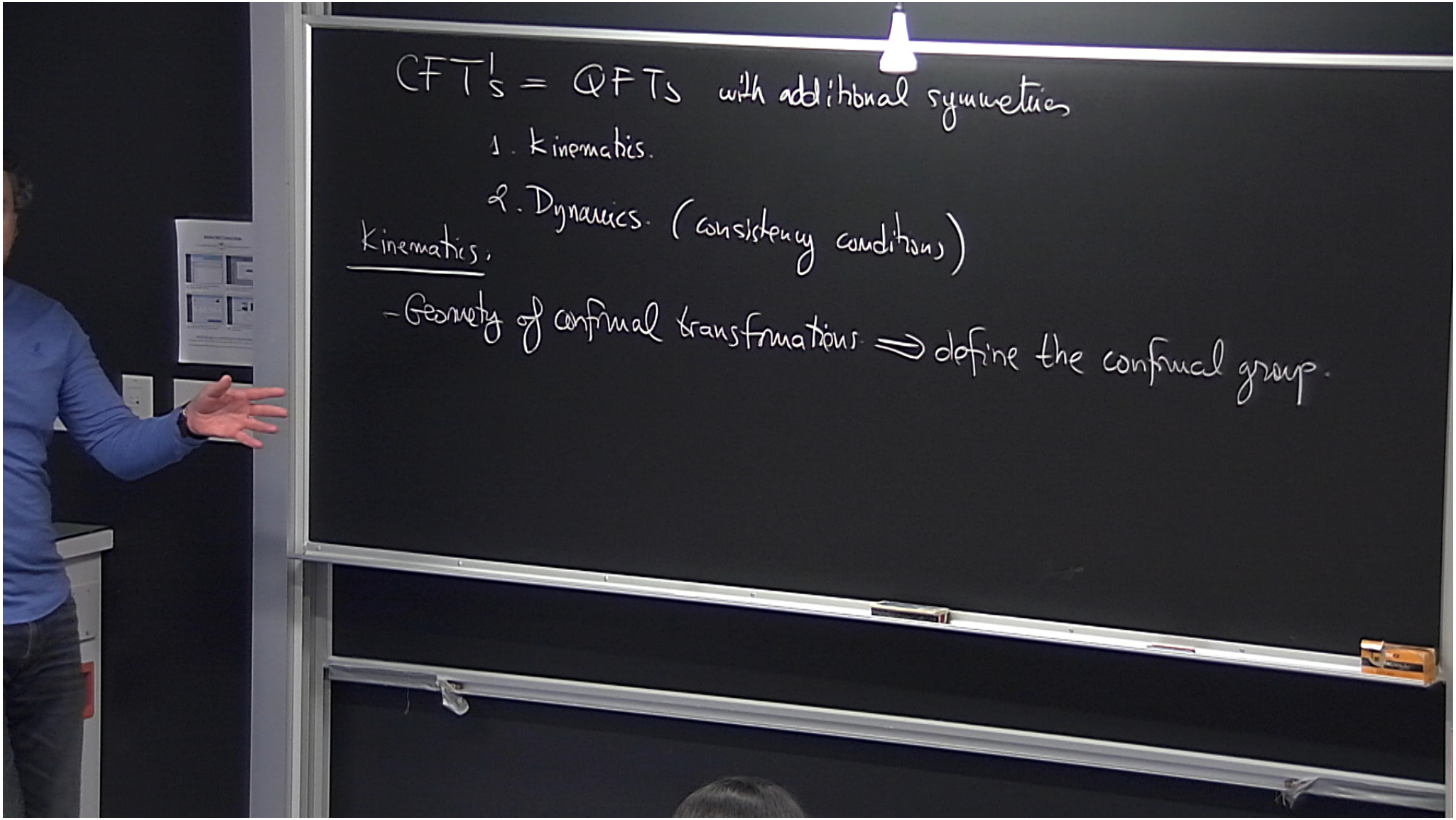
Title: PSI 2019/2020 - QFT III - Lecture 8

Speakers: Jaume Gomis

Collection: PSI 2019/2020 - QFT III

Date: February 14, 2020 - 11:15 AM

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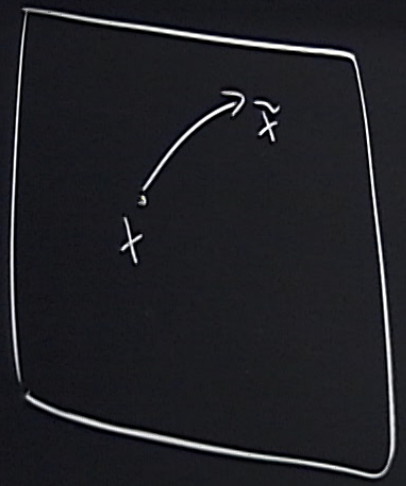

CFT's = QFTs with additional symmetries
1. kinematics.
2. Dynamics. (consistency conditions)
Kinematics:
- Geometry of conformal transformations ⇒ define the conformal group.

- Geometry of conformal transformations $\Rightarrow$ define the conformal group G.
- Discuss continous transformations connected to identity $\Rightarrow$ Lie algebra of G.

$[T_a, T_b] = i f_{ab}^{\ c} T_c$. $\Rightarrow$ conformal algebra.

spacetime - transformations

$R^{1,D-1}$ = Minkowski space

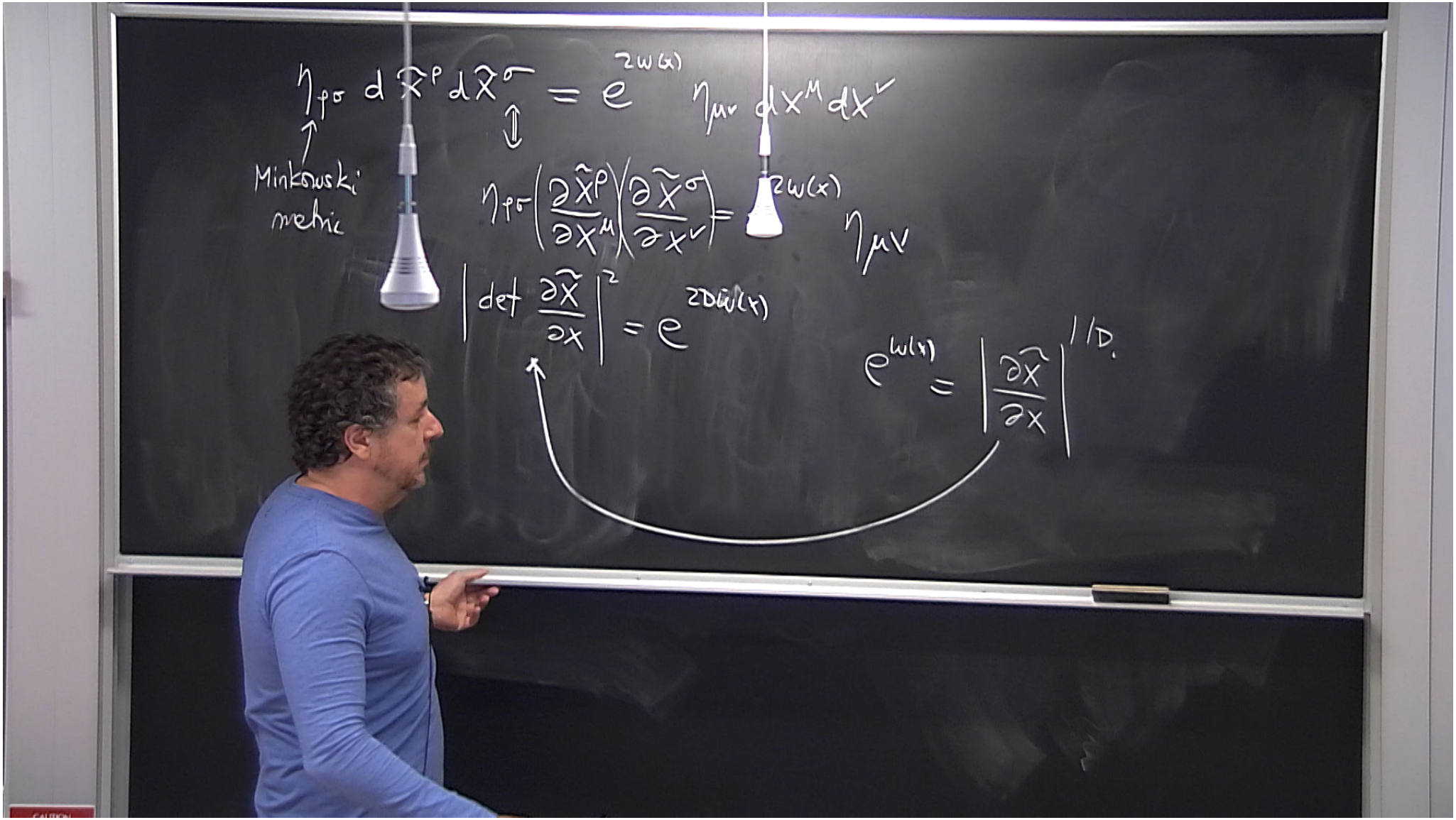


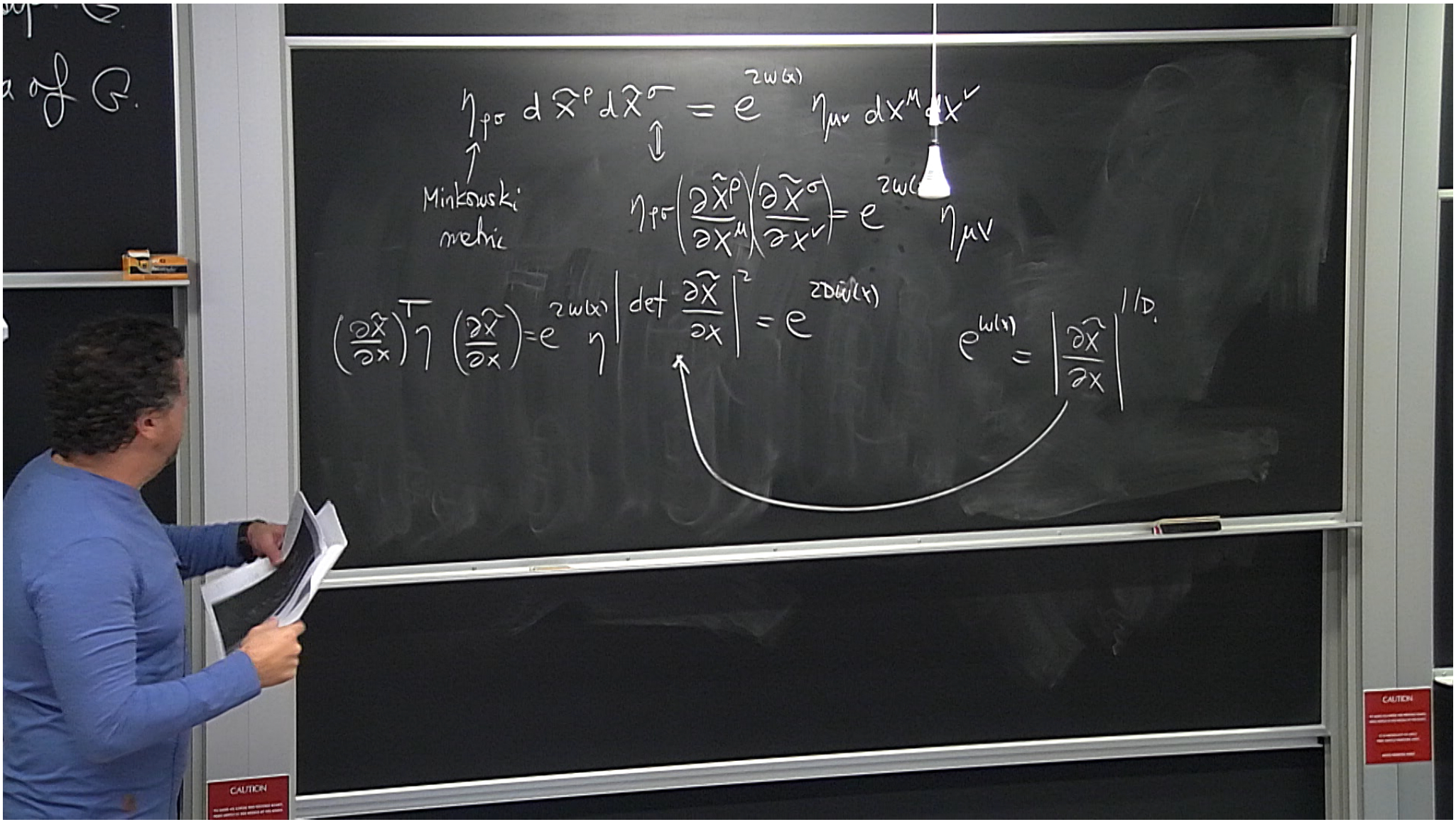
- Coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$

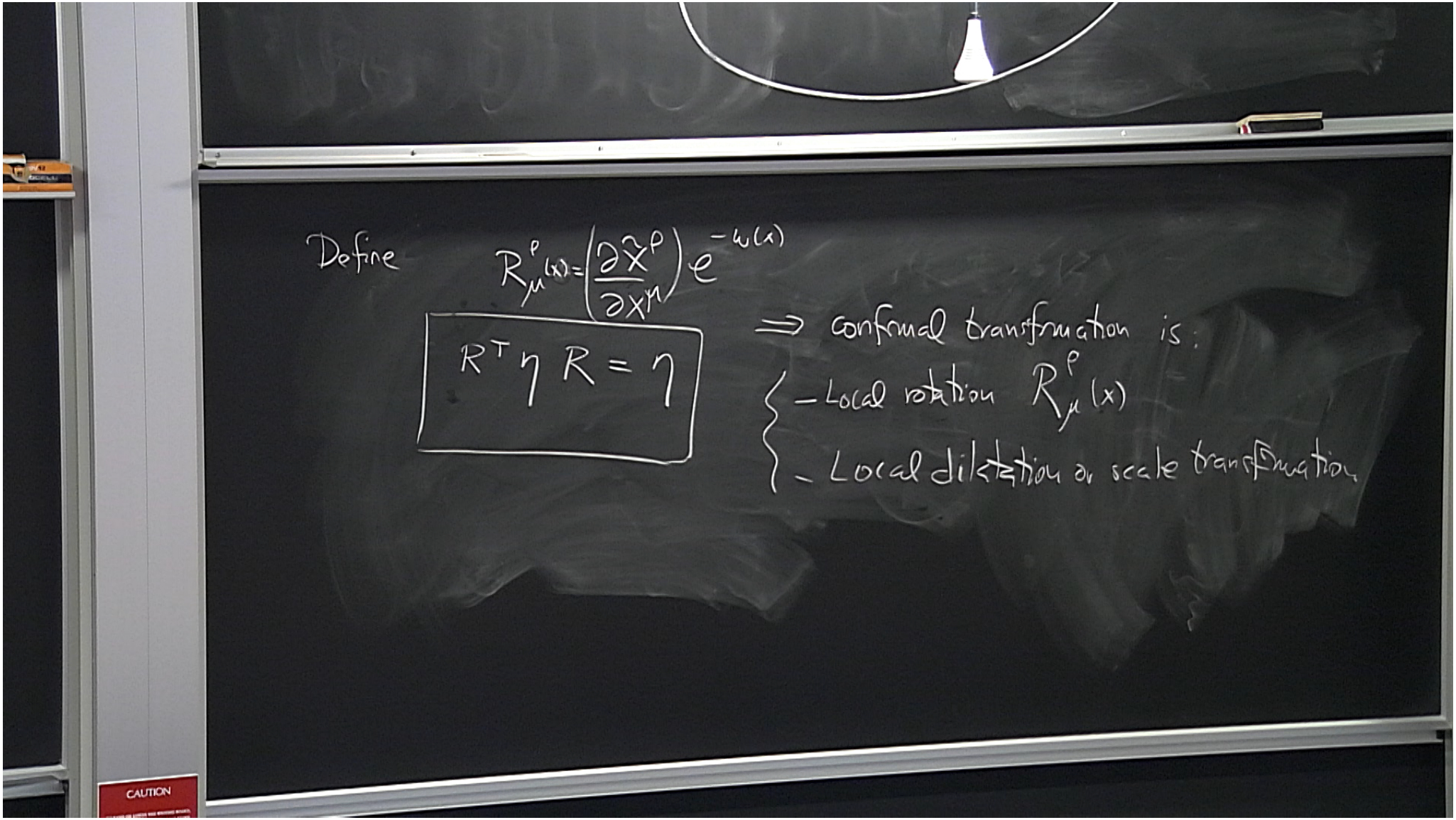
1. Preserves the lightcones
2. $\|$ angles between vectors

$x^\mu \to \tilde{x}^\mu$

$ds^2 \to e^{2\omega(x)} ds^2$


$\eta_{\rho\sigma}\, d\tilde{x}^\rho d\tilde{x}^\sigma = e^{2w(x)}\, \eta_{\mu\nu}\, dx^\mu dx^\nu$
Minkowski metric
$\eta_{\rho\sigma}\left(\frac{\partial \tilde{x}^\rho}{\partial x^\mu}\right)\left(\frac{\partial \tilde{x}^\sigma}{\partial x^\nu}\right) = e^{2w(x)}\, \eta_{\mu\nu}$
$\left|\det \frac{\partial \tilde{x}}{\partial x}\right|^2 = e^{2D w(x)}$
$e^{w(x)} = \left|\frac{\partial \tilde{x}}{\partial x}\right|^{1/D}$


$\eta_{\rho\sigma}\, d\tilde{x}^\rho d\tilde{x}^\sigma = e^{2w(x)}\, \eta_{\mu\nu}\, dx^\mu dx^\nu$
Minkowski metric
$\eta_{\rho\sigma}\left(\frac{\partial \tilde{x}^\rho}{\partial x^\mu}\right)\left(\frac{\partial \tilde{x}^\sigma}{\partial x^\nu}\right) = e^{2w(x)}\, \eta_{\mu\nu}$
$\left(\frac{\partial \tilde{x}}{\partial x}\right)^T \eta \left(\frac{\partial \tilde{x}}{\partial x}\right) = e^{2w(x)}\, \eta \left| \det \frac{\partial \tilde{x}}{\partial x} \right|^2 = e^{2D w(x)}$
$e^{w(x)} = \left| \frac{\partial \tilde{x}}{\partial x} \right|^{1/D}$
CAUTION
CAUTION

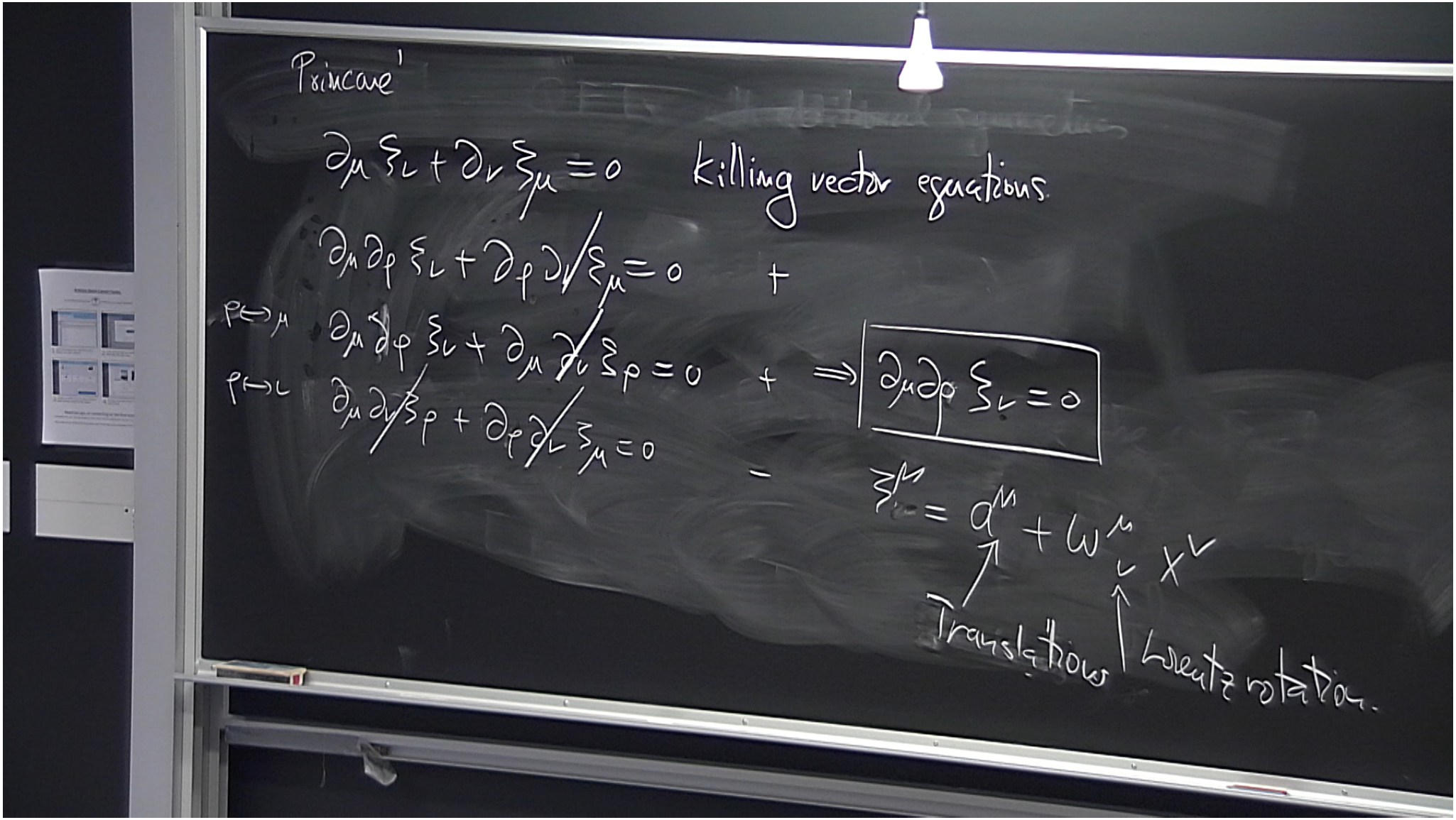

Define
$R^{\rho}_{\mu}(x) = \left(\frac{\partial x^{\rho}}{\partial x^{\mu}}\right) e^{-\omega(x)}$
$R^T \eta R = \eta$
$\Rightarrow$ conformal transformation is:
- Local rotation $R^{\rho}_{\mu}(x)$
- Local dilatation or scale transformation
CAUTION

Determine the conformal algebra. . Look at ∞'al transf

$\tilde{x}^\mu = x^\mu + \xi^\mu(x)$ (ξ^μ small) $\qquad \xi^\mu(x)$: spacetime vector field $\qquad \xi = \xi^\mu \partial_\mu$

$$\eta_{\rho\sigma}\left(\delta^\rho_\mu + \partial_\mu \xi^\rho\right)\left(\delta^\sigma_\nu + \partial_\nu \xi^\sigma\right) = \left(1 + \frac{2}{D}\partial\cdot\xi\right)\eta_{\mu\nu}$$

$O(\xi)$: $\boxed{\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{D}\partial\cdot\xi\ \eta_{\mu\nu}}$ conformal Killing equation


Poincare'
$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0$ killing vector equations
$\partial_\mu \partial_\rho \xi_\nu + \partial_\rho \partial_\nu \xi_\mu = 0$ +
$\rho \leftrightarrow \mu$ $\partial_\mu \partial_\rho \xi_\nu + \partial_\mu \partial_\nu \xi_\rho = 0$ + $\Rightarrow \boxed{\partial_\mu \partial_\rho \xi_\nu = 0}$
$\rho \mapsto \nu$ $\partial_\mu \partial_\nu \xi_\rho + \partial_\rho \partial_\nu \xi_\mu = 0$ −
$\xi^\mu = a^\mu + \omega^\mu{}_\nu x^\nu$
Translations
Lorentz rotation.

ξ is a CkV if

$$\partial_\mu \partial_\nu \partial_\rho \xi_\sigma = 0$$

- Finite dimensional space for $D > 2$
- ∞'al for $D = 2$

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V if

$\xi_\sigma = 0$

sional spaa for $D > 2$

$= 2$

$D = 2$

$\mu = \nu = 1$

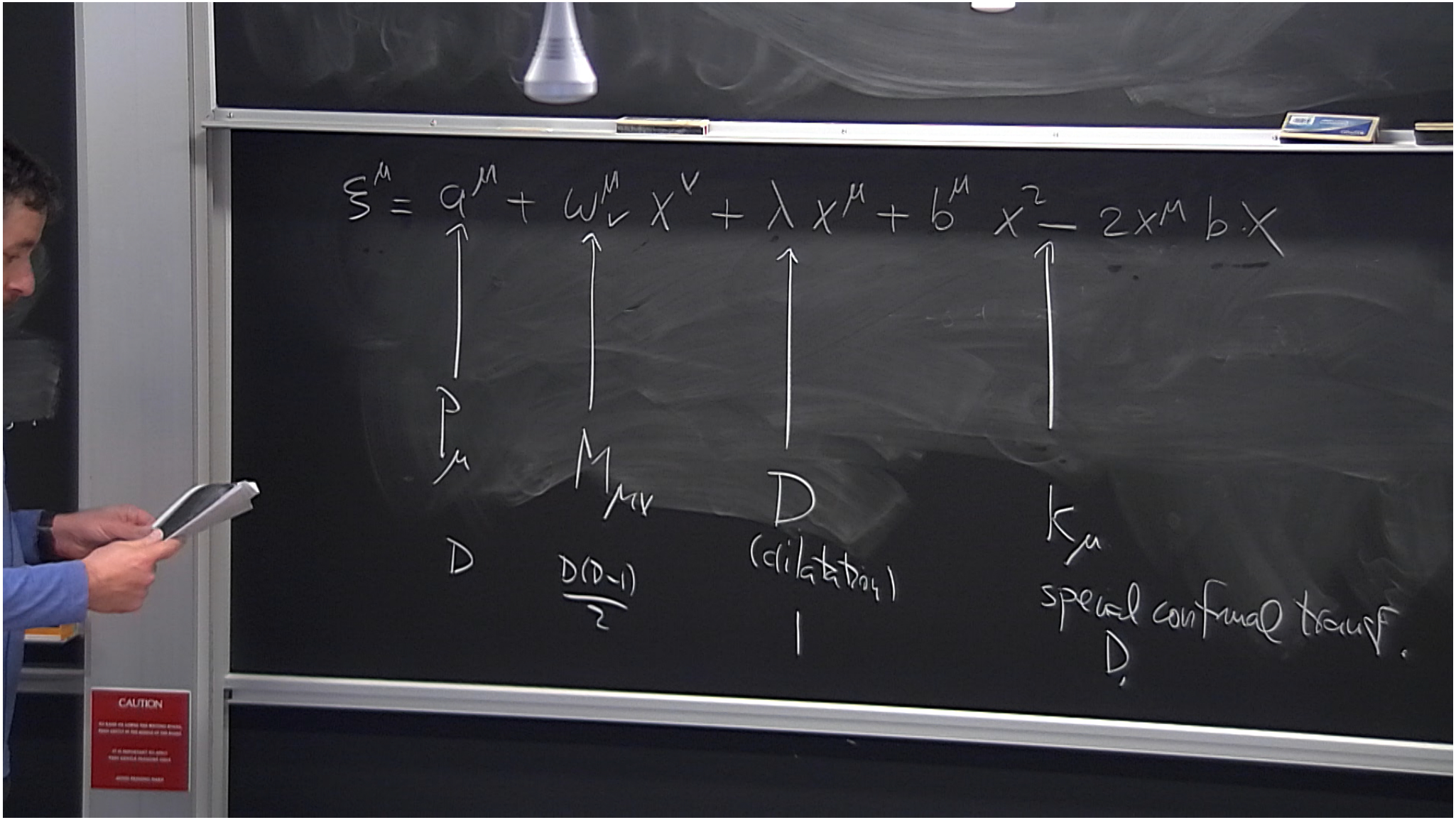
$\mu = 1$
$\nu = 2$

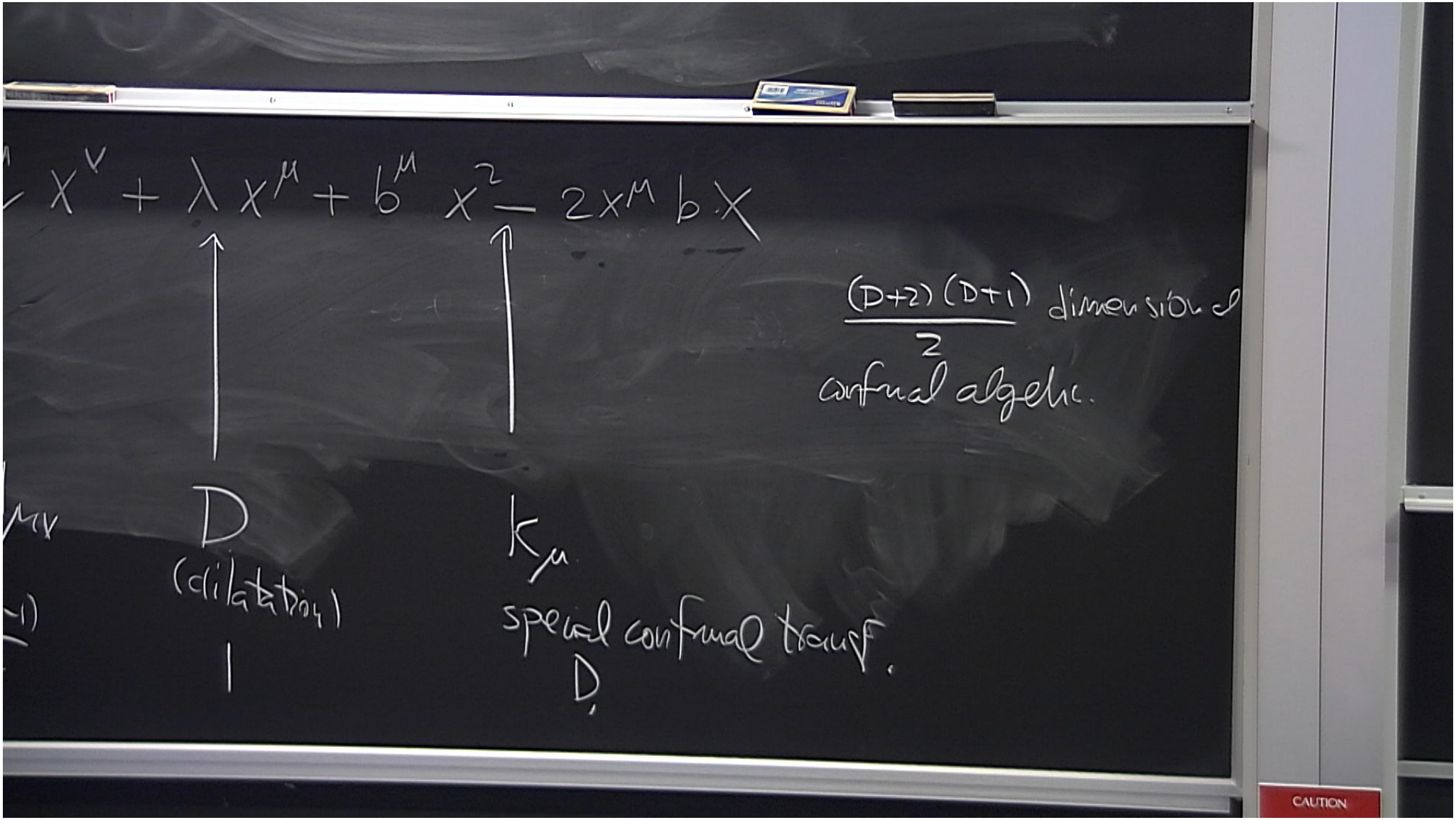
$$\left.\begin{aligned} \partial_1 \xi_1 &= \partial_2 \xi_2 \\ \partial_1 \xi_2 &= -\partial_2 \xi_1 \end{aligned}\right\}$$ Cauchy Riemann

$z = x^1 + i x^2$

$$\partial_{\bar{z}} \xi = 0$$

$D = 2$ any holomorphic vector field is a CKV


$\xi^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu b\cdot x$
P_μ
$M_{\mu\nu}$
D
K_μ
D
$\frac{D(D-1)}{2}$
(dilatations)
1
special conformal transf.
D.


$x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu b\cdot x$
D
(dilatations)
1
K_μ
special conformal transf.
D
$\frac{(D+2)(D+1)}{2}$ dimensional
conformal algebra.
CAUTION

Conformal algebra by looking

$\xi_{(1)}$, $\xi_{(2)}$ CKV

CKV

$[\xi_{(1)}, \xi_{(2)}] = \xi_{(3)}$ $\qquad \xi = \xi^\mu \partial_\mu$

Poincaré

$$\begin{cases} [M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho} M^{\nu\sigma} + \dots) \\ [M^{\mu\nu}, P^\rho] = i(\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu) \end{cases}$$

← is antisymmetric in $\mu \leftrightarrow \nu$, $\rho \leftrightarrow \sigma$

Conformal algebra by looking

$\xi_{(1)}$, $\xi_{(2)}$ CKV $\Rightarrow$ CKV

$[\xi_{(1)}, \xi_{(2)}] = \xi_{(3)}$ $\qquad \xi = \xi^\mu \partial_\mu$

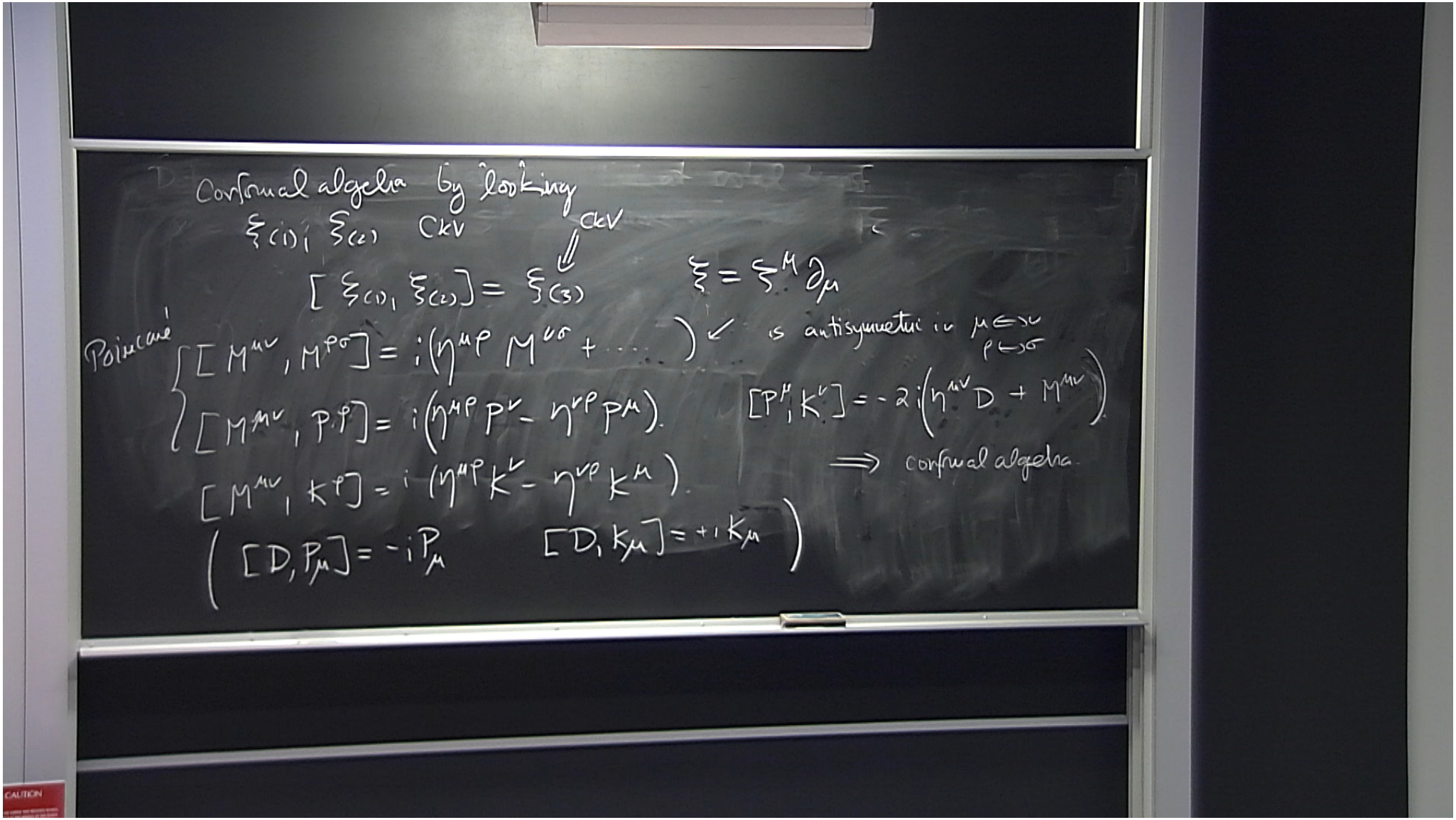
Poincaré

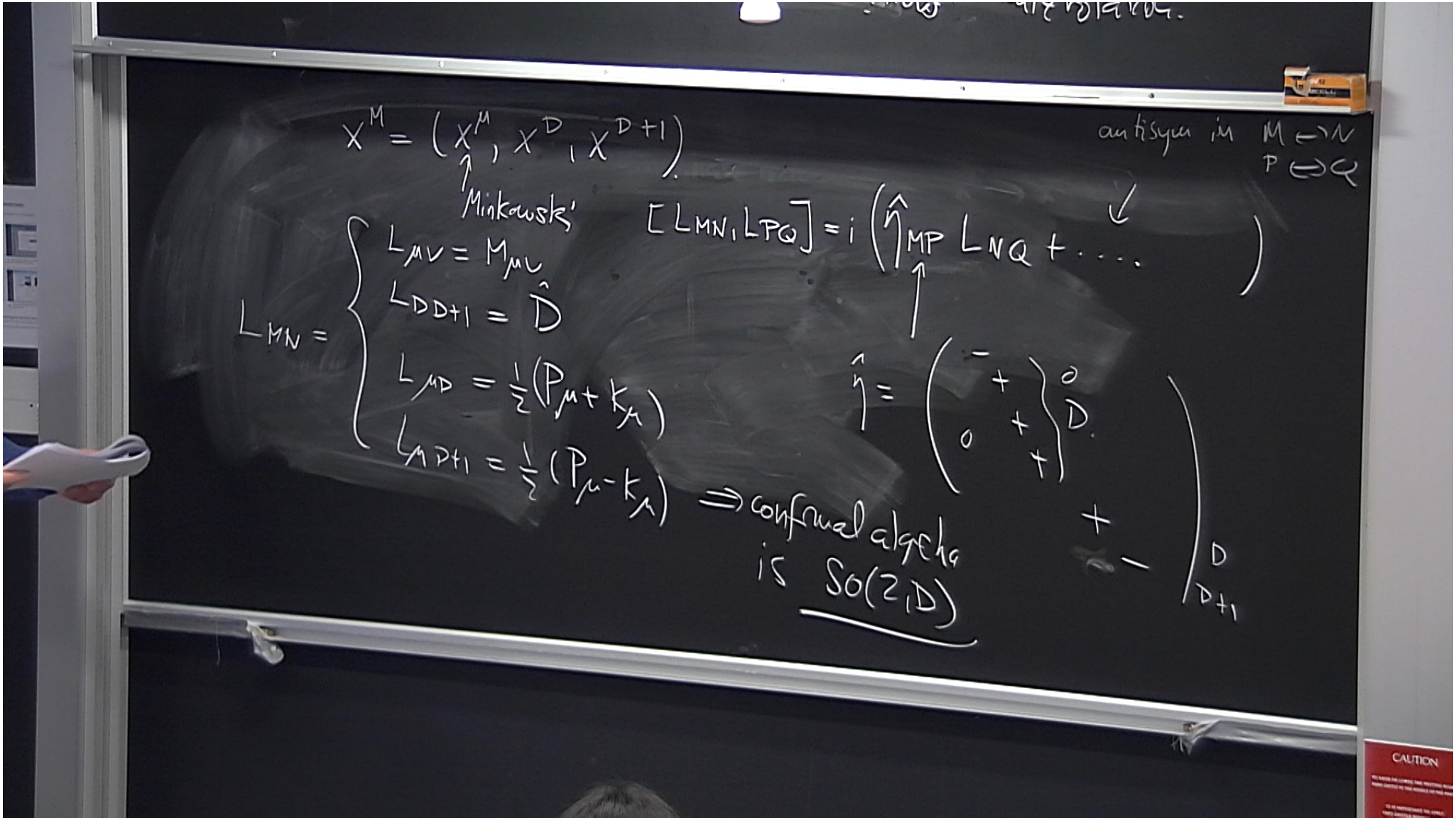
$$\begin{cases} [M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho} M^{\nu\sigma} + \dots) \leftarrow \text{is antisymmetric in } \mu \leftrightarrow \nu,\ \rho \leftrightarrow \sigma \\ [M^{\mu\nu}, P^\rho] = i(\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu) \\ [M^{\mu\nu}, K^\rho] = i(\eta^{\mu\rho} K^\nu - \eta^{\nu\rho} K^\mu) \end{cases}$$

$[P^\mu, K^\nu] = -2i(\eta^{\mu\nu} D + M^{\mu\nu})$

$\Rightarrow$

$([D, P_\mu] = -iP_\mu \qquad [D, K_\mu] = +iK_\mu)$

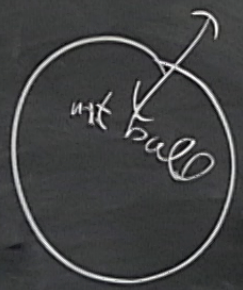

Conformal algebra by looking
$\xi_{(1)}, \xi_{(2)}$ CKV
CKV
$[\xi_{(1)}, \xi_{(2)}] = \xi_{(3)}$
$\xi = \xi^\mu \partial_\mu$
Poincaré
$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho} M^{\nu\sigma} + \dots)$ ← is antisymmetric in $\mu \leftrightarrow \nu$, $\rho \leftrightarrow \sigma$
$[M^{\mu\nu}, P^\rho] = i(\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu)$
$[M^{\mu\nu}, K^\rho] = i(\eta^{\mu\rho} K^\nu - \eta^{\nu\rho} K^\mu)$
$[D, P_\mu] = -i P_\mu$
$[D, K_\mu] = +i K_\mu$
$[P^\mu, K^\nu] = -2i(\eta^{\mu\nu} D + M^{\mu\nu})$
$\Longrightarrow$ conformal algebra


$X^M = (X^\mu, X^D, X^{D+1})$
Minkowski
antisym in $M \leftrightarrow N$, $P \leftrightarrow Q$
$[L_{MN}, L_{PQ}] = i(\hat{\eta}_{MP} L_{NQ} + \dots)$
$L_{MN} = \begin{cases} L_{\mu\nu} = M_{\mu\nu} \\ L_{DD+1} = \hat{D} \\ L_{\mu D} = \frac{1}{2}(P_\mu + K_\mu) \\ L_{\mu D+1} = \frac{1}{2}(P_\mu - K_\mu) \end{cases}$
$\hat{\eta} = \begin{pmatrix} - & & & 0 \\ & + & & \\ & & + & D \\ 0 & & & \\ & & & + \\ & & & & - \end{pmatrix}_{D+1}$
$\Rightarrow$ conformal algebra is $SO(2,D)$
CAUTION

Discrete conformal transformations:

I : inversion $x^\mu \longrightarrow \frac{x^\mu}{x^2}$ $I^2 = 1$ $\mathbb{Z}_2$ transformation.

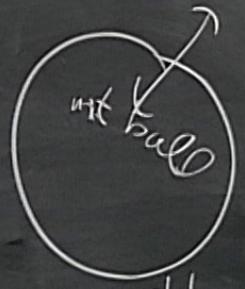
1. conformal $e^{\omega(x)} = \frac{1}{x^2}$

2. 

Discrete conformal transformations:

I : inversion $\quad x^\mu \longrightarrow \frac{x^\mu}{x^2}$ $\quad I^2 = 1$ $\quad \mathbb{Z}_2$ transformation.

1. conformal $\quad e^{\omega(x)} = \frac{1}{x^2}$

2. 

3. changes orientation

Conformal group is generated P_μ with I?

$I P_\mu I = k_\mu$

$[M^{\mu\nu}, K^\rho] = i(\eta^{\mu\rho}K^\nu - \eta^{\nu\rho}K^\mu)$. $\Rightarrow$ conformal algebra.

$([D, P_\mu] = -iP_\mu \qquad [D, K_\mu] = +iK_\mu)$

conformal group has 4 disconnected components

det 2x2	det D	
+	+	$\wedge$ conformal transf. connected to Identity
−	+	$I\wedge$, $P\wedge$, CFT is inv under I iff inv under P.
+	−	$T\wedge$
−	−	$TI\wedge$