Title: PSI 2019/2020 - Chern-Simons Theory Part 1 - Lecture 7

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Collection: PSI 2019/2020 - Chern-Simons Theory (Part 1)

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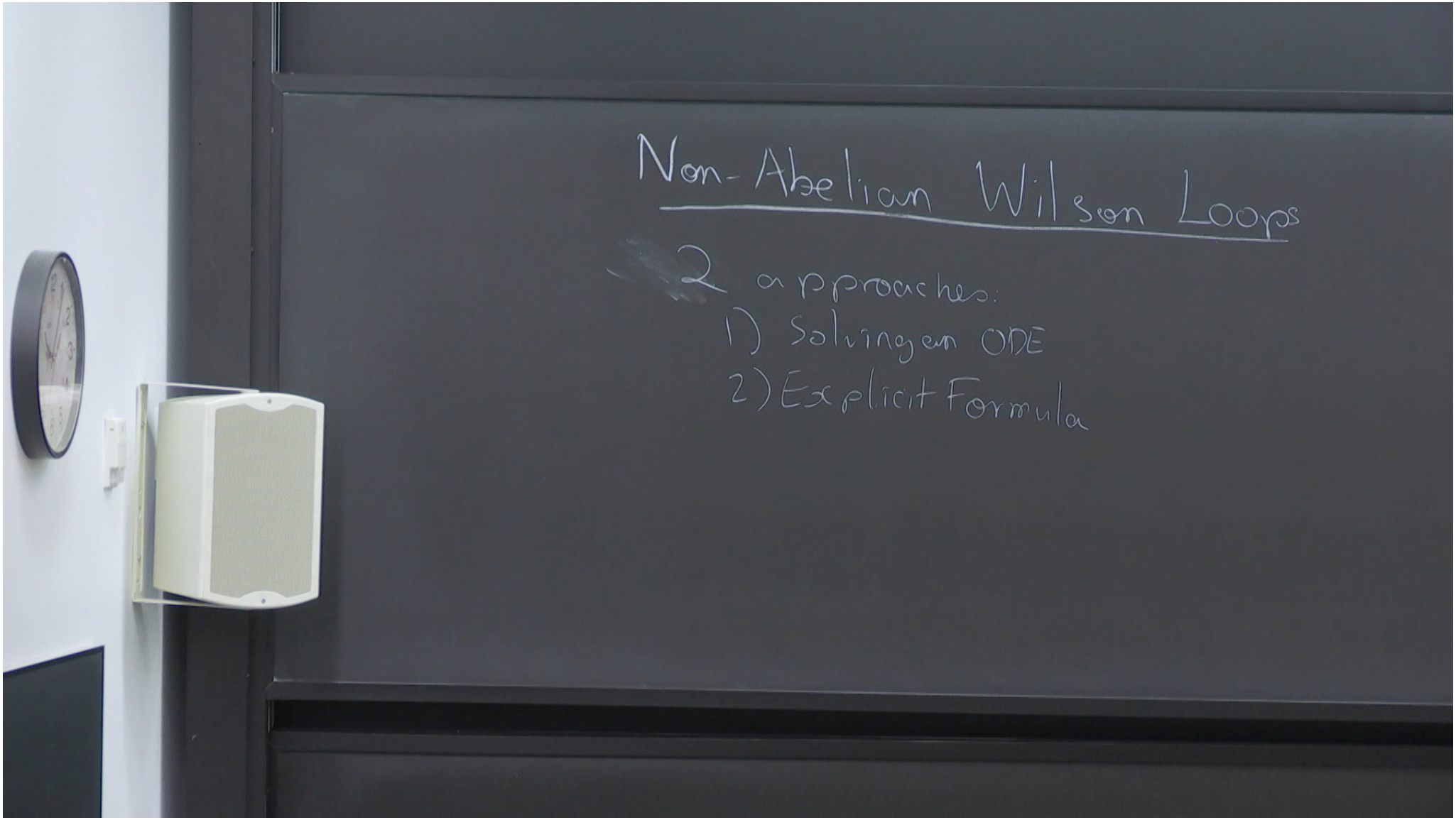
1) Parallel Transport

Given a $u(n)$ (or $gl(n,\mathbb{C})$) gauge field A on $\mathbb{R}^m$ and a path

$$\gamma : [0,1] \longrightarrow \mathbb{R}^m$$

the parallel transport is an $n \times n$ matrix

$$P(\gamma, A) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$


Non-Abelian Wilson Loops
2 approaches:
1) Solving an ODE
2) Explicit Formula

Define this as follows.

Fix an initial vector $V_0 \in \mathbb{C}^n$

Find a family of vectors $V(t)$, with

- $V(0) = V_0$

$$-\frac{\partial}{\partial t}V(t) + (\gamma^* A)V(t) = 0$$

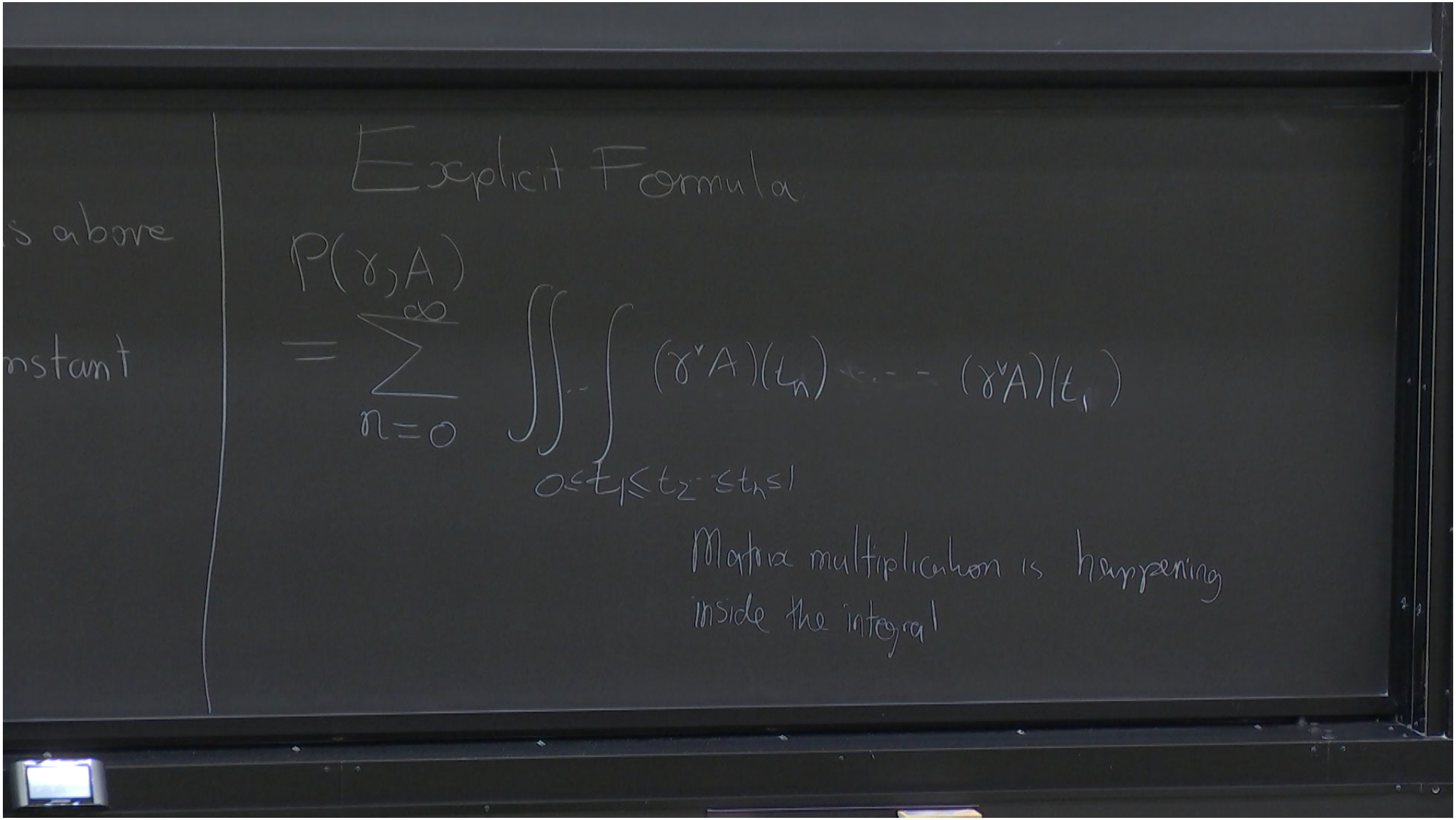
[Recall, if $A = \sum A^\mu dx_\mu$ - A^μ $n \times n$ matrices

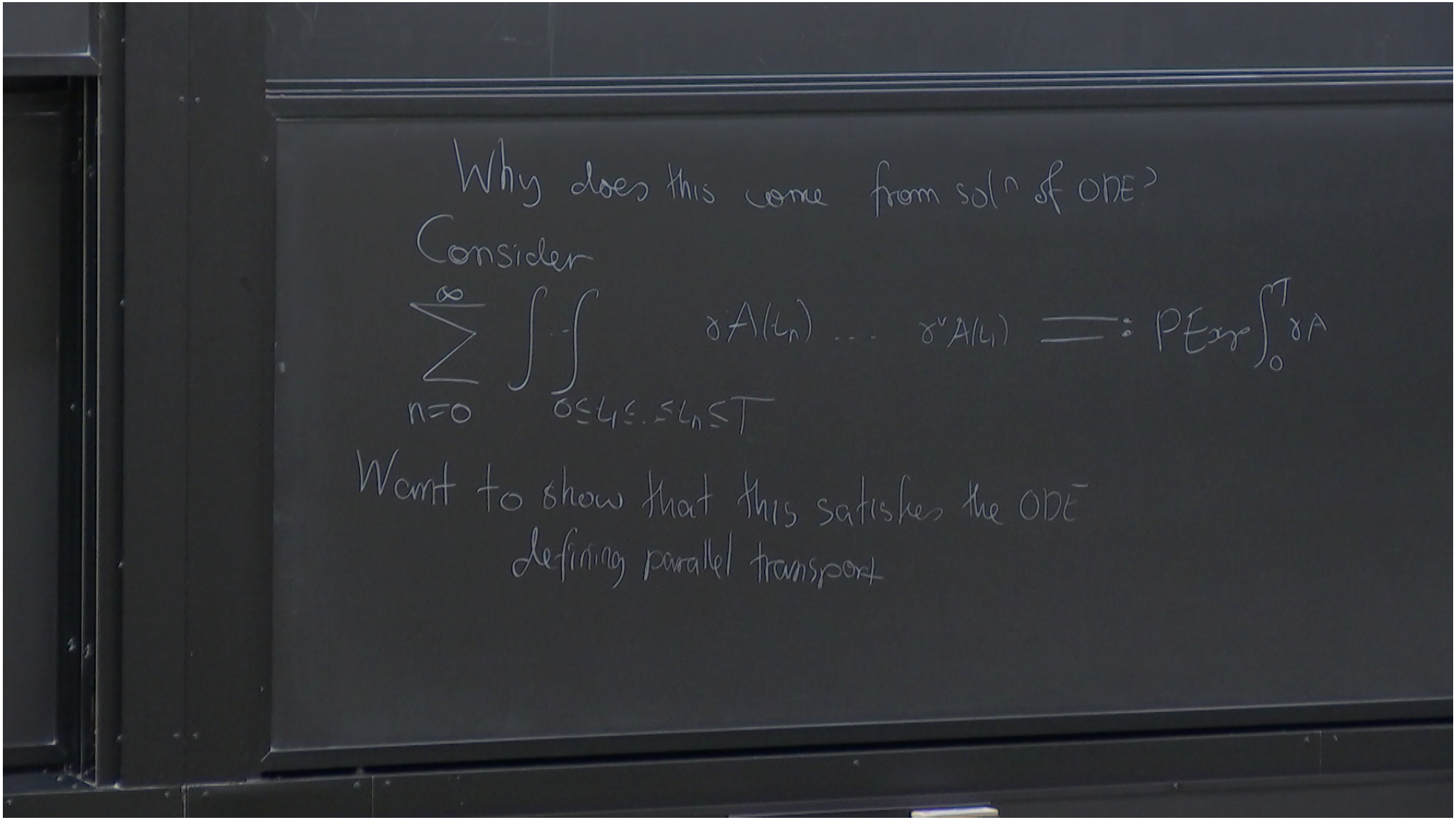
$$\gamma^* A = \left(\sum A^\mu(\gamma(t)) \frac{\partial \gamma_\mu}{\partial t}\right) dt$$

a matrix depends on t

Define $P(\gamma, A)(v_0) = V(1)$ where V is as above

The ODE for $V(t)$ just says V is covariantly constant along this path.


s above
nstant
Explicit Formula
$P(\gamma, A) = \sum_{n=0}^{\infty} \iint\cdots\int_{0 \le t_1 \le t_2 \cdots \le t_n \le 1} (\gamma^{v} A)(t_n) \cdots (\gamma^{v} A)(t_1)$
Matrix multiplication is happening inside the integral


Why does this come from soln of ODE?
Consider
$\sum_{n=0}^{\infty} \int\cdots\int_{0\le t_1\le\cdots\le t_n\le T} \gamma A(t_n)\cdots\gamma^{\nu} A(t_1) =: P\mathrm{Exp}\int_0^T \gamma A$
Want to show that this satisfies the ODE defining parallel transport

Definition

Given a closed loop $\gamma : [0,1] \longrightarrow \mathbb{R}^m$

$\gamma(0) = \gamma(1)$, the non-Abelian Wilson loop is the function of A

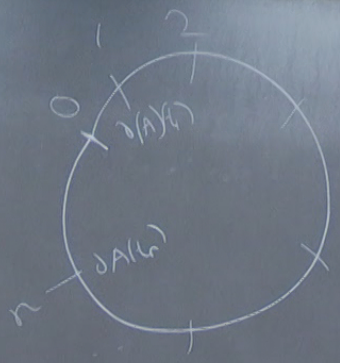
$$A \longmapsto \mathrm{Tr}\, P(\gamma, A) = \sum_n \mathrm{Tr} \int \cdots \int \gamma A(t_n) \cdots \gamma A(t_1)$$

$\gamma : [0,1] \longrightarrow \mathbb{R}^m$

the non-Abelian Wilson loop

A

$$(\gamma, A) = \sum_n \mathrm{Tr} \int \cdots \int \gamma A(t_n) \cdots \gamma A(t_1)$$



from soln of ODE?

$$\frac{\partial}{\partial t} P \exp \int^{\gamma} \gamma^{\nu} A$$

Proposition

The Wilson loop is gauge invariant

Proof We need to first show that $P(\gamma, A)$ on an open path is gauge covariant.

If we change A by a gauge trans. to

$$A^X = X^{-1} dX + X^{-1} A X$$

then

$$P(\gamma, A^X) = X(\gamma(1))\, P(\gamma, A)\, X(\gamma(0))^{-1}$$

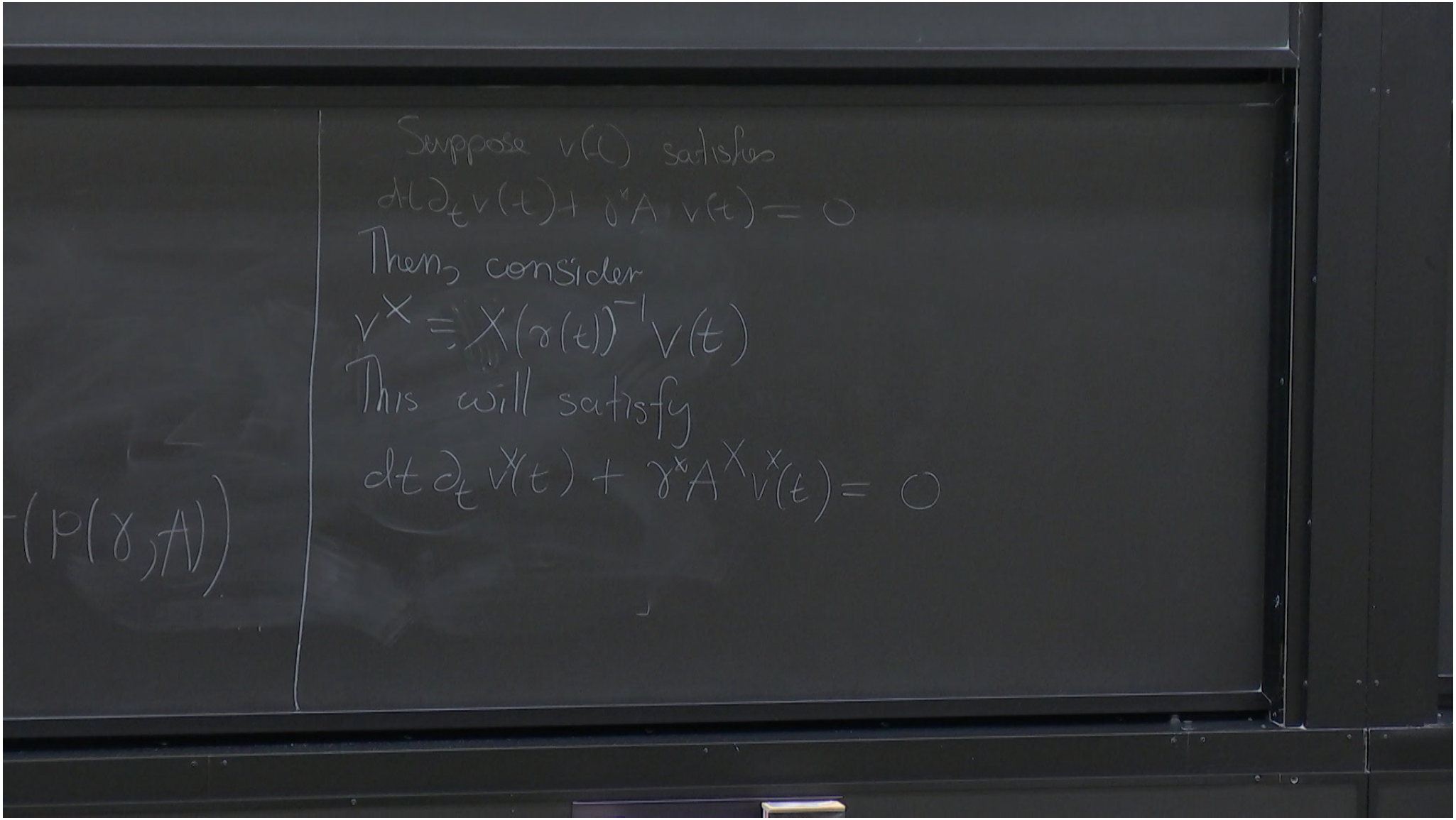
[or perhaps $X(\gamma(1))^{-1} P(\gamma, A) X(\gamma(0))$]

If this holds, and $\gamma(0) = \gamma(1)$,

then

$$\mathrm{Tr}\, P(\gamma, A^X) = \mathrm{Tr}\left(X(\gamma(1))\, P(\gamma, A)\, X(\gamma(0))^{-1} \right)$$

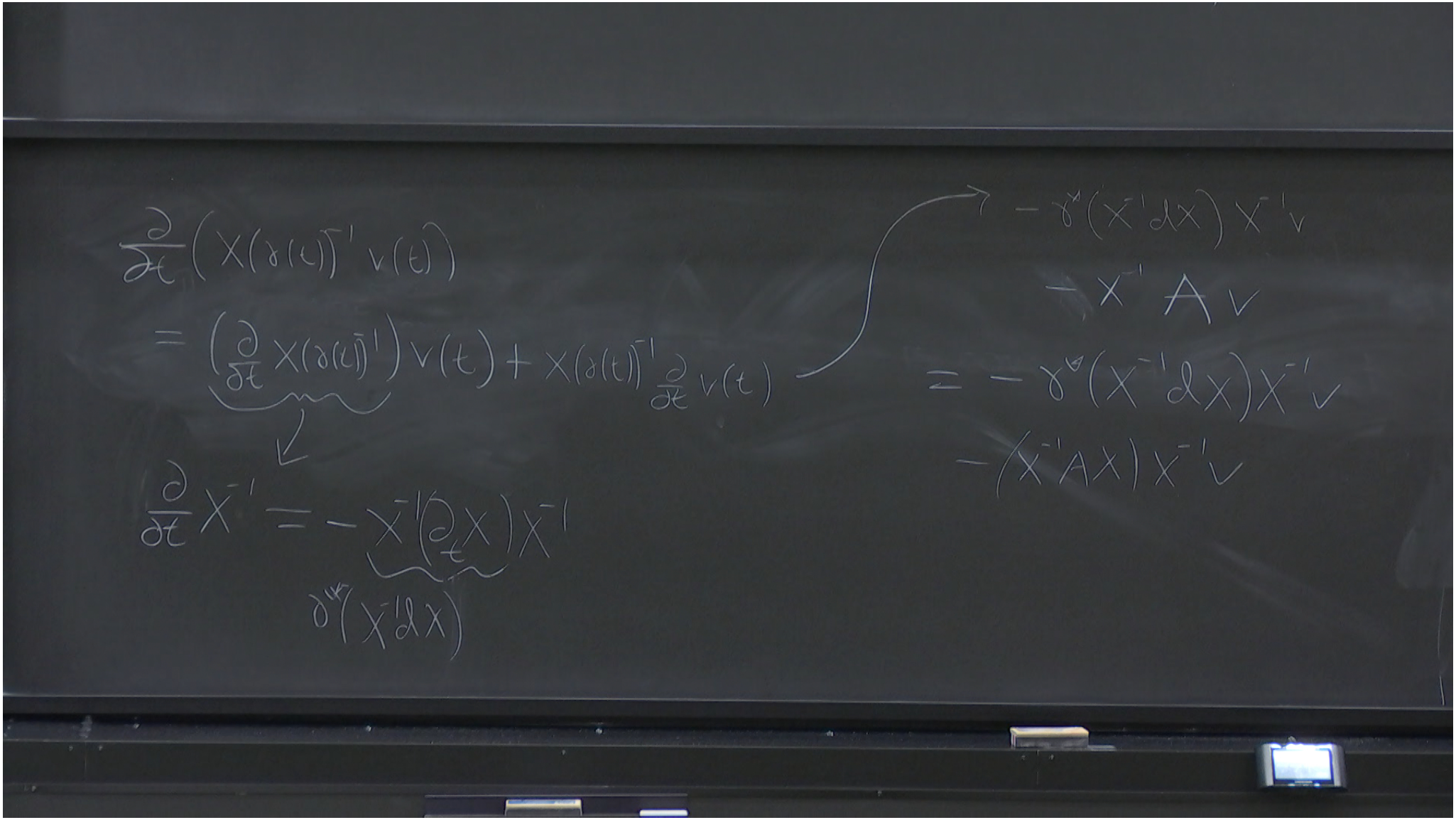
$$= \mathrm{Tr}\left(X(\gamma(0))^{-1} X(\gamma(1))\, P(\gamma, A) \right) = \mathrm{Tr}\left(P(\gamma, A) \right)$$

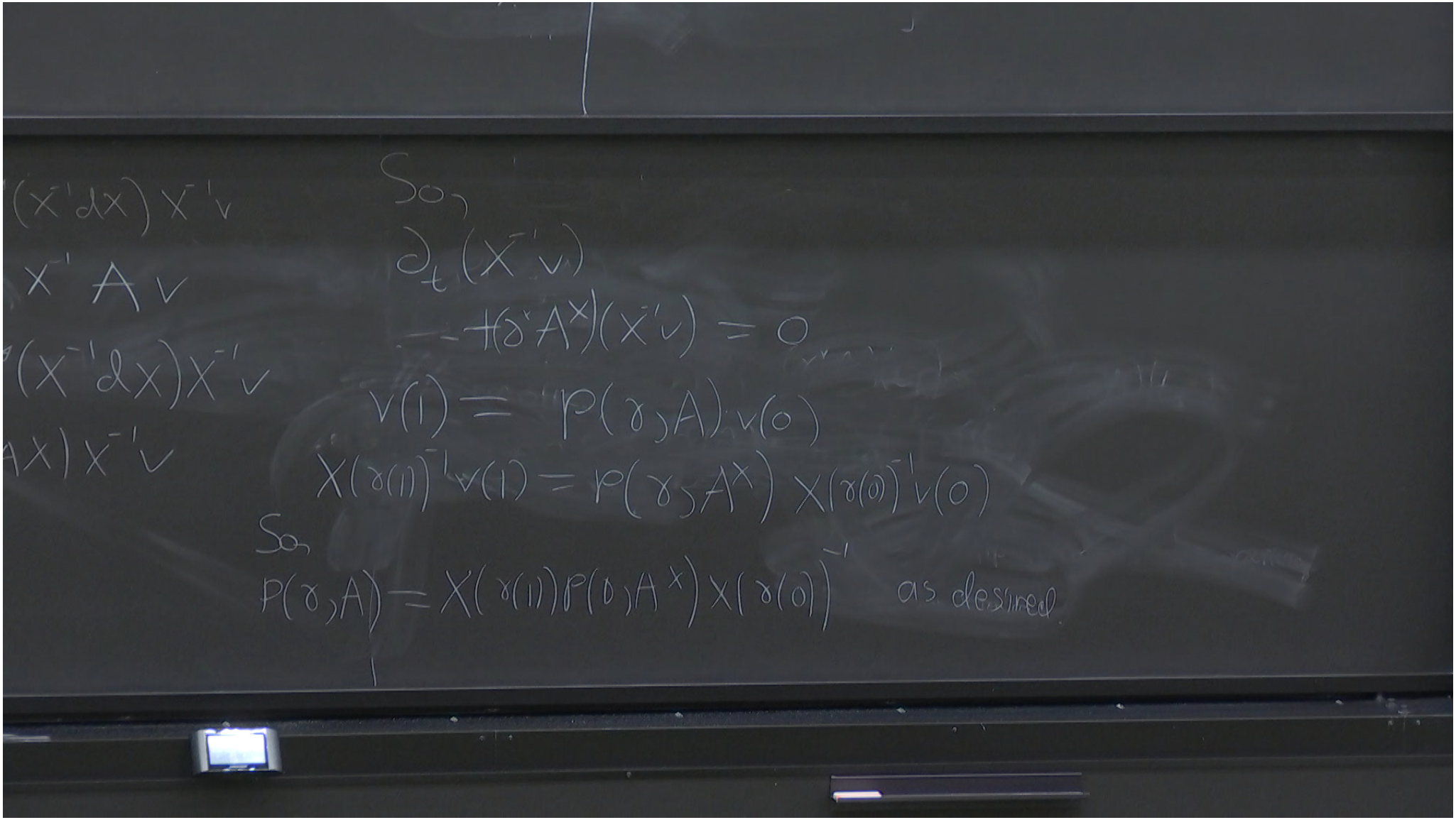

Suppose v(t) satisfies
$dt(\partial_t v(t) + \gamma^* A\, v(t) = 0$
Then, consider
$v^X = X(\gamma(t))^{-1} v(t)$
This will satisfy
$dt\, \partial_t v^X(t) + \gamma^* A^X v^X(t) = 0$
$(P(\gamma, A))$

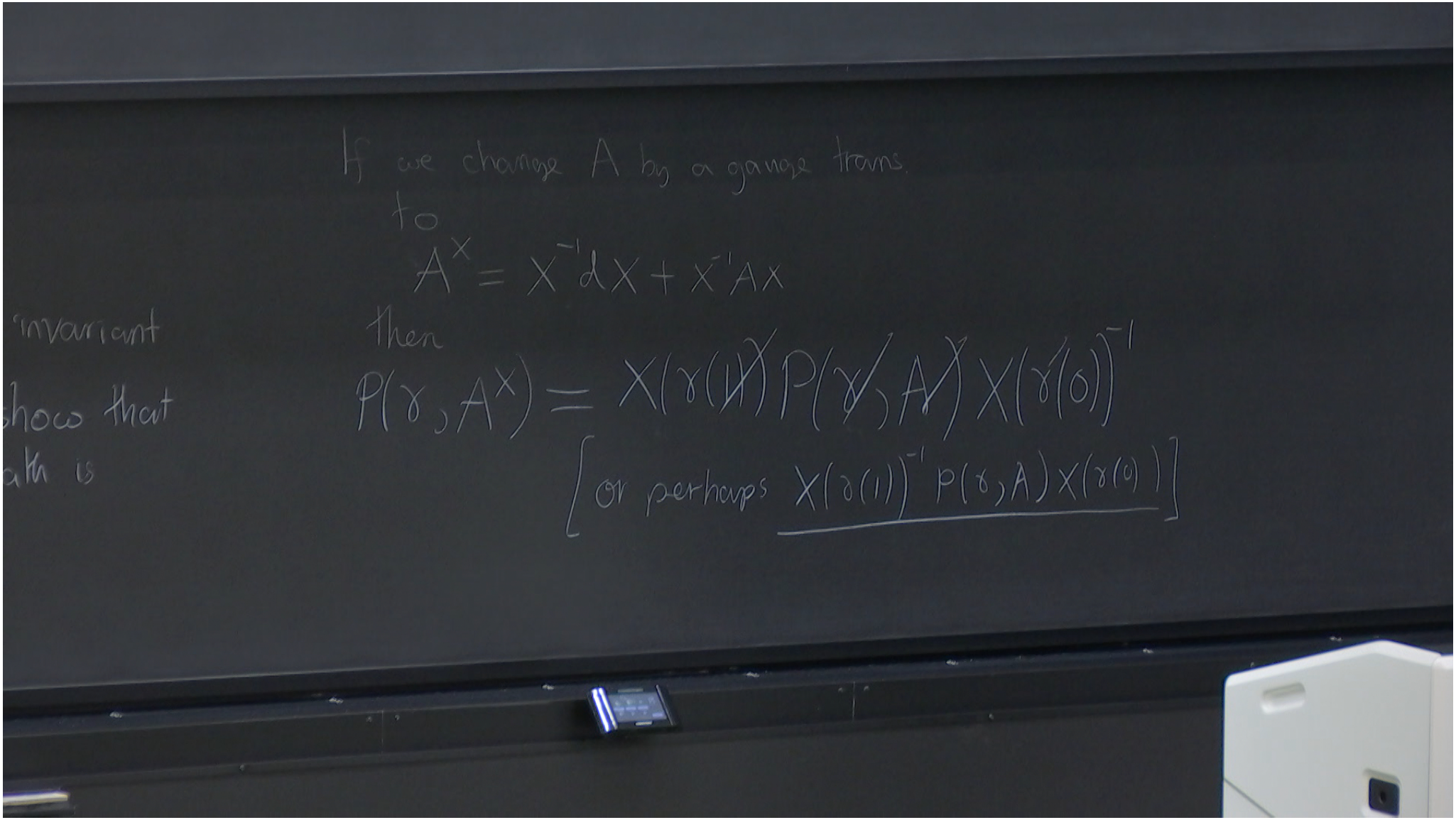
Explicit Formula

$$P(\gamma, A) = \sum_{n=0}^{\infty} (-1)^n \int\cdots\int_{0 \le t_1 \le t_2 \cdots \le t_n \le 1} (\gamma^* A)(t_n) \cdots (\gamma^* A)(t_1)$$

Matrix multiplication is happening inside the integral


$\frac{\partial}{\partial t}\left(X(\gamma(t))^{-1} v(t)\right)$
$= \left(\frac{\partial}{\partial t} X(\gamma(t))^{-1}\right) v(t) + X(\gamma(t))^{-1} \frac{\partial}{\partial t} v(t)$
$\frac{\partial}{\partial t} X^{-1} = -X^{-1}(\partial_t X) X^{-1}$
$\gamma^*(X^{-1} dX)$
$-\gamma^*(X^{-1} dX) X^{-1} v$
$-X^{-1} A v$
$= -\gamma^*(X^{-1} dX) X^{-1} v$
$-(X^{-1} A X) X^{-1} v$


$(X^{-1}dX)X^{-1}v$
$X^{-1}AV$
$(X^{-1}dX)X^{-1}v$
$AX)X^{-1}v$
So,
$\partial_t(X^{-1}v)$
$-t(\partial^* A^X)(X^{-1}v) = 0$
$v(1) = P(\gamma, A)\, v(0)$
$X(\gamma(1))^{-1} v(1) = P(\gamma, A^X)\, X(\gamma(0))^{-1} v(0)$
So,
$P(\gamma, A) = X(\gamma(1))\, P(0, A^X)\, X(\gamma(0))^{-1}$ as desired


If we change A by a gauge trans.
to
$A^X = X^{-1}dX + X^{-1}AX$
then
$P(\gamma, A^X) = X(\gamma(1))\,P(\gamma, A)\,X(\gamma(0))^{-1}$
[or perhaps $X(\gamma(1))^{-1}\,P(\gamma, A)\,X(\gamma(0))$]
'invariant
show that
ath is

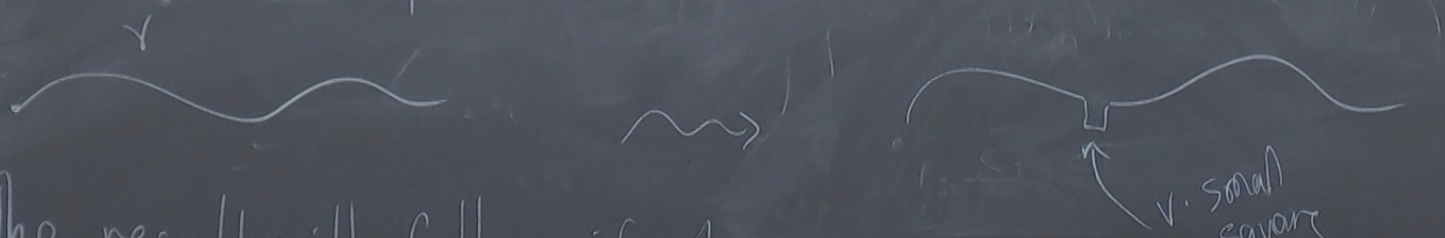
Proposition

If $F(A) = 0$, then

$P(\gamma, A)$ does not change if we change γ continuously while fixing the endpoints.

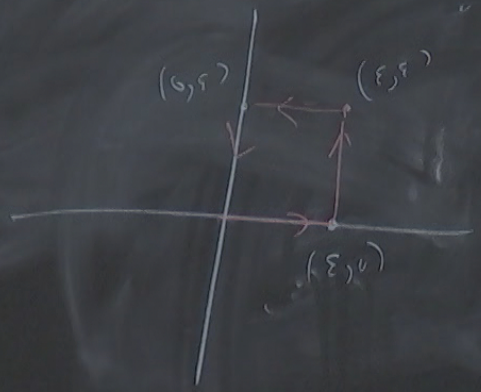
Proof:

Any continuous variation is built by repeated modifications of the form:



The result will follow, if when we go around a small square of size ε, we get $Id + \varepsilon^2 F + O(\varepsilon^3)$ from parallel transport.

Work on $\mathbb{R}^2$, gauge field A, coords x, y consider $P(\gamma, A)$ where γ is



$\mathbb{R}^2$, gauge field A, coords x, y consider

here γ is

Plug in PExp

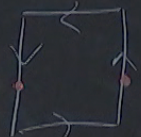
Easy to check to order ε,

parallel paths cancel

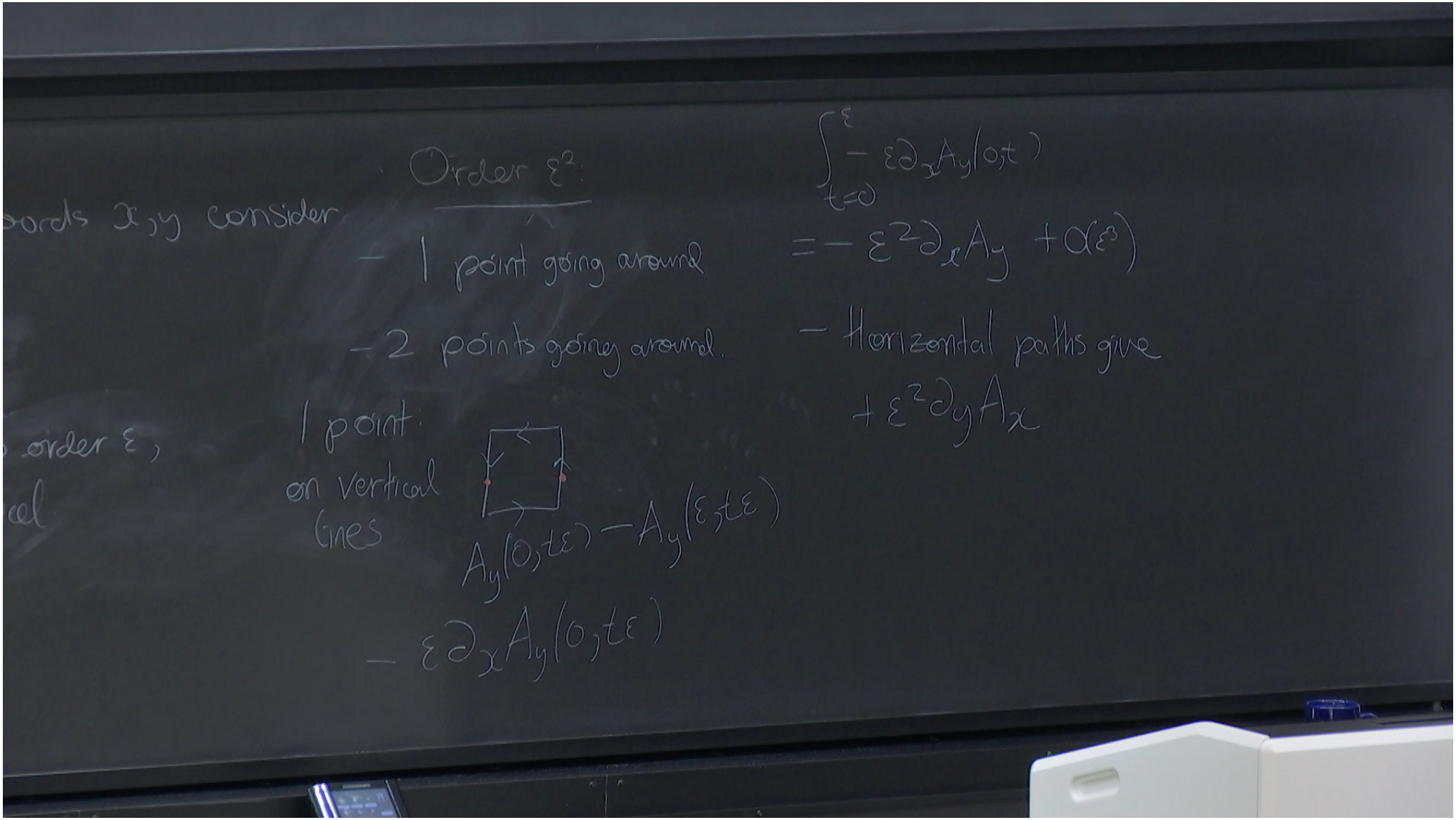
Order ε^2:

- 1 point going around
- 2 points going around.

1 point on vertical lines



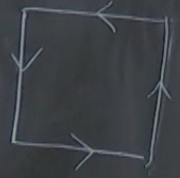
$A_y(0, t\varepsilon) - A_y(\varepsilon, t\varepsilon)$


ords x, y consider
order ε,
Order ε²
- 1 point going around
- 2 points going around.
1 point on vertical lines
$A_y(0,t\varepsilon) - A_y(\varepsilon,t\varepsilon)$
$-\varepsilon\partial_x A_y(0,t\varepsilon)$
$\int_{t=0}^{\varepsilon} -\varepsilon\partial_x A_y(0,t)$
$= -\varepsilon^2\partial_x A_y + O(\varepsilon^3)$
- Horizontal paths give
$+\varepsilon^2\partial_y A_x$

2 points on the square

- Order ε^2 from 2 integrals on $[0,\varepsilon]$
- Gauge field is $A = A(0,0) + O(\varepsilon)$

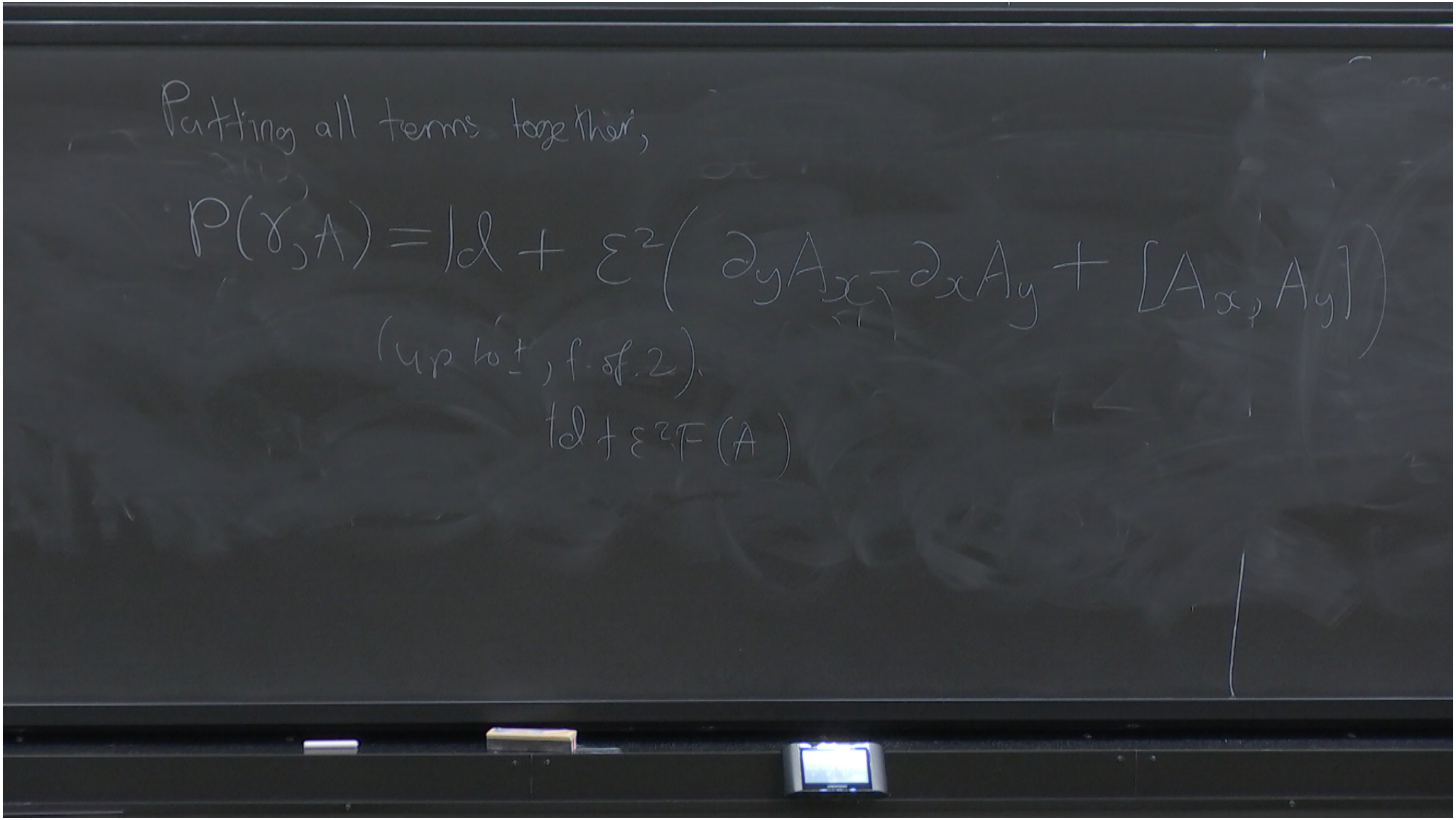
so to order ε^2, only $A(0,0)$, not derivatives, will appear



2 points, P_1, P_2

$P_1 < P_2$

If P_1 is on x-axis, P_2 on y axis } get A_x A_y with a sign


Putting all terms together,
$P(\gamma, A) = 1d + \varepsilon^2 (\partial_y A_x - \partial_x A_y + [A_x, A_y])$
(up to $\frac{1}{2}$, f. of. 2).
$1d + \varepsilon^2 F(A)$