Title: PSI 2019/2020 - Chern-Simons Theory Part 1 - Lecture 9

Speakers: Kevin Costello

Collection: PSI 2019/2020 - Chern-Simons Theory (Part 1)

Date: March 02, 2020 - 11:30 AM

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Last time

- Defined Kauffmann bracket
- Claim:

$$\int e^{\frac{1}{4\pi\hbar} CS(A)} W_K(A) = \langle K \rangle (x)$$

afte

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after:

1) A substitution of form $x = e^{\frac{c}{4\pi\hbar}}$

2) Possibly normalizing by $x^{3wr(K)}$

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Witten remarks:

Every knot in S^3 or $\mathbb{R}^3$ has a canonical framing so that the self-linking vanishes.

If we use this framing, then QFT $\rightsquigarrow$ a knot inv. (without framing worries)

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Witte

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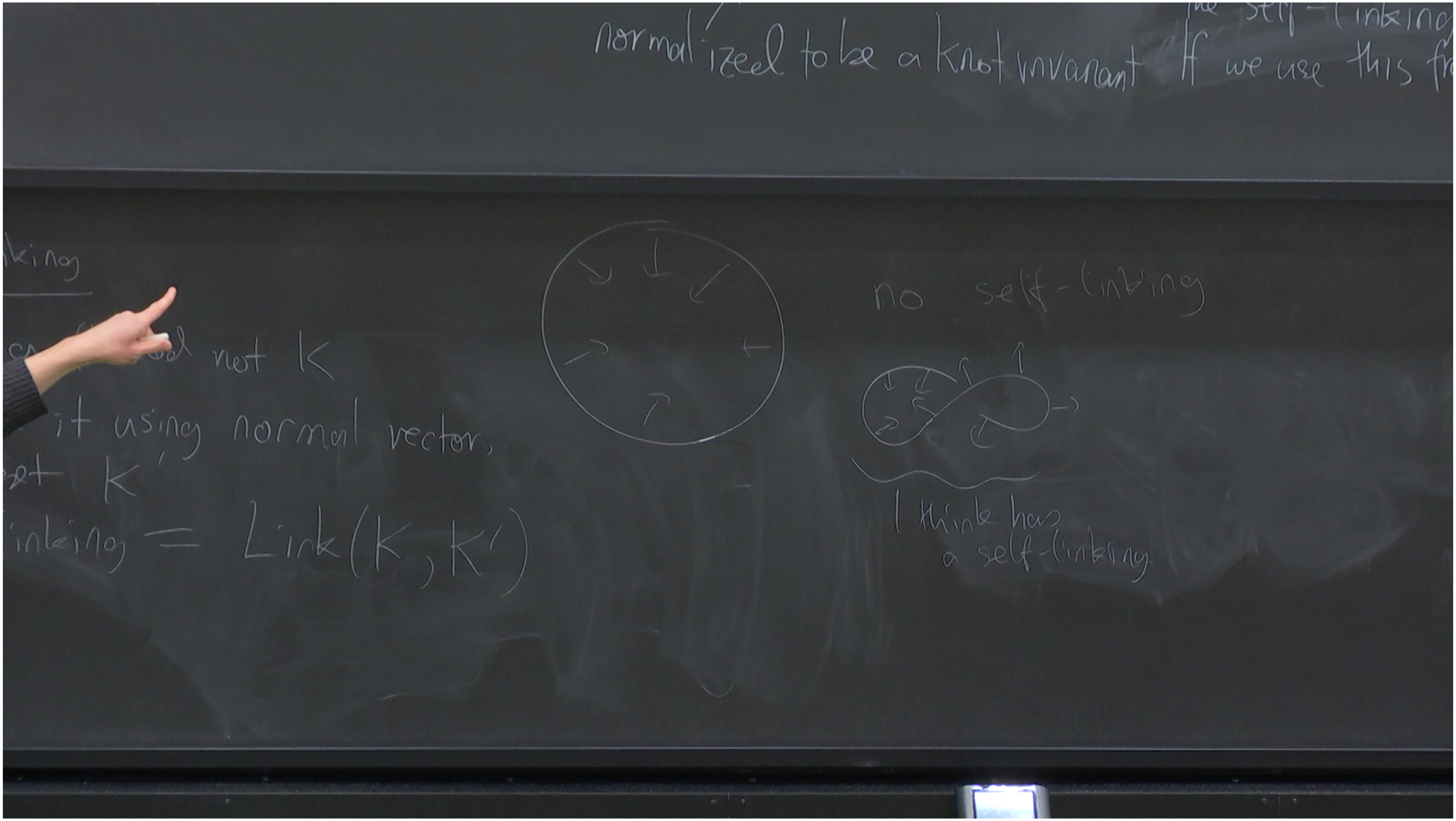
Self-linking

- Take a framed not K
- Displace it using normal vector, to get K'
- Self-linking = $\mathrm{Link}(K, K')$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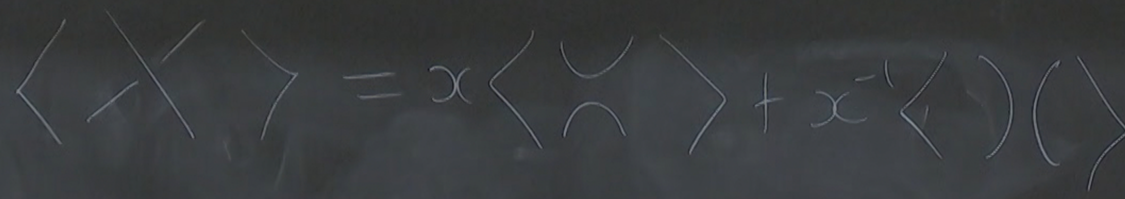

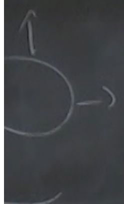

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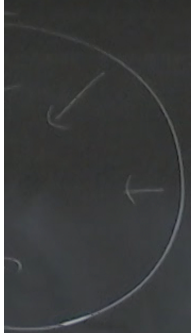

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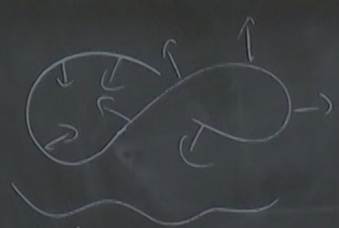

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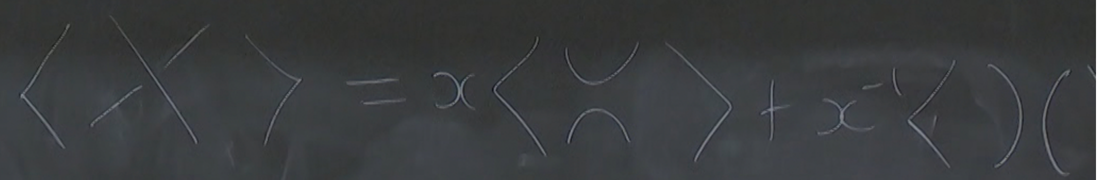
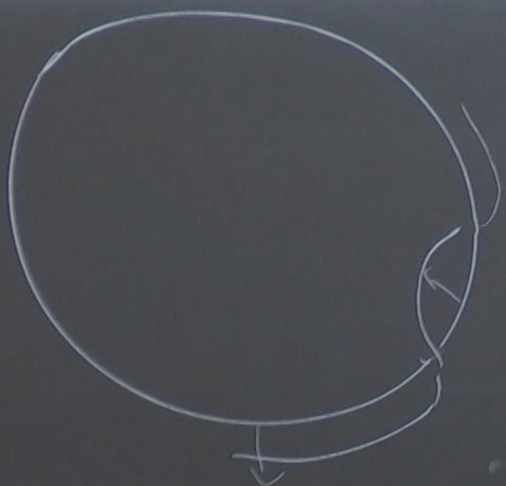


Regularized version of

for $U(1)$

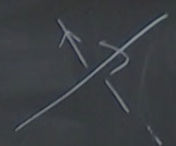


Qn

Why does QFT have any relation
(like

$$\langle \times \rangle = x\langle)(\rangle + x^{-1}\langle \asymp \rangle \text{ ???}$$

It turns out we get an equivalent knot invariant as follows.

Let $L_+ =$  $L_- =$  $L_0 =$ 

The Jones polynomial satisfies the local rule

$$t^{-1}J(L_+) - t\,J(L_-) = (t^{1/2} - t^{-1/2})\,J(L_0)$$

This is the same as $(-x^{3wr})\langle\ \rangle$ if $t^{-1/4} = x$

Every
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We would like to show that QFT invariants satisfy same relation

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Witten-Atiyah-Segal axioms for a 3d oriented TFT.

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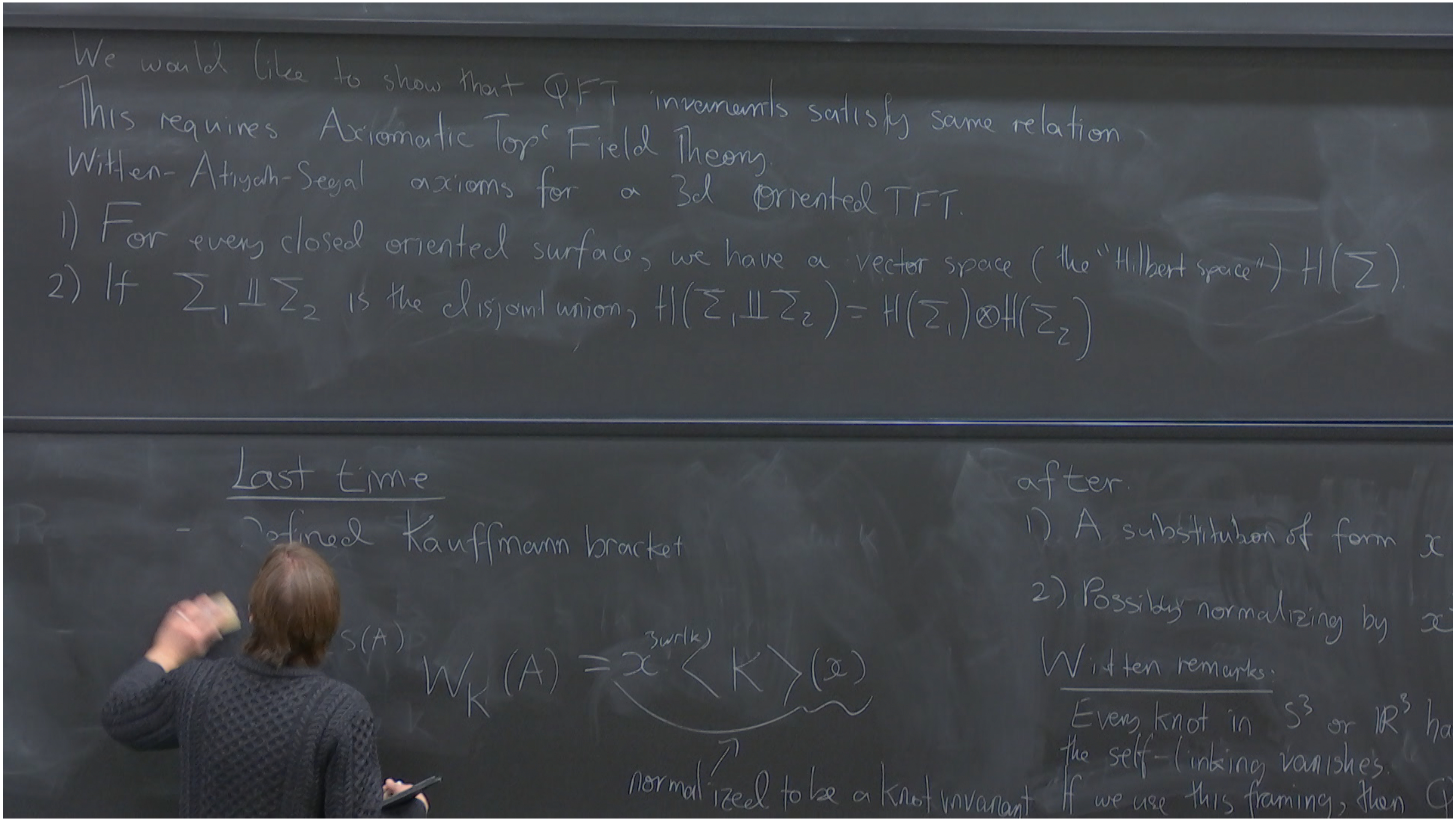

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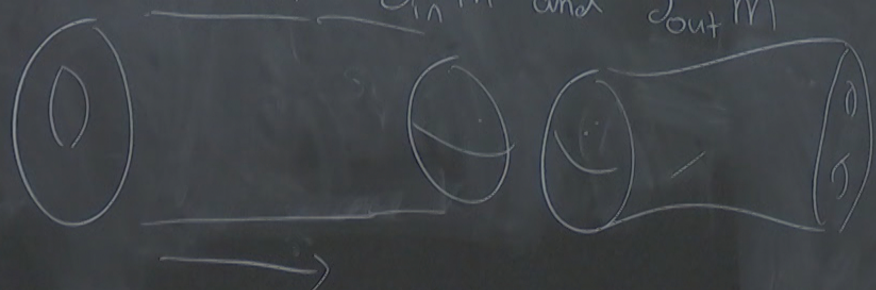

then, $Z(M): H(\partial_{in} M) \longrightarrow H(\partial_{out} M)$

4) Most important axiom: If M_1, M_2, $\partial_{out} M_1 = \partial_{in} M_2$, then we can glue $M_1 \amalg_{\partial_{out} M_1} M_2$

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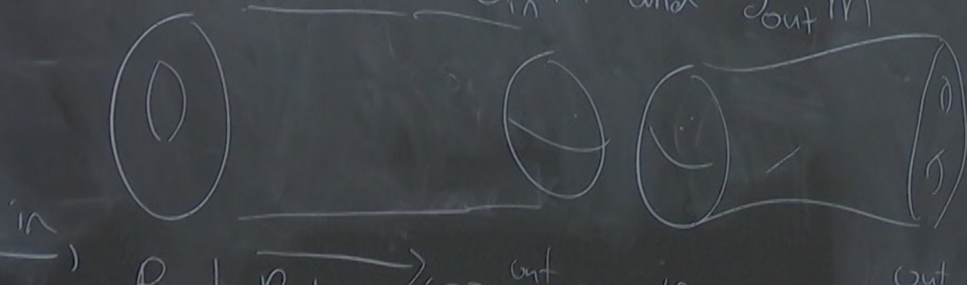

Real Picture: $(\overset{out}{D^2 \times S^1}) \setminus (\overset{in}{a\ ball})$

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$M_1 \underset{\partial_{out} M_1}{\amalg} M_2$ and,

$$Z(M_1 \underset{\partial_{out} M_1}{\amalg} M_2) = Z(M_2) \circ Z(M_1)$$

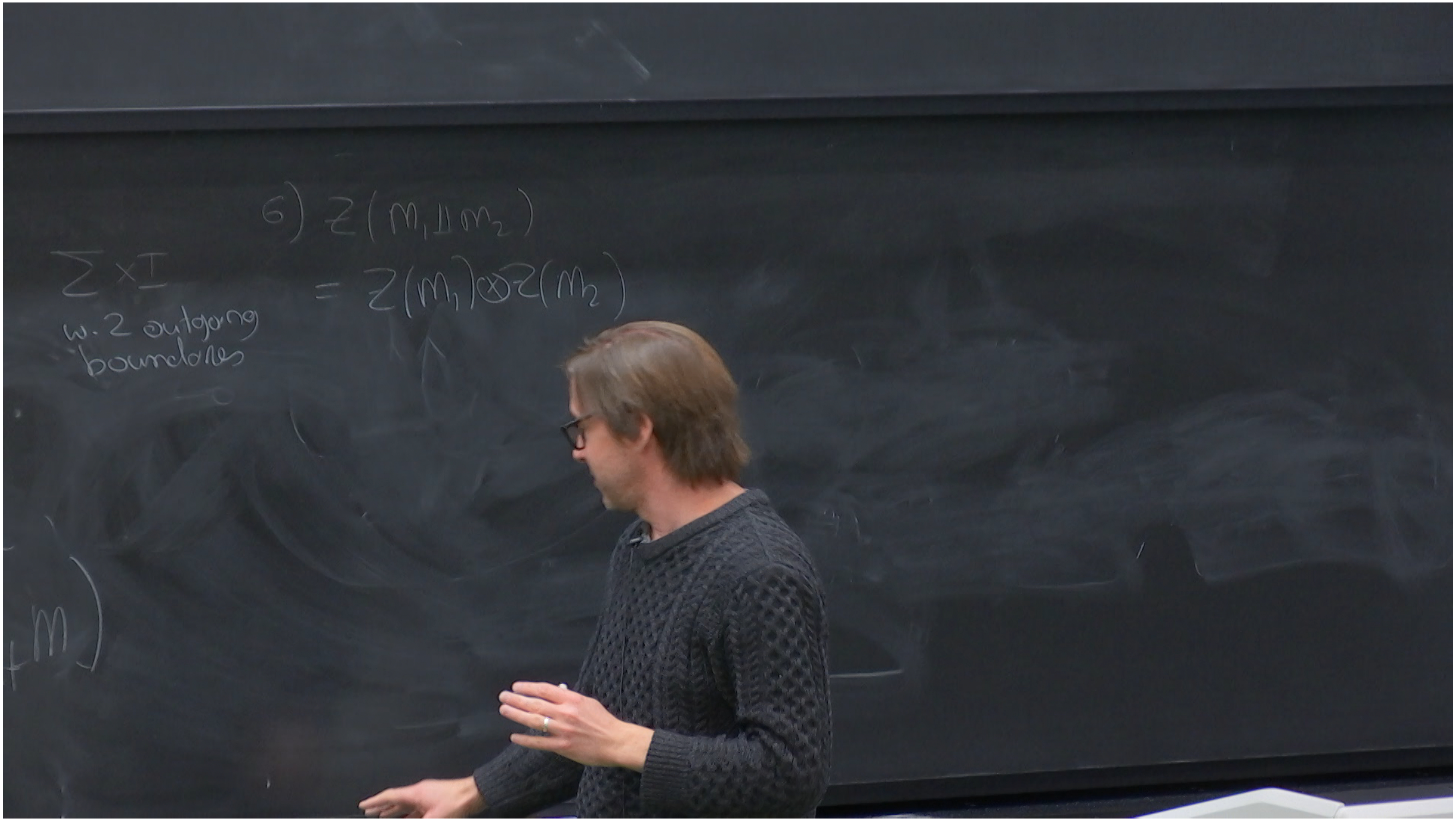
$$: H(\partial_{in} M_1) \longrightarrow H(\partial_{out} M_1) = H(\partial_{in} M_2) \longrightarrow H(\partial_{out} M_2)$$

If $\bar{\Sigma} = \Sigma$ w. opposite orientation,

5) $H(\bar{\Sigma}) = H(\Sigma)^*$

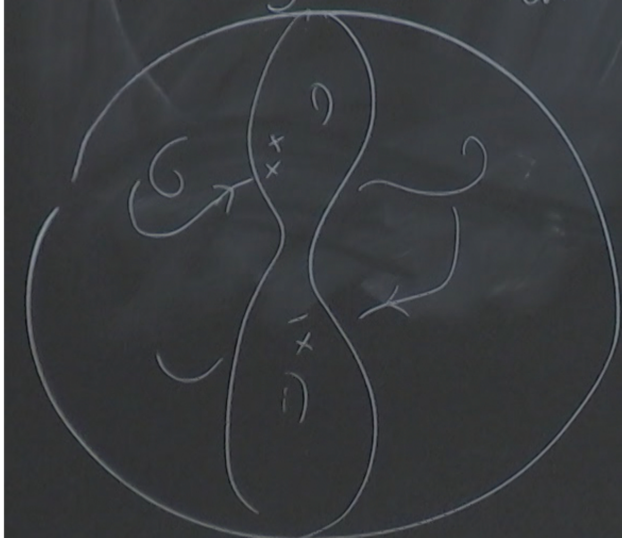
If we like, can view all boundaries as outgoing; this reverses orientation on $\partial_{in} M$, so

$$Z(M) \in H(\partial_{in} M)^* \otimes H(\partial_{out} M)$$


6) $Z(M_1 \sqcup M_2)$
$= Z(M_1) \otimes Z(M_2)$
$\Sigma \times I$
w. 2 outgoing
boundaries

Axioms for 3d TFT w. a 1d oriented defect:

Very similar except:



A link intersects Σ at points which are labelled $\pm$, depending on orientation of the link.

efect:

st

epending

In this setting

- $\mathcal{H}$ is defined for a surface Σ with $n+m$ points, n $+$, m $-$

- M has a link w. boundary on ∂M

$$Z(M,L) : H(\partial_{in} M, \partial_{in} L) \longrightarrow H(\partial_{out} M, \partial_{out} L)$$

as before. All other axioms are the same,

$$H\left(\Sigma, \begin{smallmatrix} n & + \\ m & - \end{smallmatrix}\right) = H\left(\overline{\Sigma}, \begin{smallmatrix} n & - \\ m & + \end{smallmatrix}\right)^*$$

Near b of m,

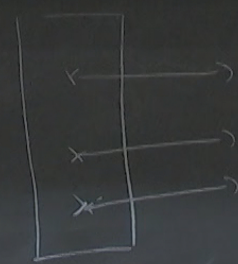
$$m = \Sigma \times [0,1)$$

Link looks like

$$p_1 \times [0,1)$$

$$\vdots$$

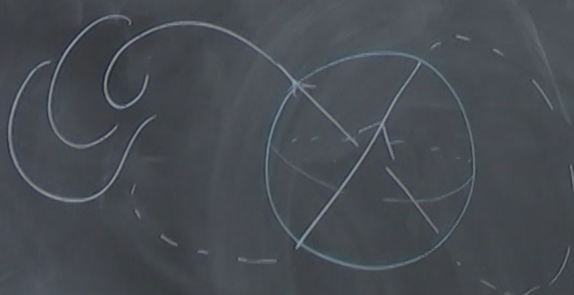
$$p_{n+m} \times [0,1)$$



$$\partial\text{Link} = \text{points in } \Sigma$$

$\ldots(\Sigma, m-) = H(\Sigma, m+)$

Suppose we have a link in S^3, focus near 1 crossing



How do we understand

$Z(S^3, L)$ in terms of local data at crossing?

$S^3 = M_1 \cup M_2$

M_1 = a ball surrounding the crossing

M_2 = complement of M_1

These are glued along

$(S^2, p_+^1, p_+^2, p_-^1, p_-^2)$

...ta at crossing.

$H(\Sigma_2)$

ta at crossing.

These are glued along

$$(S^2, p_+^1, p_+^2, p_-^1, p_-^2)$$

$$Z(S^3, L)$$

$$= \langle Z(M_2, L'), Z(M_1, \ )\rangle$$

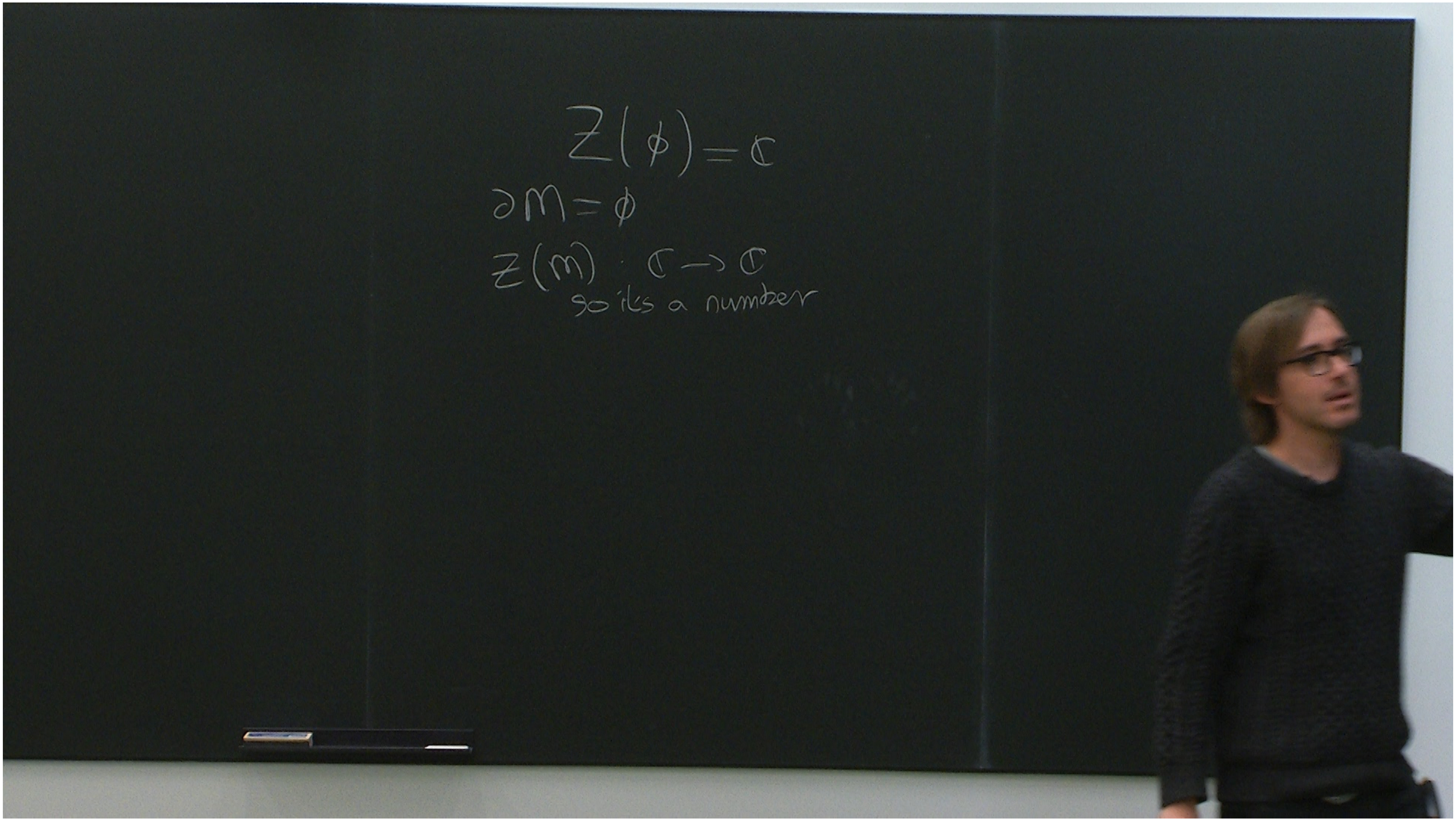
where

$$Z(M_1 = \text{ball}, \cancel{X'}) \in \mathcal{H}(S^2, p_+^1, p_+^2, p_-^1, p_-^2)$$

$$Z(M_2, L) \in \text{dual vector space}$$

If we modify our L locally to

then $Z(M, L_-)$, $Z(M, L_0)$ has the same expression where we replace $Z(\mathrm{Ball}, \;)$ by $Z(\mathrm{Ball}, \;)$ or $Z(\mathrm{Ball}, \;)$


$Z(\phi) = \mathbb{C}$
$\partial M = \phi$
$Z(M) : \mathbb{C} \to \mathbb{C}$
so it's a number

A Skein relation like in Jones poly will hold.

If there are functions $F_+(\hbar), F_-(\hbar), F_0(\hbar)$

so that

$$F_+(\hbar)\, Z(\mathrm{Ball}, \;) + F_-(\hbar)\, Z(\mathrm{Ball}, \;) + F_0(\hbar)\, Z(\mathrm{Ball}, \;) = 0$$

$$\in \mathcal{H}(S^2, p_+^1, p_+^2, p_-^1, p_-^2)$$

6) $Z(M_1 \sqcup M_2)$

Theorem

$\mathcal{H}(S^2, P_1^+, P_2^+, P_1^-, P_2^-)$ is 2 dimensional

if $G = SU(n)$

and the defect is a Wilson line in fundamental representation