Title: PSI 2019/2020 - Chern-Simons Theory Part 1 - Lecture 9

Speakers: Kevin Costello

Collection: PSI 2019/2020 - Chern-Simons Theory (Part 1)

Date: March 02, 2020 - 11:30 AM

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## Last time

- Defined Kauffmann bracket
- Claim

$$\int e^{\frac{1}{4\pi\hbar} CS(A)} W_K(A) = \langle K \rangle (x)$$

after.

1) A substitution of form $x = e^{\frac{c}{4\pi\hbar}}$

2) Possibly normalizing by $x^{3\omega r(K)}$

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2) Possibly normalizing by $x^{3w_r(K)}$ $\infty$

Witten remarks:

Every knot in $S^3$ or $\mathbb{R}^3$ has a canonical framing so that the self-linking vanishes.
If we use this framing, then QFT $\rightsquigarrow$ a knot inv. (without framing worries)

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Witte

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## Self-linking

- Take a framed knot K
- Displace it using normal vector, to get K'
- Self-linking = Link(K, K')

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not K

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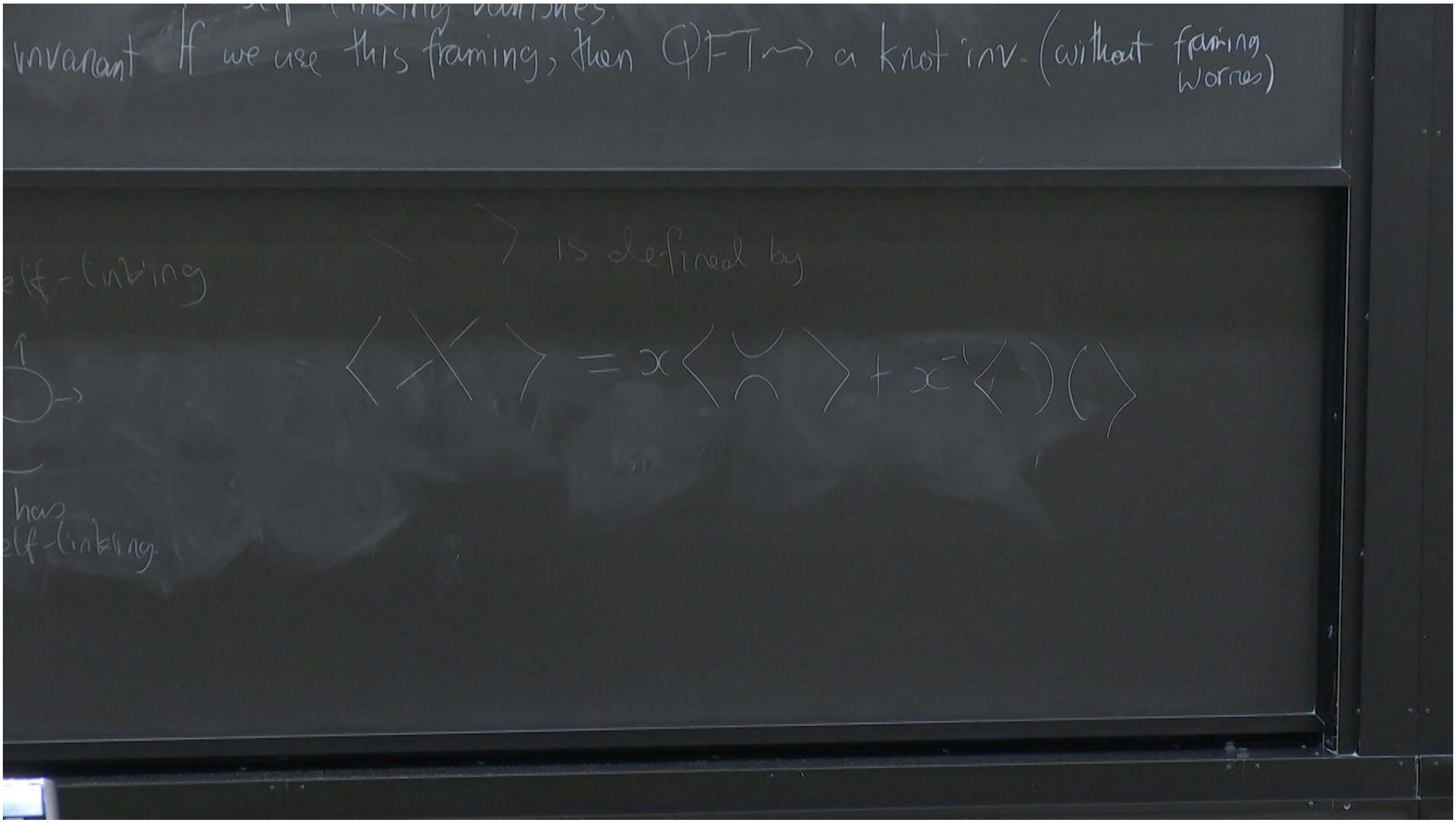
no self-linking

I think has a self-linking.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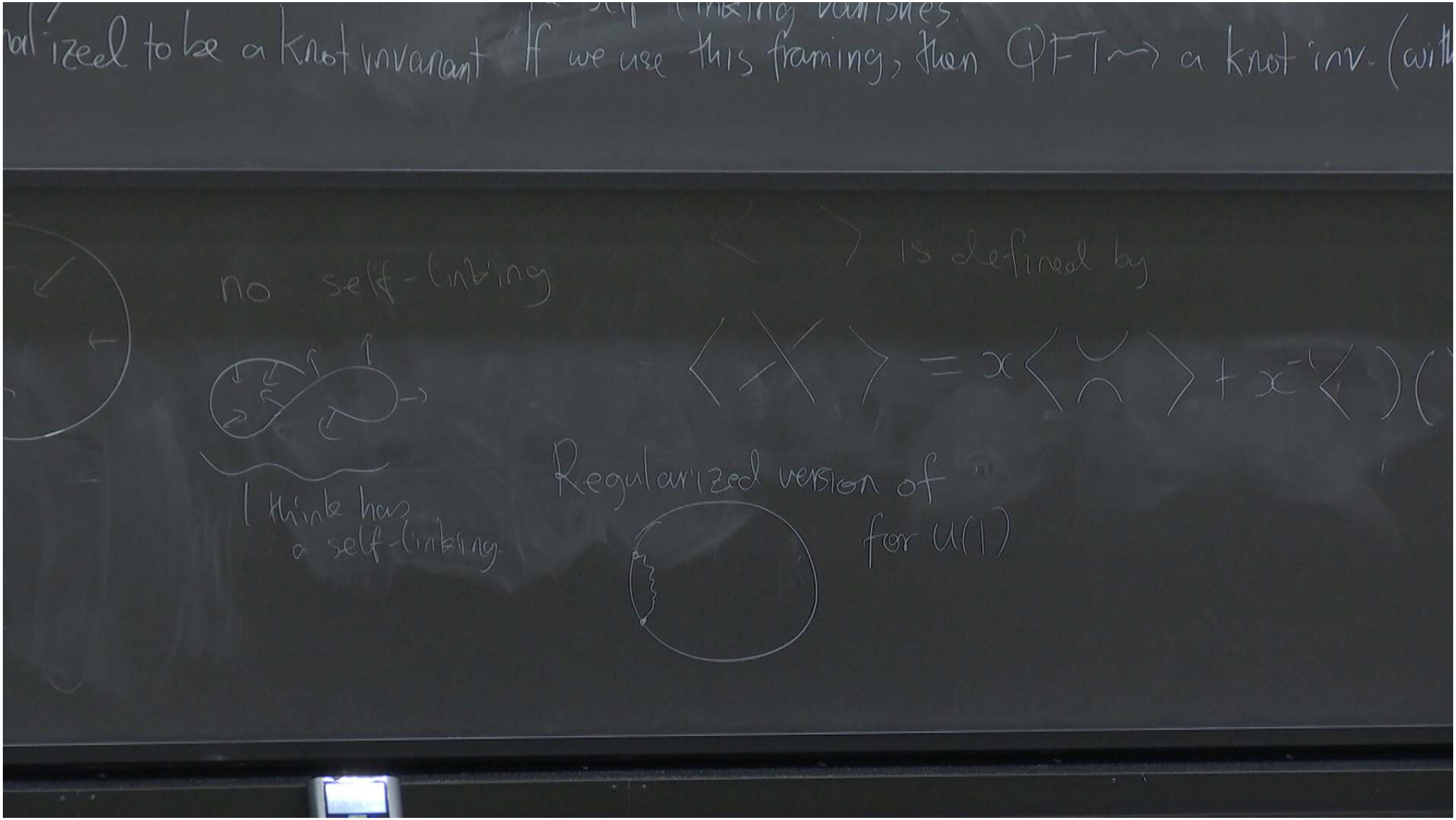

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Regularized version of
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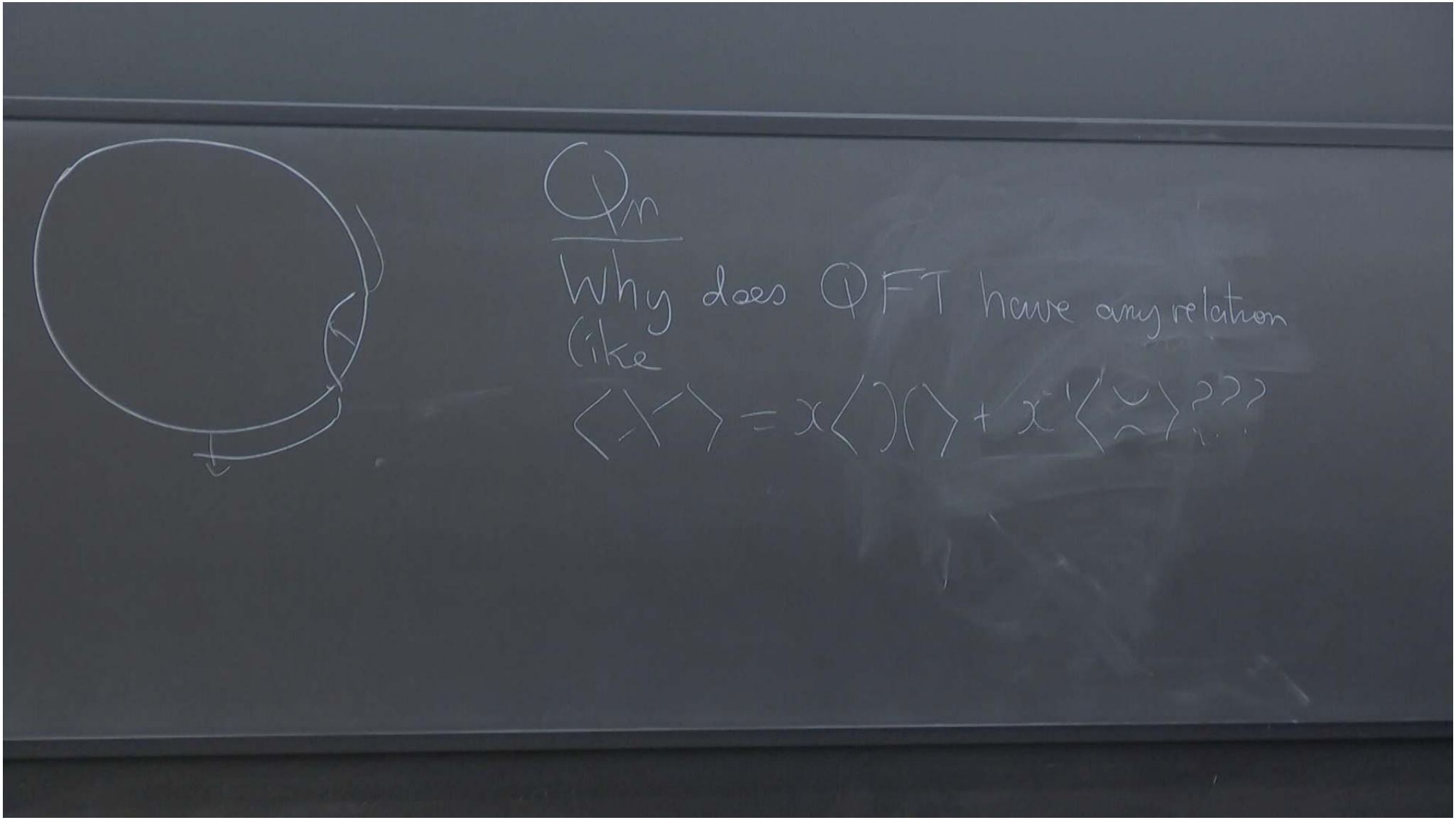

Qn
Why does QFT have any relation like
???

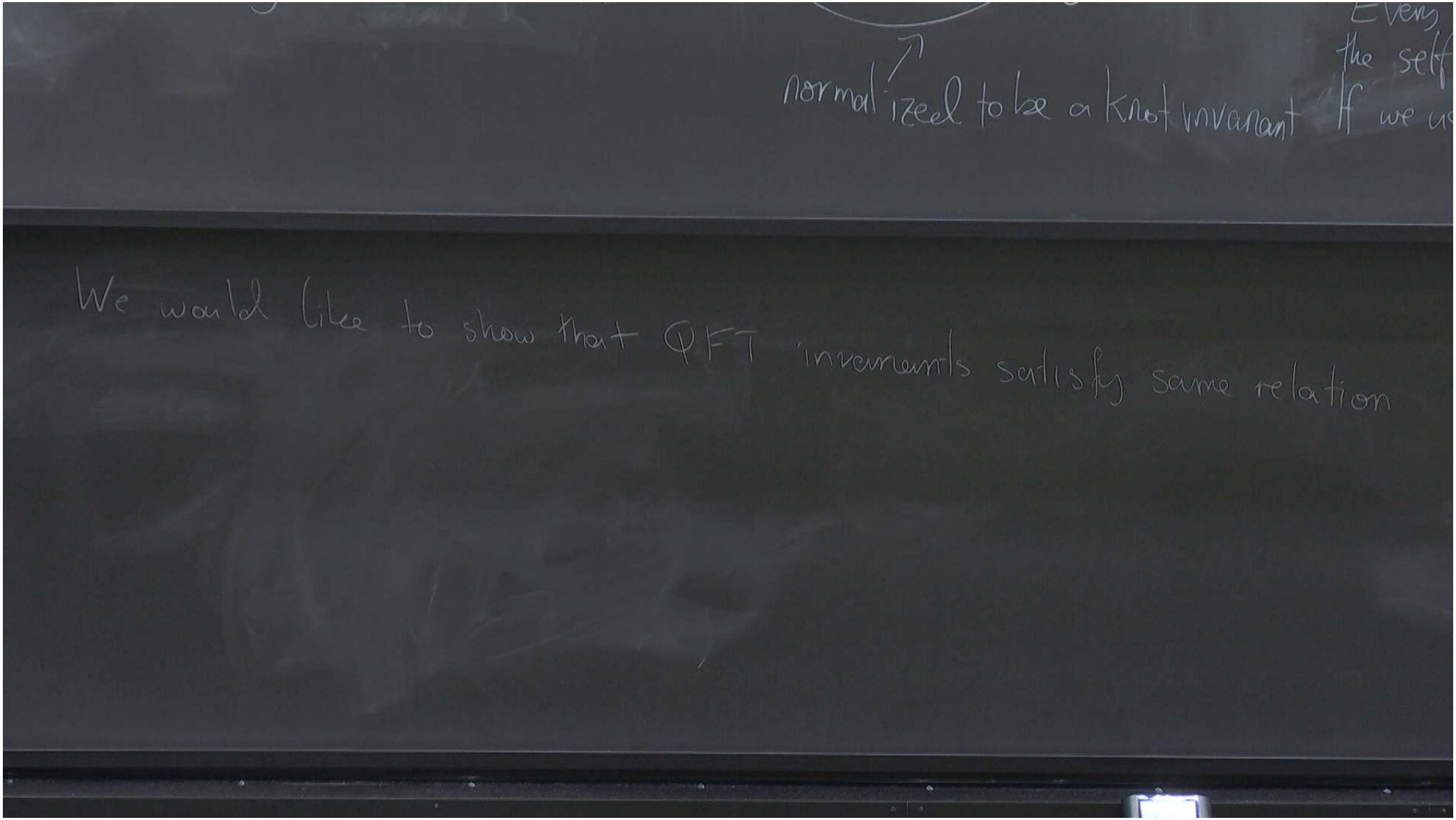
It turns out we get an equivalent knot invariant as follows.

Let $L_+ =$ $L_- =$ $L_0 =$

The Jones polynomial satisfies the local rule

$$t^{-1}J(L_+) - tJ(L_-) = (t^{1/2} - t^{-1/2})J(L_0)$$

This is the same as $(-x^{3w_r})\langle\ \rangle$ if $t^{-1/4} = x$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We would like to show that QFT invariants satisfy same relation

$$\int e^{\frac{1}{4\pi\hbar}(S(A)} W_K(A) = \underbrace{x^{\mathrm{sw}(K)} \langle K \rangle(x)}_{\text{normalized to be a knot invariant}}$$

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This requires Axiomatic Top$^c$ Field Theory.

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$S(A)$

$$W_K(A) = x^{3w(K)} \langle K \rangle (x)$$

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3) If $M$ is a 3-manifold (oriented) then, if $\partial M = \partial_{in} M$ and $\partial_{out} M$

then, $Z(M): H(\partial_{in} M) \longrightarrow H(\partial_{out} M)$

4) Most important axiom:
If $M_1, M_2$, $\partial_{out} M_1 = \partial_{in} M_2$,
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$M_1 \amalg_{\partial_{out} M_1} M_2$

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Real Picture: $(D^2 \times S^1) \setminus (\text{a ball})$

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$M_1 \underset{\partial_{out} M_1}{\amalg} M_2$ and,

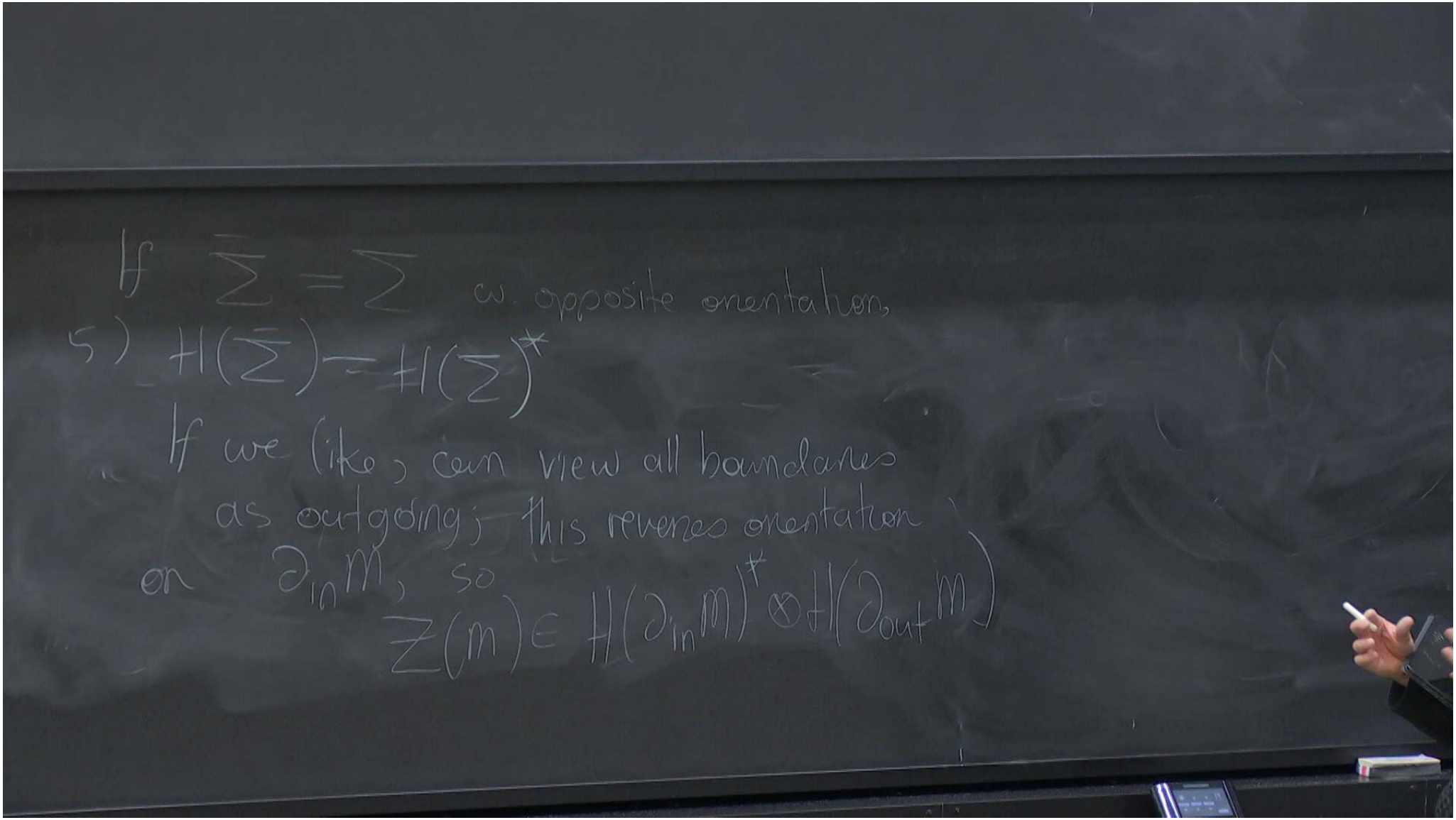
$Z(M_1 \underset{\partial_{out} M_1}{\amalg} M_2)$

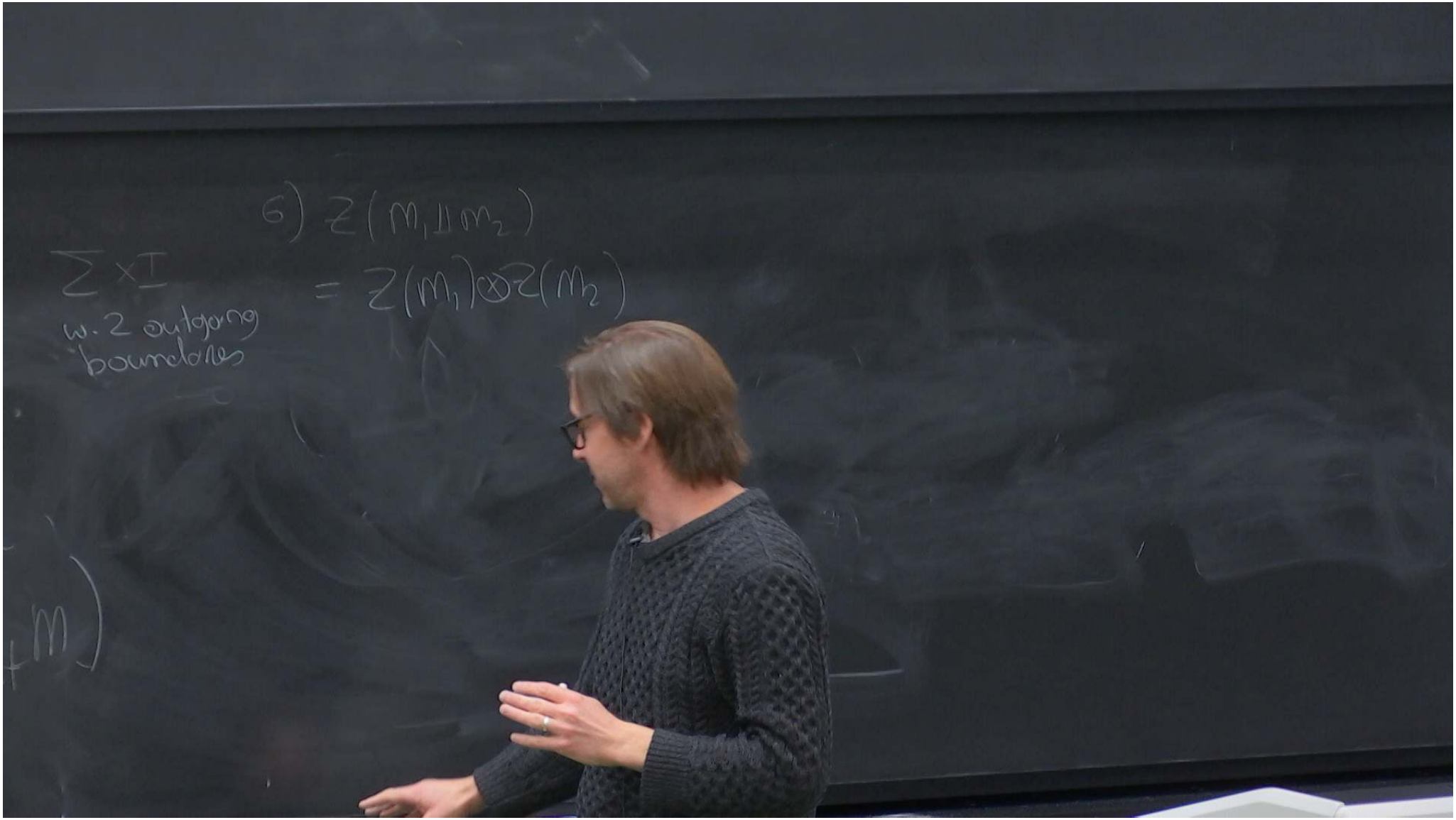
$= Z(M_2) \circ Z(M_1)$

$: H(\partial_{in} M_1)$

$\longrightarrow H(\partial_{out} M_1) = H(\partial_{in} M_2)$

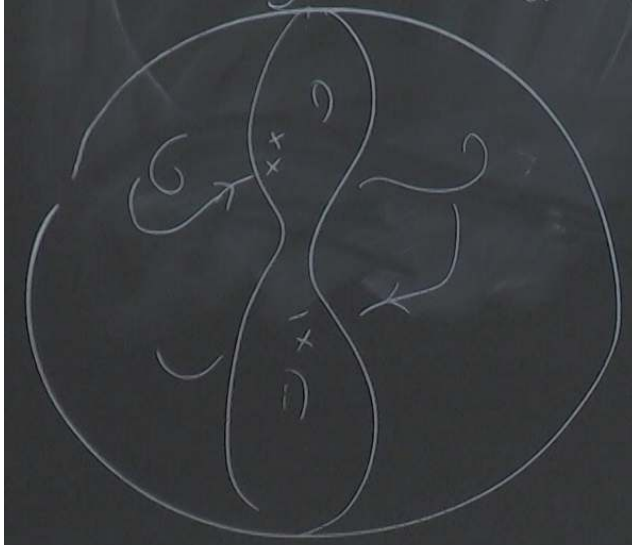
$\longrightarrow H(\partial_{out} M_2)$


If $\bar{\Sigma} = \Sigma$ w. opposite orientation,
5) $H(\bar{\Sigma}) = H(\Sigma)^*$
If we (like, can view all boundaries as outgoing; this reverses orientation on $\partial_{in} M$, so
$Z(M) \in H(\partial_{in} M)^* \otimes H(\partial_{out} M)$


6) $Z(M_1 \amalg M_2)$
$\Sigma \times I$
$= Z(M_1) \otimes Z(M_2)$
w. 2 outgoing boundaries

Axioms for 3d TFT w. a 1d oriented defect:

Very similar except:



A link intersects $\Sigma$ at points which are labelled $\pm$, depending on orientation of the link.

efect:

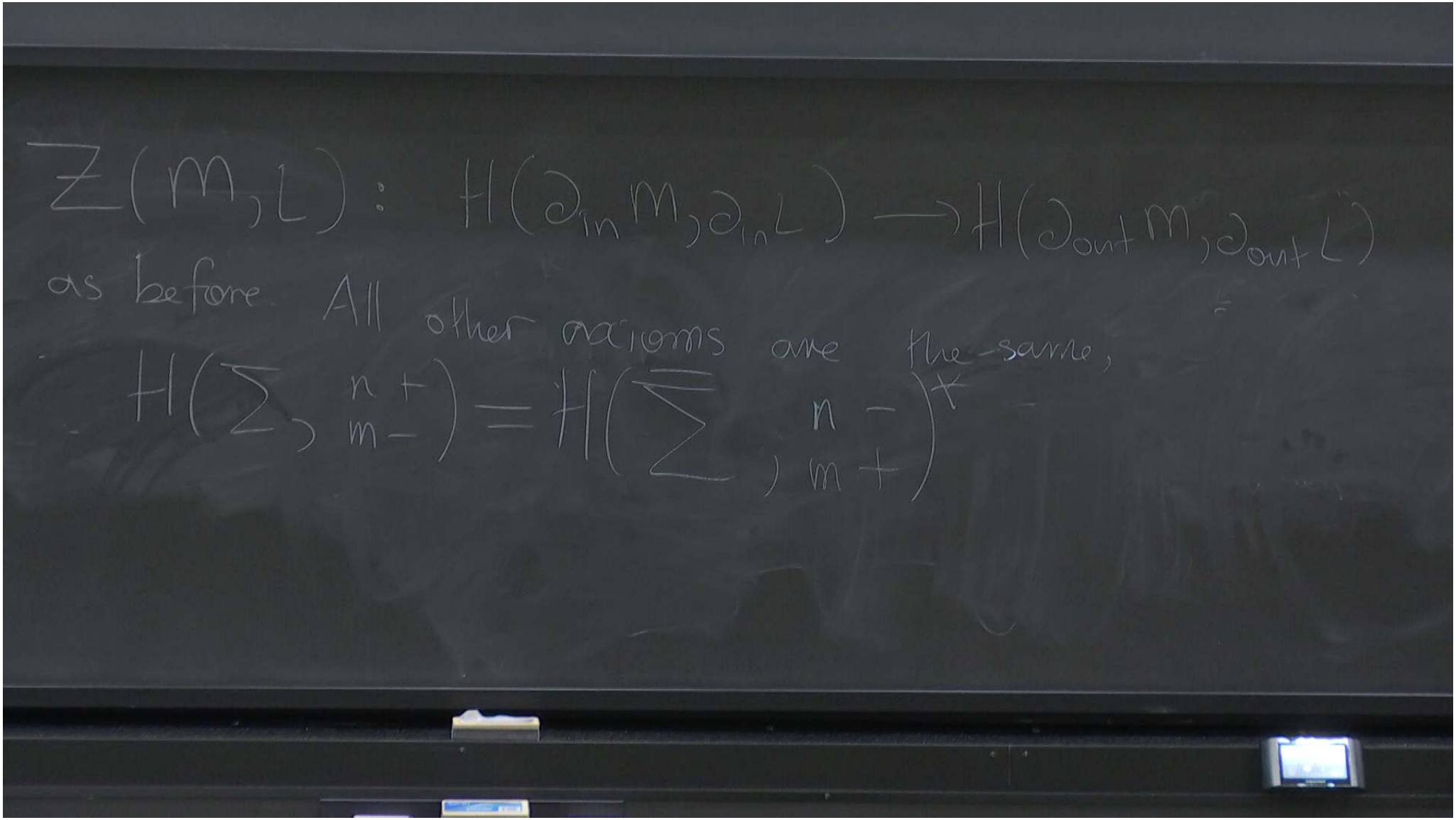
s

epending

In this setting

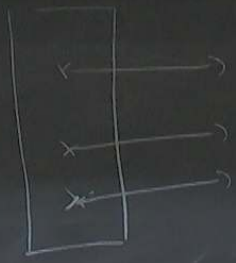
- $\pm 1$ is defined for a surface $\Sigma$ with $n+m$ points, $n$ $+$, $m$ $-$

- $M$ has a link w. boundary on $\partial M$


$Z(M,L) : H(\partial_{in} M, \partial_{in} L) \longrightarrow H(\partial_{out} M, \partial_{out} L)$
as before. All other axioms are the same,
$H(\Sigma, \begin{matrix} n_+ \\ m_- \end{matrix}) = H(\overline{\Sigma}, \begin{matrix} n_- \\ m_+ \end{matrix})^*$

Near b of $M_3$

$M = \Sigma \times [0,1)$



Link looks like

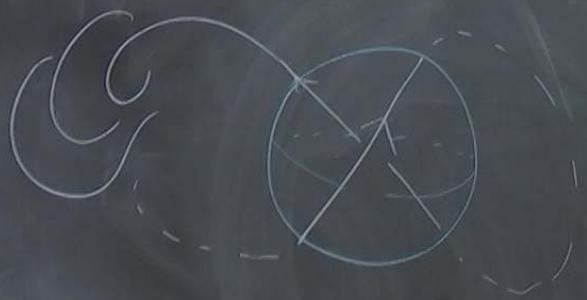
$p_1 \times [0,1)$

$\vdots$

$p_{n+m} \times [0,1)$

$\partial(\text{link}) = \text{points in } \Sigma$

Suppose we have a link in $S^3$, focus near 1 crossing



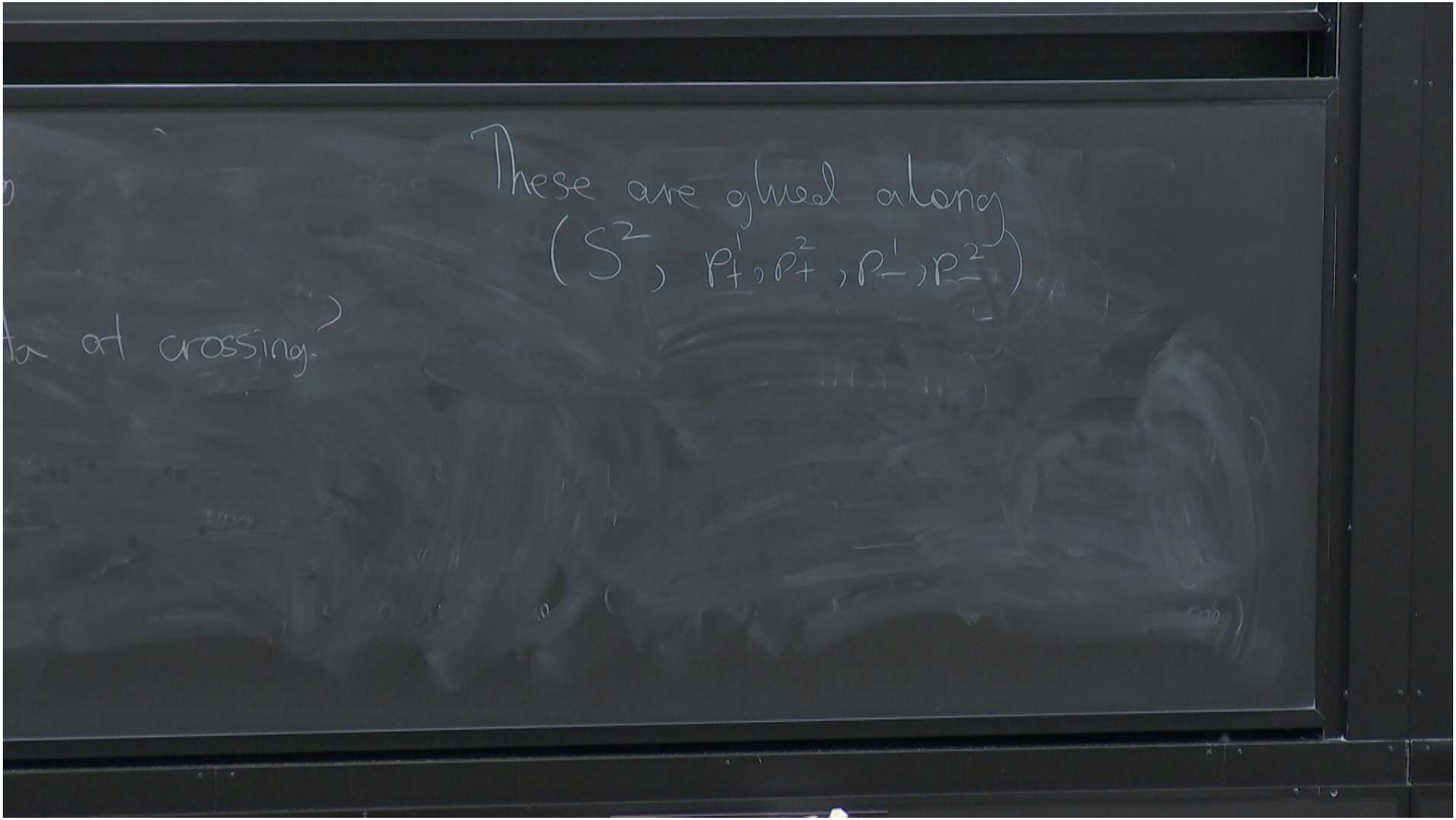
How do we understand

$Z(S^3;L)$ 'in terms of local data at crossing?

$S^3 = M_1 \cup M_2$

$M_1$ = a ball surrounding the crossing

$M_2$ = complement of $M_1$


These are glued along
$(S^2, p_+^1, p_+^2, p_-^1, p_-^2)$
ta at crossing.

$H(\Sigma_2)$

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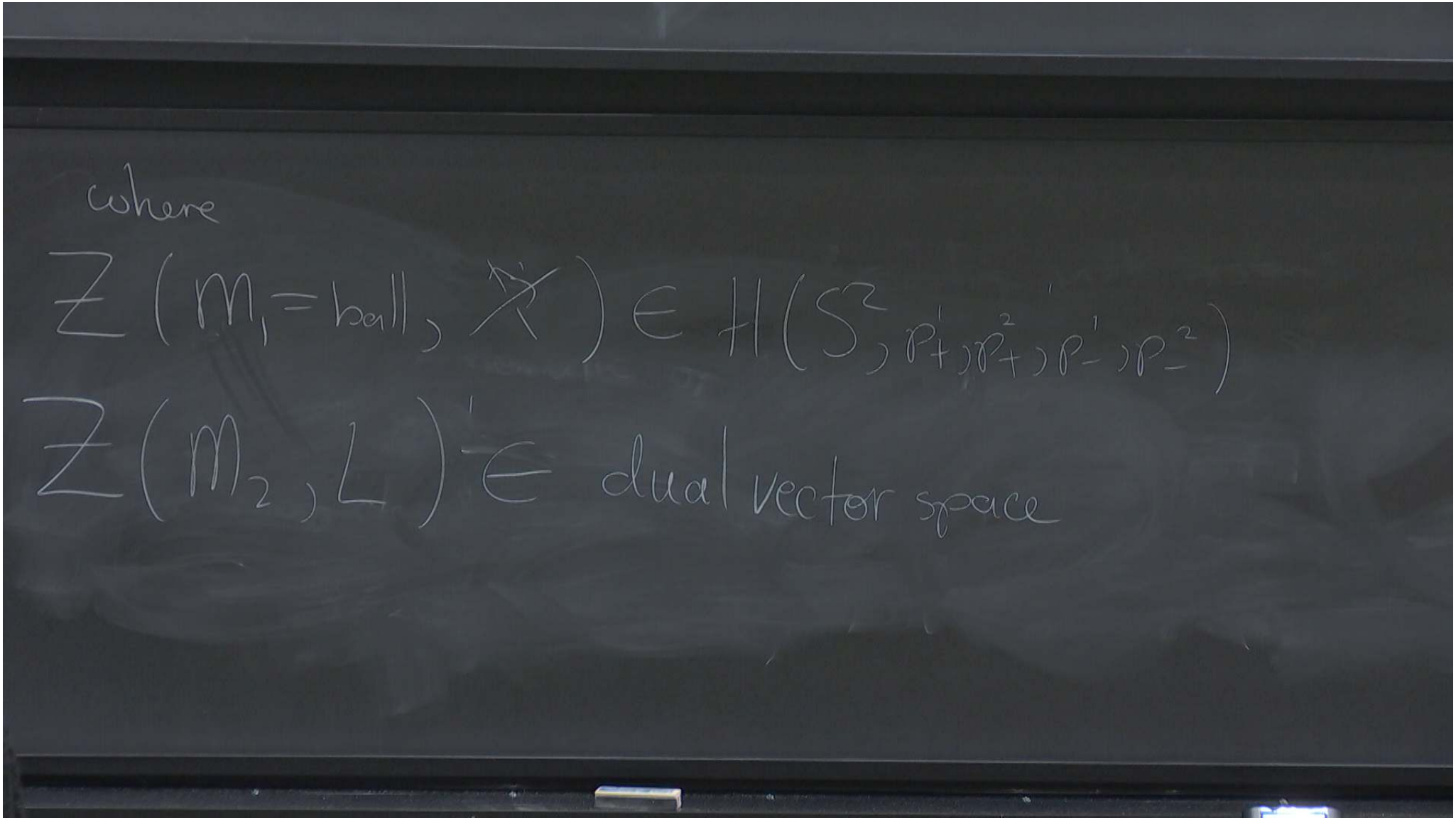
at crossing.

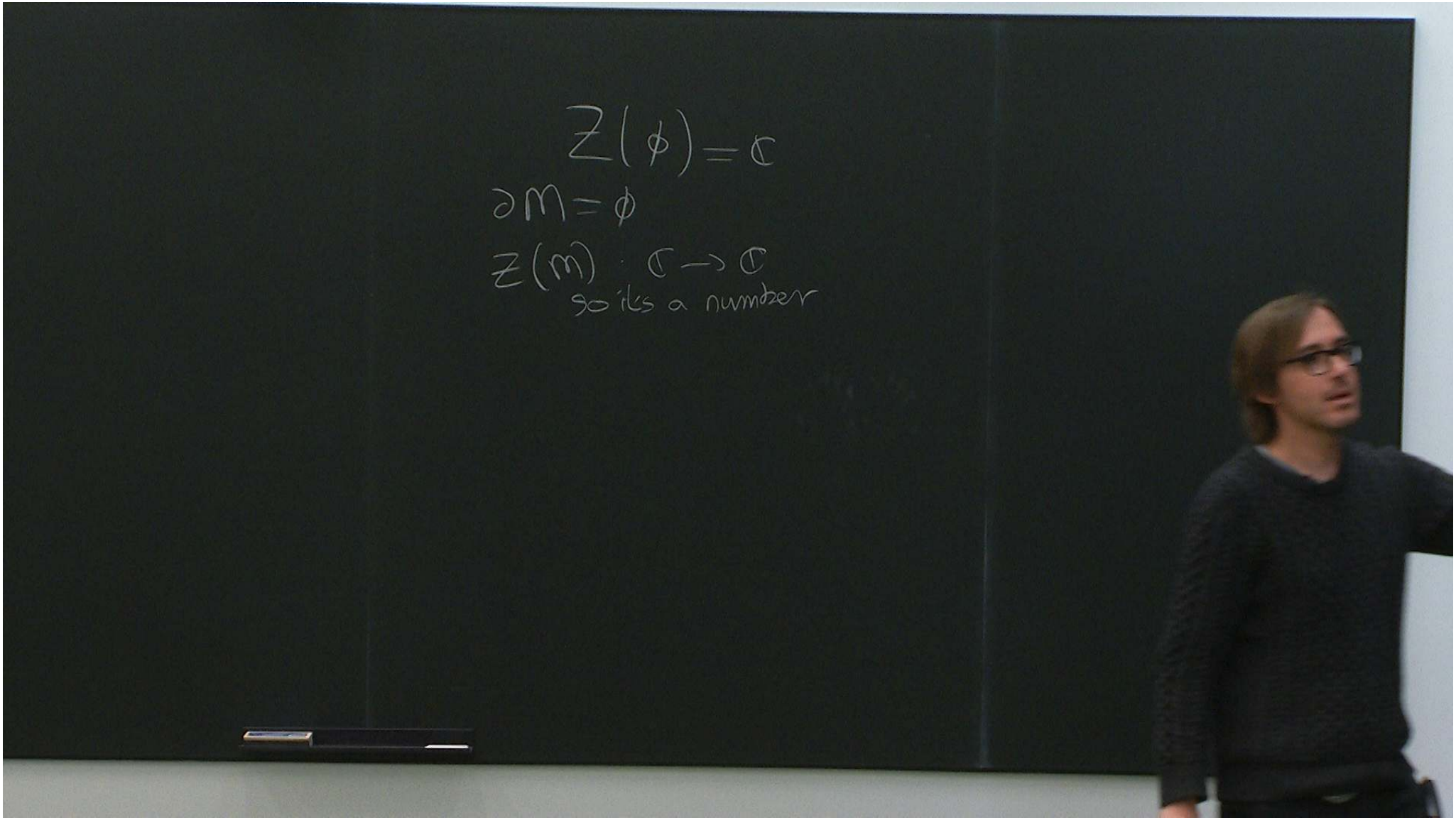
$$Z(S^3, L)$$
$$= \langle Z(M_2, L'), Z(M_1, \text{~~X~~}) \rangle$$


where
$Z(M_1 = \text{ball}, \cancel{X}) \in H(S^2, p_+^1, p_+^2, p_-^1, p_-^2)$
$Z(M_2, L) \in$ dual vector space

If we modify our $L$ locally to

or

then $Z(M, L_-)$, $Z(M, L_0)$ has the same expression where we replace $Z(\text{Ball}, \nearrow\nearrow)$ by $Z(\text{Ball}, \nearrow\nearrow)$ or $Z(\text{Ball}, \nearrow\uparrow)$


$Z(\phi) = \mathbb{C}$
$\partial M = \phi$
$Z(M) : \mathbb{C} \to \mathbb{C}$
so it's a number

A Skein relation like in Jones poly will hold.

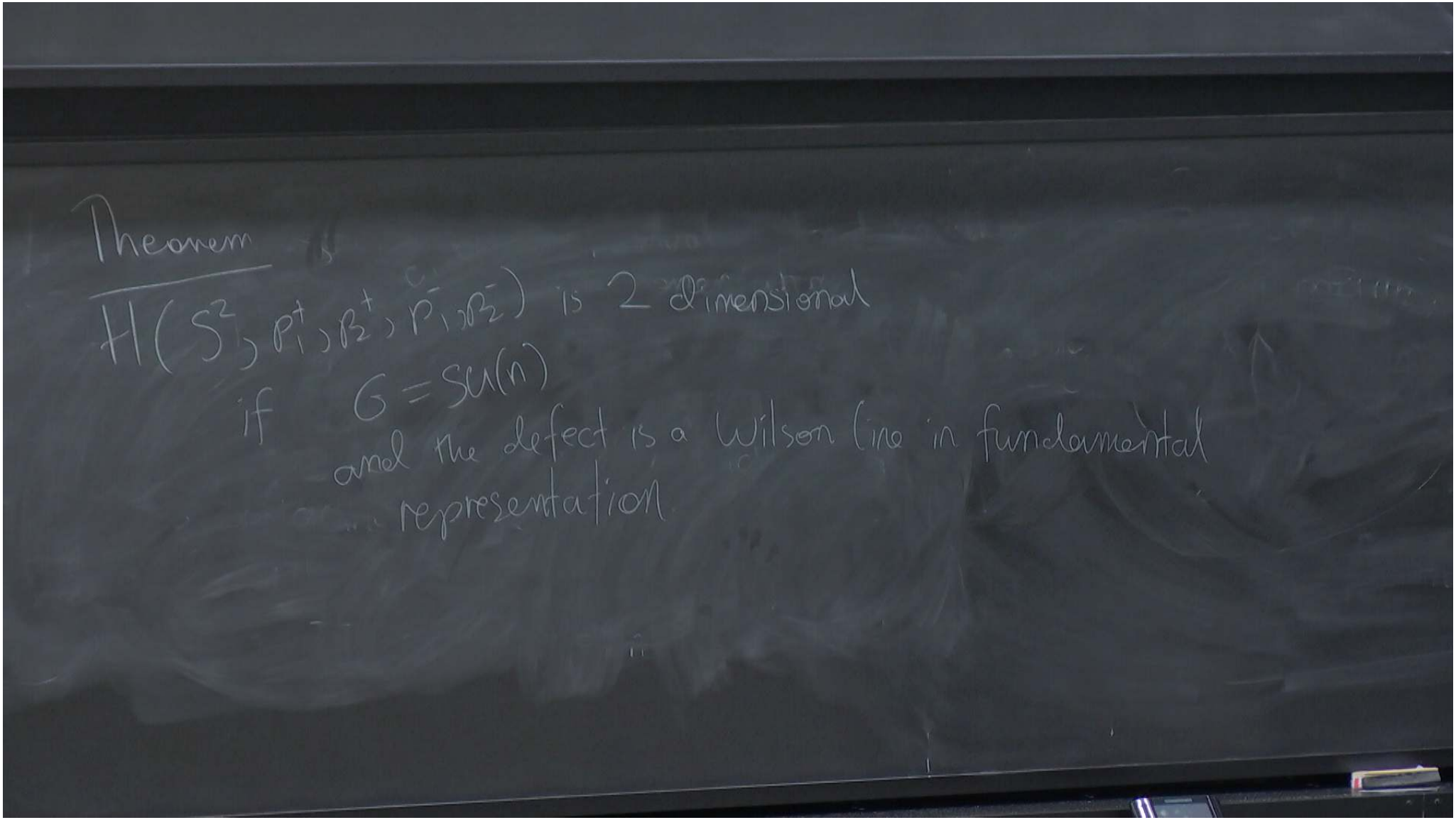
If there are functions $F_+(h), F_-(h), F_0(h)$

so that

$$F_+(h)Z(\text{Ball}, \nearrow) + F_-(h)Z(\text{Ball}, \nearrow) + F_0(h)Z(\text{Ball}, )( ) = 0$$

$$\in H(S^2, P_+, P_+^2, P_-, P_-^2)$$

6) $Z(M_1 \amalg M_2)$


Theorem
$\mathcal{H}(S^2, P_1^+, P_2^+, P_1^-, P_2^-)$ is 2 dimensional
if $G = SU(n)$
and the defect is a Wilson line in fundamental representation