Title: PSI 2019/2020 - Chern-Simons Theory Part 1 - Lecture 8

Speakers: Kevin Costello

Collection: PSI 2019/2020 - Chern-Simons Theory (Part 1)

Date: February 28, 2020 - 10:00 AM

URL: http://pirsa.org/20020031

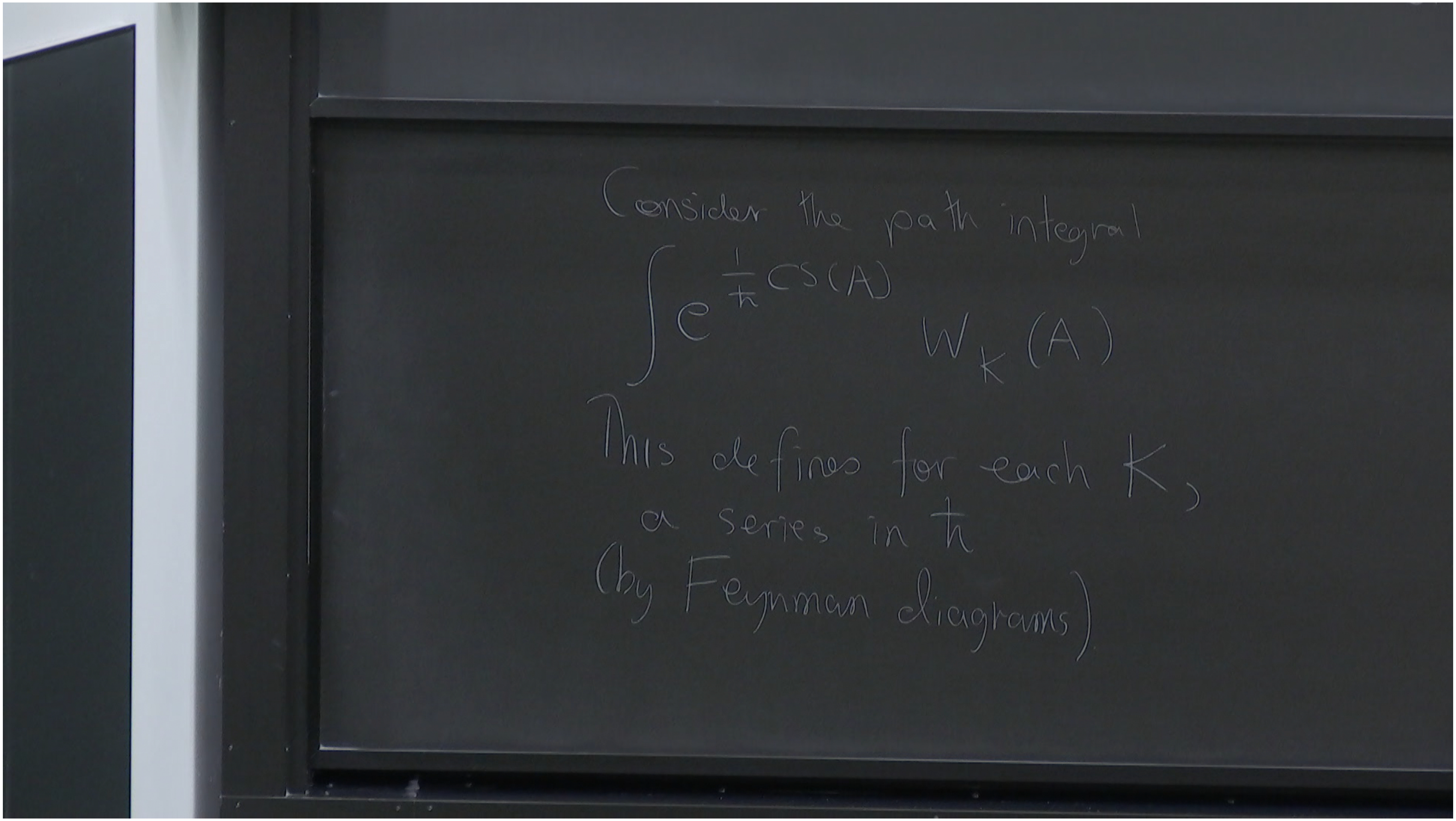
Chern-Simons Perturbation Theory and the Jones Polynomial

Last time, we defined Wilson loops for non-abelian gauge groups:

If $K \subseteq \mathbb{R}^3$ is a closed loop, then

$$W_K(A) = \mathrm{Tr}\, P\mathrm{Exp}_K A = \mathrm{Tr}\, P(K, A)$$

This is gauge invariant


Consider the path integral
$\int e^{\frac{1}{\hbar} CS(A)} W_K(A)$
This defines for each K,
a series in $\hbar$
(by Feynman diagrams)

This is (almost) a knot invariant: i.e. something that doesn't change if we continuously vary K.

"Proof"

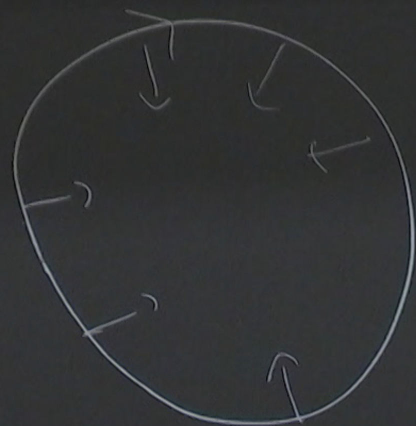
- Variations of K can be realized by diffeos. φ

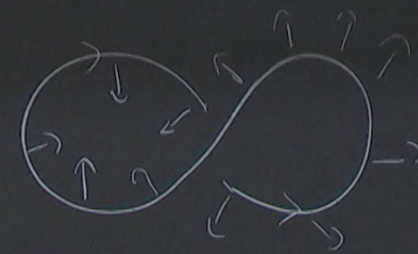
$$W_K(\varphi^* A) = W_{\varphi(K)}(A)$$

Further, path integral measure is inv. under $A \longrightarrow \varphi^* A$

– On shell, $F(A)=0$
so that $W_k(A)$ on-shell
does not change if we vary k,
this should persist at quantum
level.

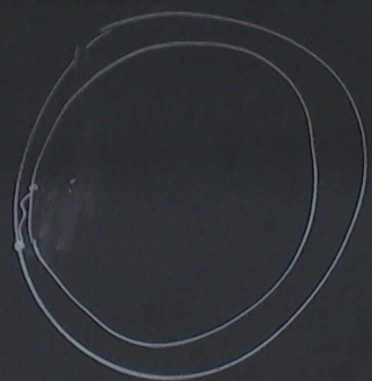
In fact,
the path integral depends
on one more piece of data:
a framing
= a choice of a normal
vector, everywhere on knot,
nowhere vanishing.





But,

$\int e^{\frac{1}{\hbar} CS(A)} W_K(A)$ does not change if we continuously vary the knot and the framing.



$$\int e^{-S(x)/\hbar}\,dx$$

$$V = f(x)\partial_x$$

$$0 = \int f(x)\partial_x\left(e^{-S/\hbar}\,dx\right)$$

if we

$$-S(x)/\hbar \quad dx$$

$$f(x)\partial_x$$

$$x\left(e^{-S/\hbar}dx\right)$$

$$= -\int \underbrace{\frac{1}{\hbar} f(x)\partial_x S'}\, e^{-S/\hbar}$$

$$+\int (\partial_x f)\, e^{-S/\hbar}\, dx$$

$\hbar \to 0$ observables

$\partial_x S'$ are zero

divergence, wrt functional measure

For S^1, how many choices of normal vector, up to continuous variation, are there?

Answer: $\mathbb{Z}$

The normal vector can wind around as we move along the knot.

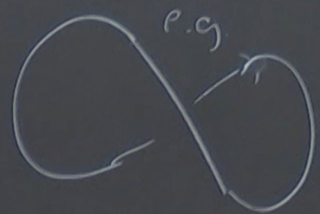
$\varphi(K)$

Further, path integral measure is inv. under $A \longrightarrow \varphi^{*}A$

Defn (Kauffman bracket)

Given a link, defined by a planar projection

e.g.



we define $\langle L \rangle (x)$ by 3 rules.

as we

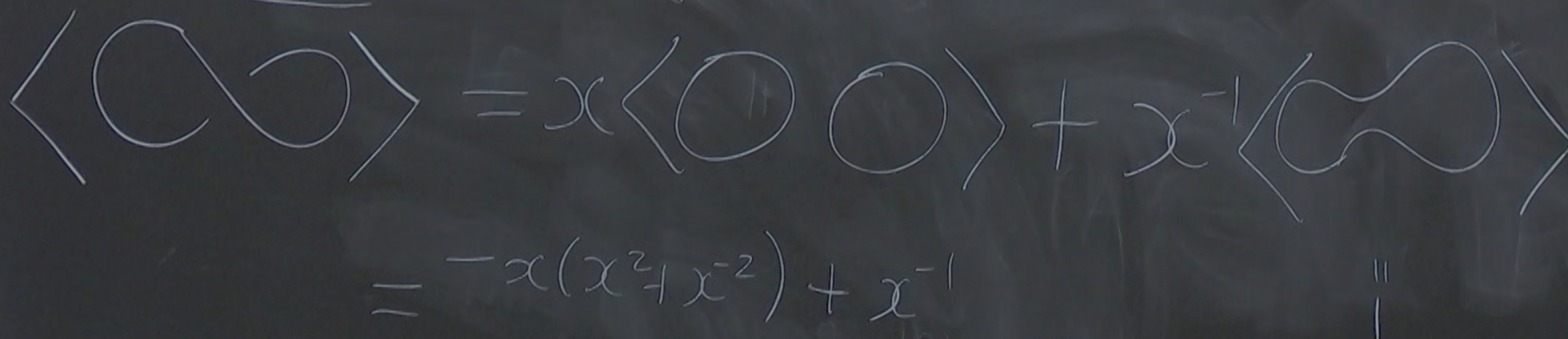
$$\langle \times \rangle = x \langle) (\rangle + x^{-1} \langle \asymp \rangle$$

$$\langle O \rangle = 1 \qquad \langle O \cup L \rangle = (-x^{2} - x^{-2}) \langle L \rangle$$

$\langle \bigcirc \rangle = 1$

These rules make $\langle L \rangle(x)$ into a Laurent polynomial of x

Example



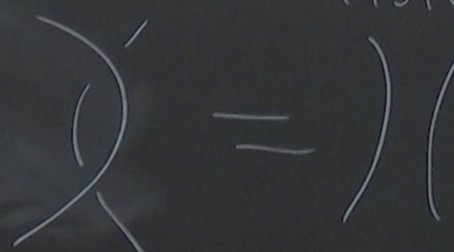
$$= -x(x^2 + x^{-2}) + x^{-1}$$

$$= -x^3$$

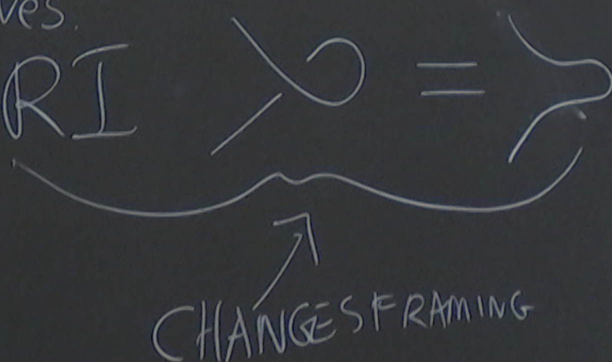
Reidemeister Moves

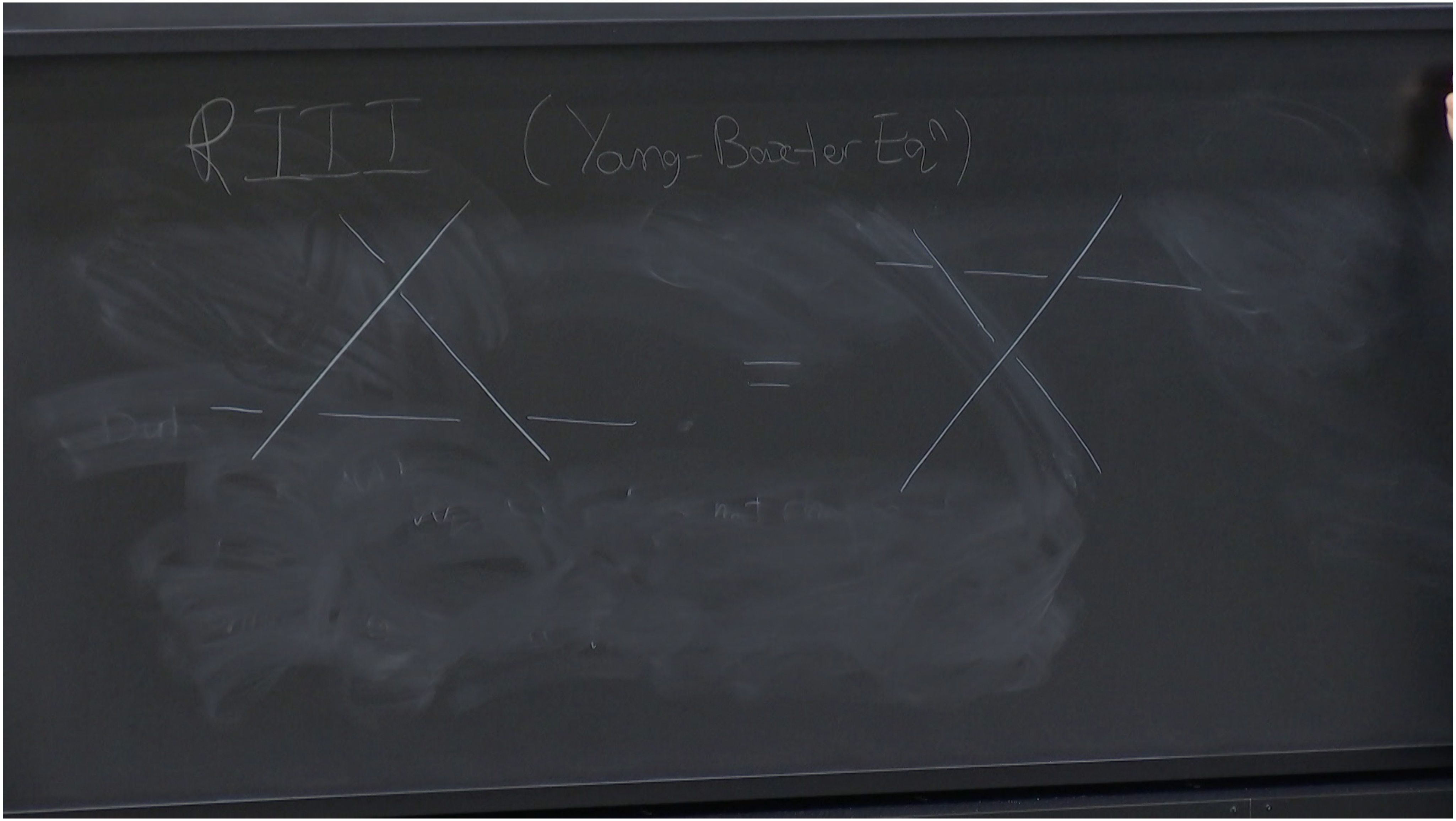
A knot can be represented by a planar projection in many ways, but these are related by the Reidemeister moves.

RII



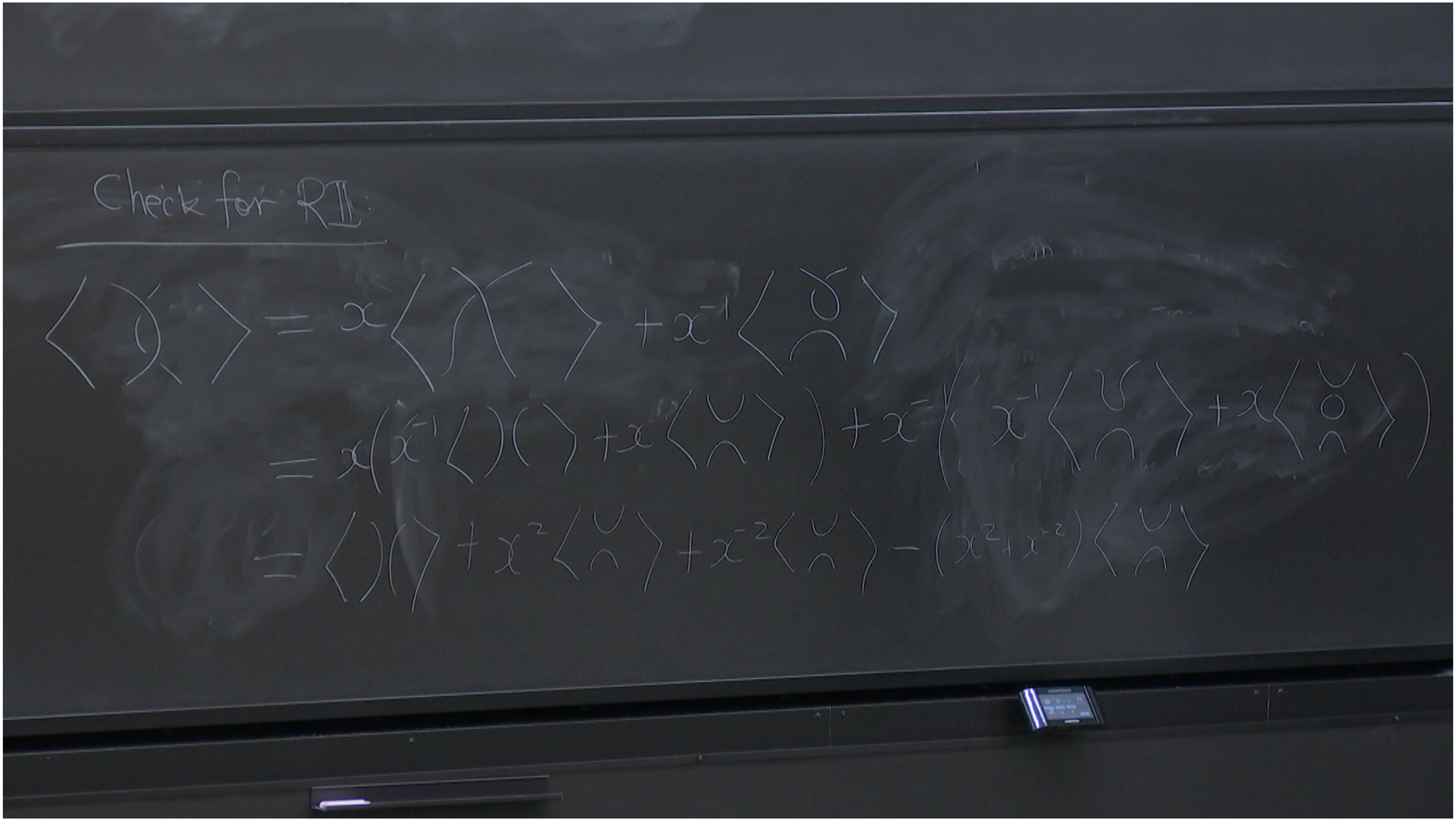
RI




RIII (Yang-Baxter Eq^n)

If we are interested in invariants of framed knots, framing comes from the planar projection, we only need to apply RII and RIII.

Theorem Kauffman bracket is invariant under RII and RIII.

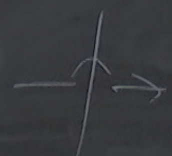

Check for RII:

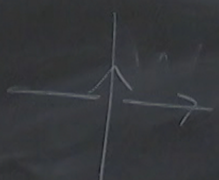
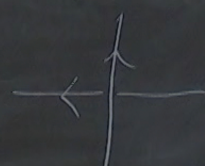
$$\langle \times \rangle = x^{-1} \langle)(\rangle + x \langle \asymp \rangle$$

by 90° rotation

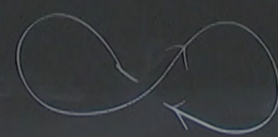
$\langle \quad \rangle$ does not satisfy RI.

But: if we have a knot (or link), and choose an orientation

 we define the writhe = # over crossings − # under crossings.

 $+1$  -1

<u>Theorem</u>



writhe $= -1$

Given a link L,

$(-x^{wr(L)})\langle L \rangle$ is an invariant under all Reidemeister moves.

This is the Jones polynomial

Theorem (Witten)

$$\int e^{\frac{i}{\hbar} CS(A)} W_K(A) = \left(\quad \right) \langle K \rangle$$

$= $ (simple prefactor) Jones polynomial

(normalizing factor only depends on writhe)

after a substitution of the form $x = e^{2\pi i \hbar}$