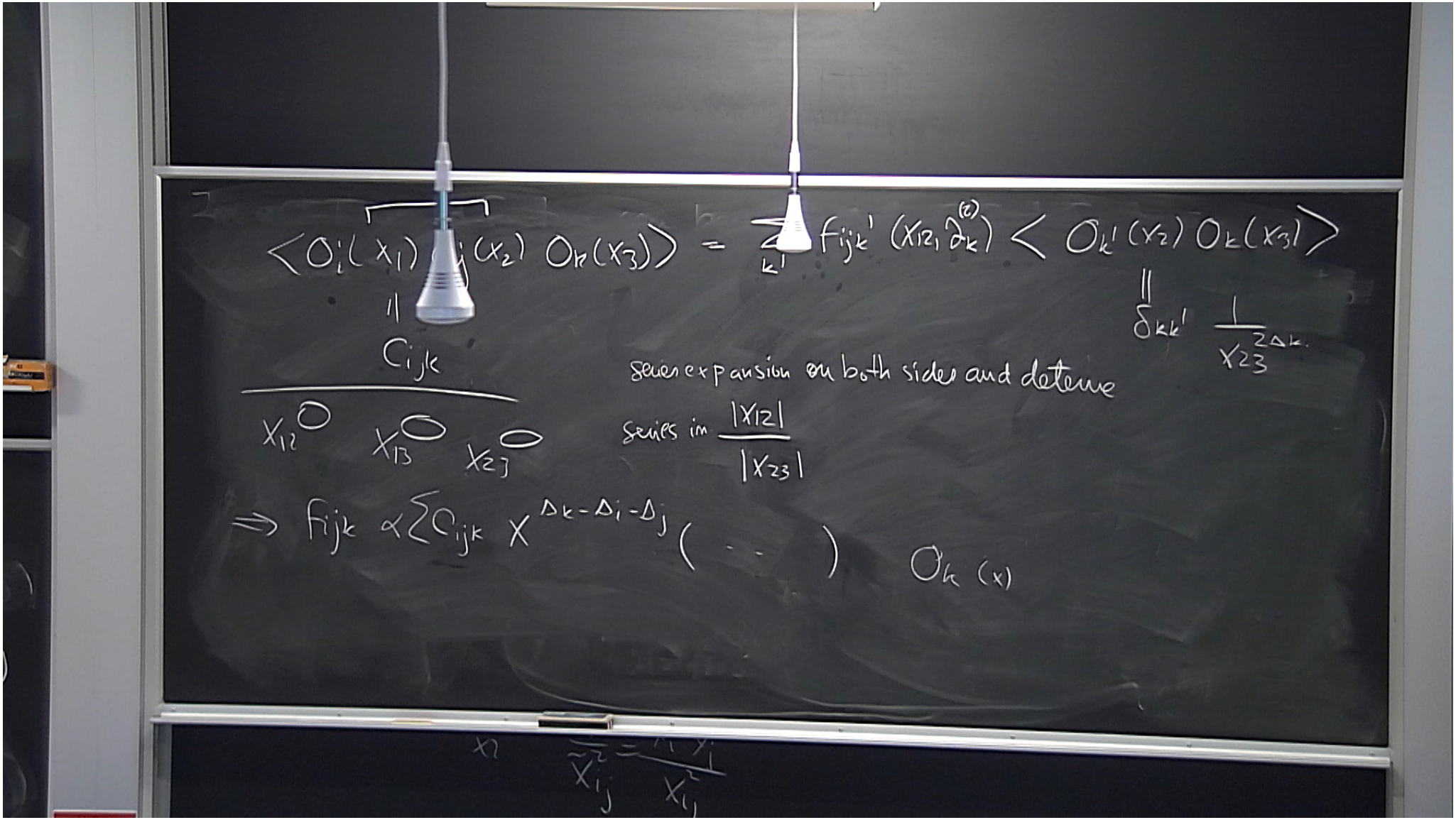
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- 4pt. function constraints $\{\Delta_I, C_{IJK}\}$,

$$v^{\Delta_\phi} g(u,v) = u^{\Delta_\phi} g(v,u)$$

4pt function of identical scalars

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$$


$\langle O_i(x_1) O_j(x_2) O_k(x_3) \rangle = \sum_{k'} f_{ijk'}(x_{12}, \partial_k^{(z)}) \langle O_{k'}(x_2) O_k(x_3) \rangle$
$\| C_{ijk}$
$\| \delta_{kk'} \frac{1}{x_{23}^{2\Delta_k}}$
x_{12} x_{13} x_{2j}
series expansion on both sides and determine
series in $\frac{|x_{12}|}{|x_{23}|}$
$\Rightarrow f_{ijk} \propto \sum C_{ijk} x^{\Delta_k - \Delta_i - \Delta_j} (\cdots) O_k(x)$

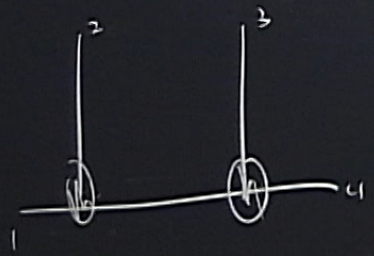
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4pt function of identical scalars

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{\mathcal{O},\mathcal{O}'} C_{\phi\phi\mathcal{O}} C_{\phi\phi\mathcal{O}'} a(x_{12},\partial_2)\, b(x_{34},\partial_4) \times \langle \mathcal{O}^a(x_2)\mathcal{O}^b(x_4)\rangle$$

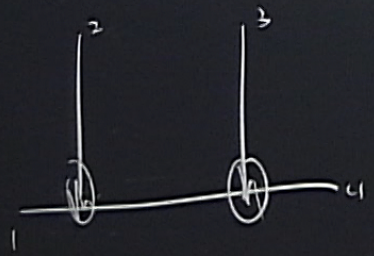


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$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{a,b} C_{\phi\phi O} C_{\phi\phi O'} C_a(x_{12},\partial_2) C_b(x_{34},\partial_4) \times \langle O^a(x_2) O^b(x_4)\rangle$$



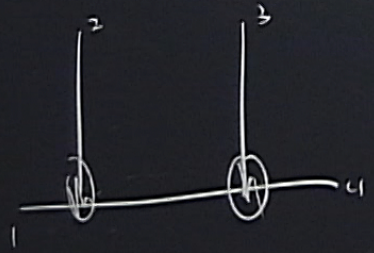
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$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{a,b} C_{\phi\phi O_a} C_{\phi\phi O_b} C_a(x_{12},\partial_2)\, C_b(x_{34},\partial_4) \times \langle O^a(x_2)\, O^b(x_4)\rangle$$

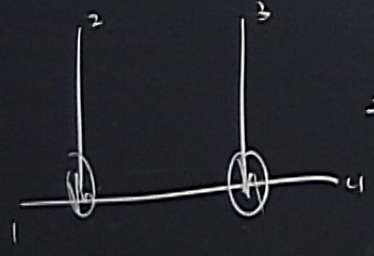


- 4pt. function constraints $\{\Delta_\mathcal{I}, C_{\mathcal{I}jk}\}$,

$$v^{\Delta_\phi} g(u,v) = u^{\Delta_\phi} g(v,u)$$

4pt function of identical scalars

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{a,b} C_{\phi\phi\mathcal{O}_a} C_{\phi\phi\mathcal{O}_b} C_a(x_{12},\partial_2) C_b(x_{34},\partial_4) \times$$



$$= \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} C_a(x_{12},\partial_2) C_b(x_{34},\partial_4) \frac{I^{ab}(x_{24})}{x_{24}^{2\Delta_\phi}} \left[\langle \mathcal{O}^a(x_2)\tilde{\mathcal{O}}^b(x_4)\rangle = \delta^{ab} \frac{I^{ab}}{x_{24}^{2\Delta_\mathcal{O}}} \right]$$

=

- 4pt. function constraints $\{\Delta_I, C_{ijk}\}$,

$$v^{\Delta_\phi} g(u,v) = u^{\Delta_\phi} g(v,u)$$

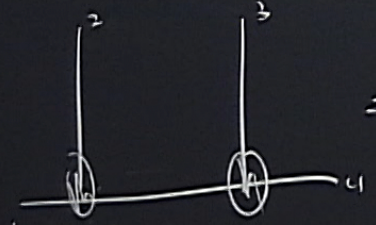
4pt function of identical scalars

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{a,b} C_{\phi\phi\mathcal{O}_a} C_{\phi\phi\mathcal{O}_b} C_a(x_{12},\partial_2) C_b(x_{34},\partial_4) \times \langle \mathcal{O}^a(x_2)\tilde{\mathcal{O}}^b(x_4)\rangle$$

$$= \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \left(C_a(x_{12},\partial_2) C_b(x_{34},\partial_4) \frac{I^{ab}(x_{24})}{x_{24}^{2\Delta_\phi}} \right)$$

$$= \frac{1}{x_{12}^{2\Delta_\phi}} \cdot \frac{1}{x_{34}^{2\Delta_\phi}} \cdot \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}}\, g_{\Delta_\mathcal{O}, \ell_\mathcal{O}}(x_i)$$

$\langle \mathcal{O}^a(x_2)\tilde{\mathcal{O}}^b(x_4)\rangle = \delta^{ab} \frac{I^{ab}}{x_{24}^{2\Delta_\mathcal{O}}}$



s-channel OPE

- 4pt. function constraints $\{\Delta_I, C_{IJK}\}$,

$$v^{\Delta_\phi} g(u,v) = u^{\Delta_\phi} g(v,u)$$

4pt function of identical scalars

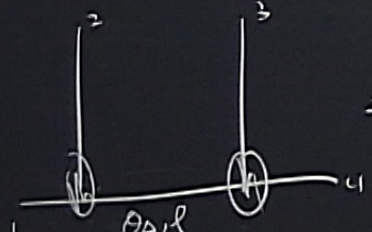
$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{a,b} C_{\phi\phi\mathcal{O}_a} C_{\phi\phi\mathcal{O}_b} C_a(x_{12},\partial_2) C_b(x_{34},\partial_4) \times \langle \mathcal{O}^a(x_2)\,\tilde{\mathcal{O}}^b(x_4)\rangle$$

$$= \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \left(C_a(x_{12},\partial_2) C_b(x_{34},\partial_4) \frac{I^{ab}(x_{24})}{x_{24}^{2\Delta_\phi}} \right)$$

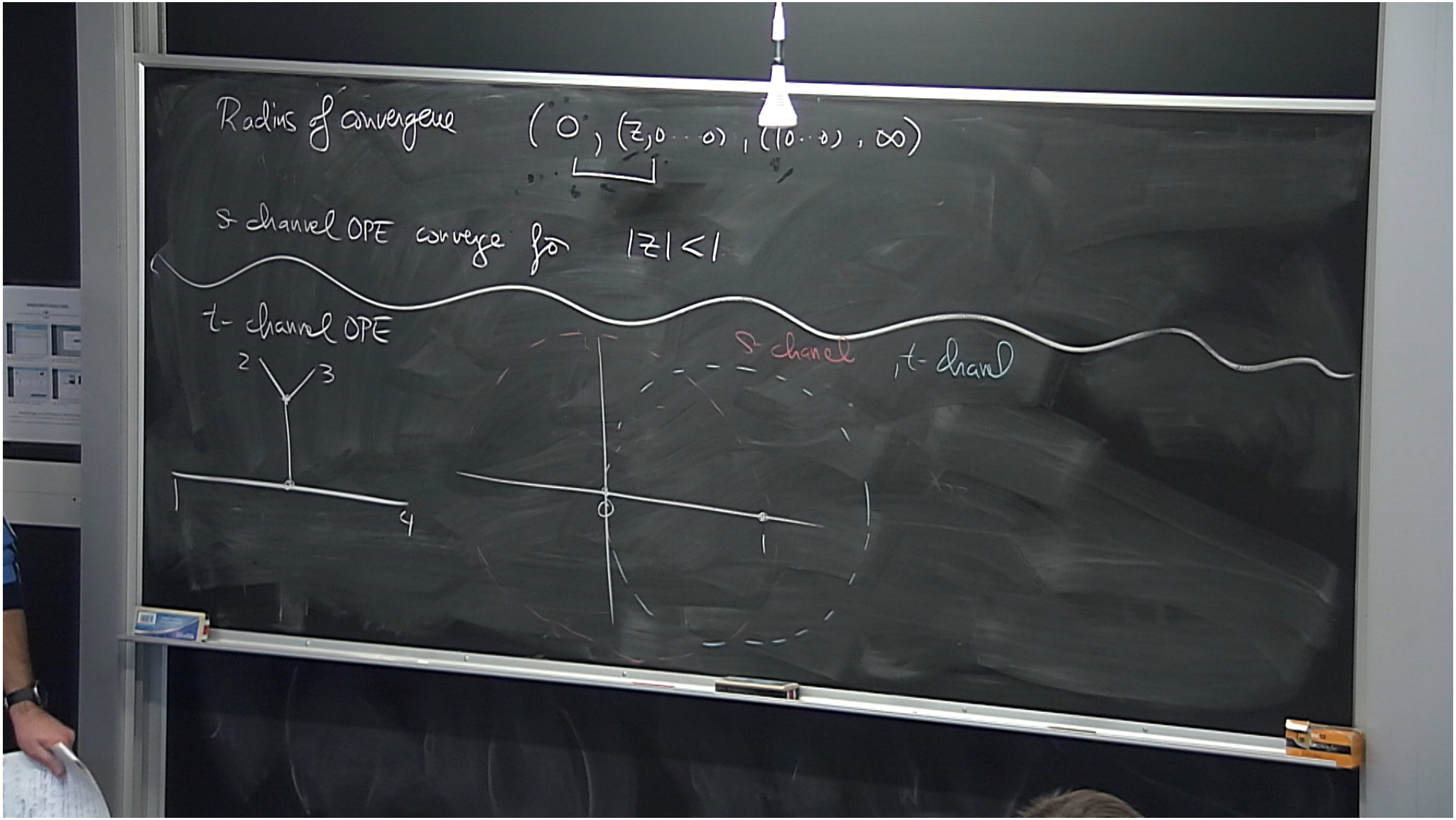
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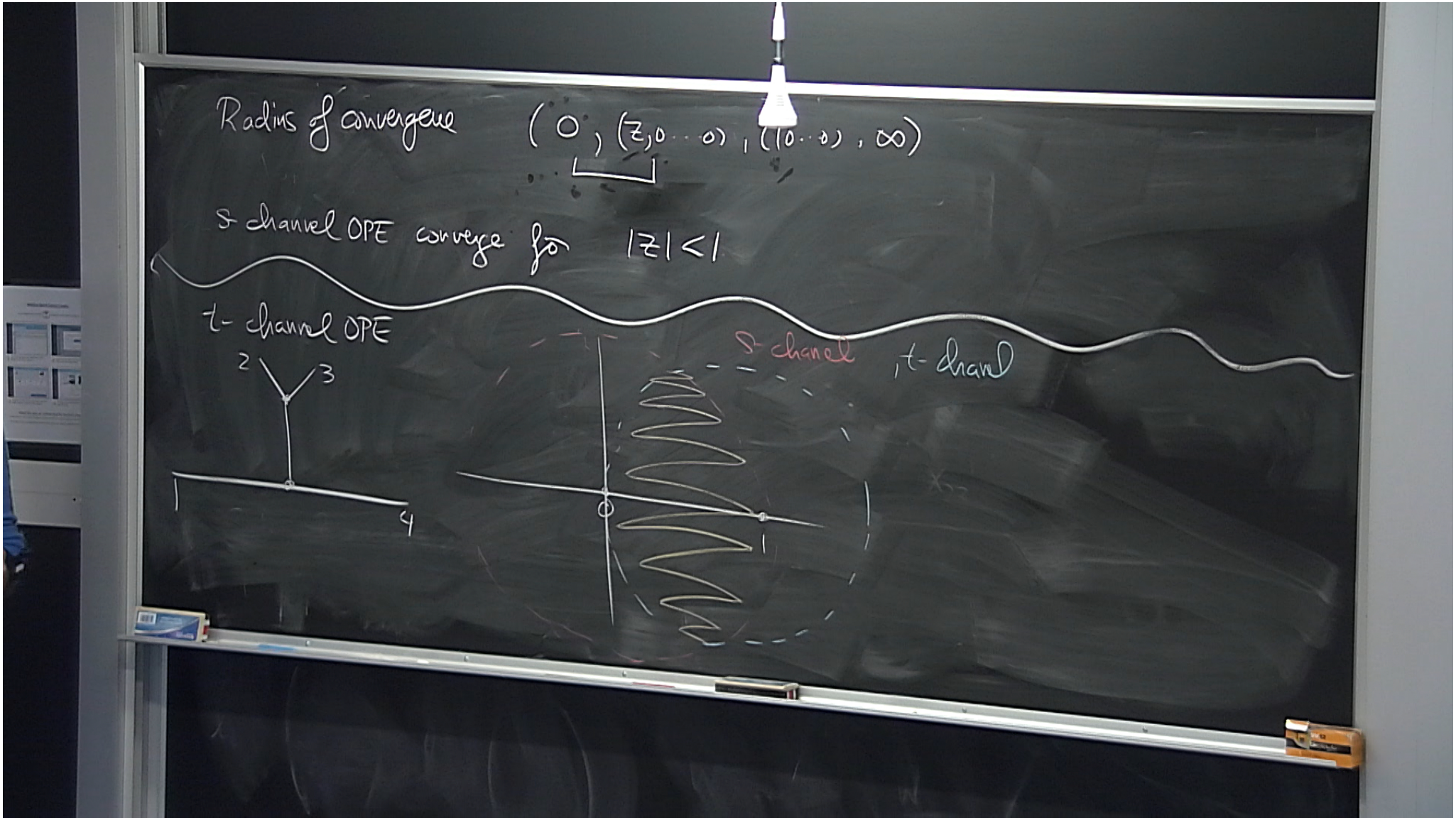
$$= \frac{1}{x_{12}^{2\Delta_\phi}} \cdot \frac{1}{x_{34}^{2\Delta_\phi}} \cdot \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}}\, g_{\Delta_\mathcal{O},\ell_\mathcal{O}}(x_i)$$

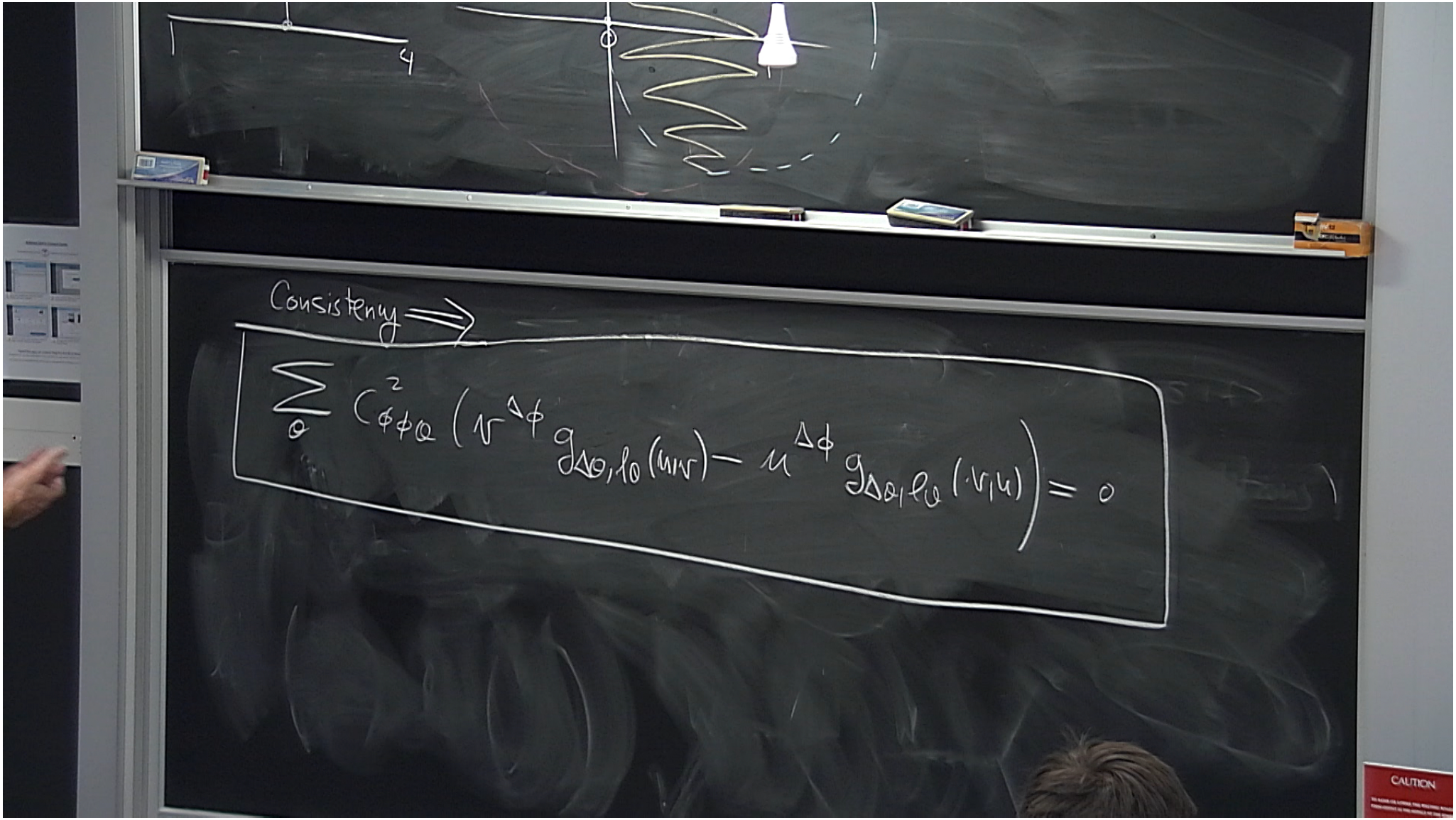
conformal blocks

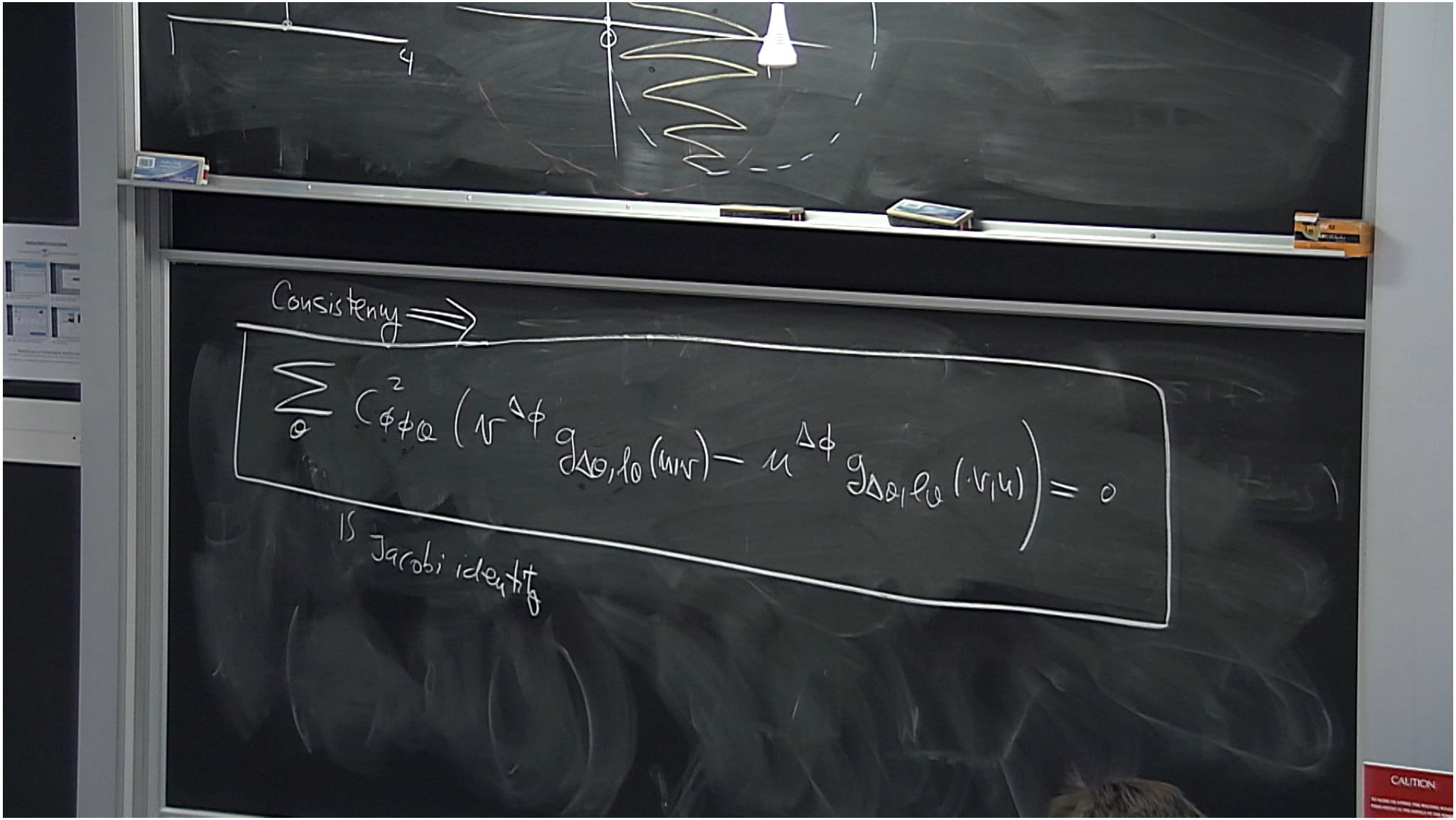


s-channel OPE


Radius of convergence (0, (z,0...0), (1...1), ∞)
s-channel OPE converge for |z|<1
t-channel OPE
2
3
4
s-channel
t-chanel


Radius of convergence (0, (z,0...0), ((10...0), ∞)
s-channel OPE converge for |z|<1
t-channel OPE
2
3
4
s-channel
,t-chanel


Consistency
$$\sum_{\alpha} C^{2}_{\phi\phi\alpha}\left(v^{\Delta_\phi} g_{\Delta_\alpha,\ell_\alpha}(u,v) - u^{\Delta_\phi} g_{\Delta_\alpha,\ell_\alpha}(v,u)\right) = 0$$
CAUTION


Consistency ⟹
$$\sum_{\alpha} C^{2}_{\phi\phi\alpha}\left(v^{\Delta_\phi} g_{\Delta_\alpha,\ell_\alpha}(u,v) - u^{\Delta_\phi} g_{\Delta_\alpha,\ell_\alpha}(v,u)\right) = 0$$
IS Jacobi identity
CAUTION