Title: PSI 2019/2020 - Chern-Simons Theory Part 1 - Lecture 6

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Collection: PSI 2019/2020 - Chern-Simons Theory (Part 1)

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Last Time:

We claimed that

$$\int_{A \in \Omega^1(\mathbb{R}^3)/\text{gauge}} e^{\int A dA} \int_{K_1} A \int_{K_2} A = \text{Gauss Linking number } L(K_1, K_2)$$

$G = U(1)$

Does not change

If we vary k_1, k_2 continuously,
without making them cross
"Link Invariant"

Consider the path integral

$$\int_{A \in \text{Gauge Fields}/\text{Equivalence}} e^{\int CS(A)} W_{K_1}(A) \cdots W_{K_n}(A)$$

$W_{K_i}(A)$ = Wilson loops on closed curves.

If we wiggle t_i, does this change?

Answer, heuristically, no!

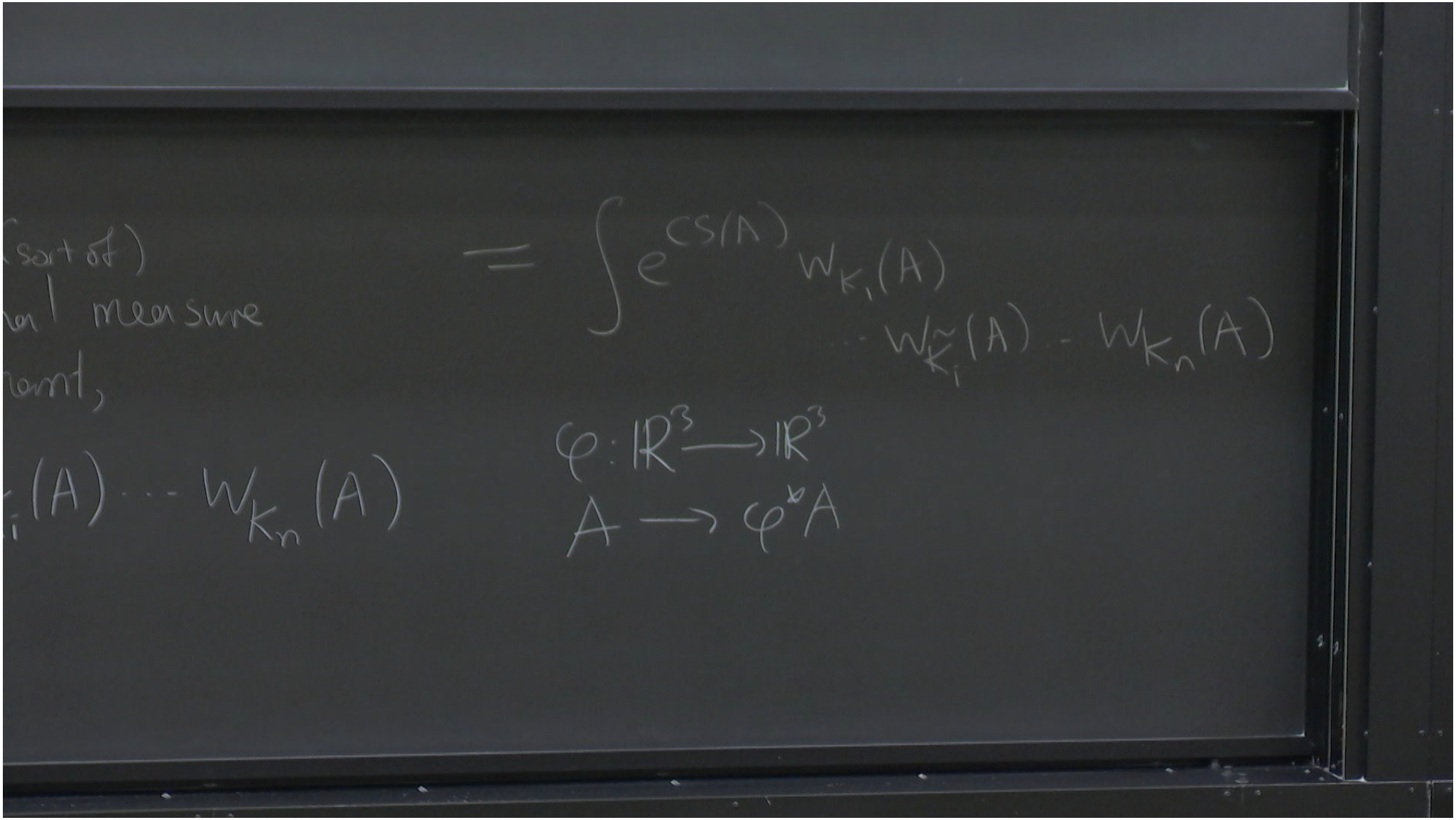
The CS Lagrangian is invariant under <u>all</u> diffeomorphisms, and, Wilson loops $W_{K_i}(A)$ are diffeo. covariant.

If φ is a diffeo. $\varphi^* W_{K_i}(A) = W_{\varphi(K_i)}(A)$

If we vary K_i to $\tilde{K}_i$
(w/o introducing any crossings)
then we can find a diffeo.
φ, so $\varphi(K_j) = K_j$, $j \neq i$
$\varphi(K_i) = \tilde{K}_i$

Since we believe (sort of)
that the functional measure
is diffeo invariant,

$$\int_A e^{CS(A)} W_{K_1}(A) \cdots W_{K_i}(A) \cdots W_{K_n}(A)$$


(sort of)
measure
$= \int e^{CS(A)} W_{K_1}(A)$
$\cdots W_{K_i}(A) \cdots W_{K_n}(A)$
$\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
$A \longrightarrow \varphi^* A$

$$A \wedge dA \to \varphi^* A \wedge d\varphi^* A$$

$$= \varphi^* (A \wedge dA)$$

$$\int_{\mathbb{R}^3} A \wedge dA = \int_{\mathbb{R}^3} \varphi^* (A \wedge dA)$$

as in coordinates, φ^* gives a Jacobian factor.

$$\omega = \sum \omega^{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

$$d\omega = \sum \frac{\partial \omega^{i_1 \dots i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \dots \wedge dx_{i_k}$$

$$\varphi^* d\omega = \sum \frac{\partial \omega^{i_1 \dots i_k}(\varphi_1(y), \dots, \varphi_n(y))}{\partial x_j} d\varphi_j \wedge d\varphi_{i_1} \wedge \dots \wedge d\varphi_{i_k}$$

$$d\varphi^* \omega = \sum \frac{\partial \omega^{i_1 \dots i_k}(\varphi_1(y) \dots \varphi_n(y))}{\partial y_\ell} dy_\ell \wedge d\varphi_{i_1} \wedge \dots \wedge d\varphi_{i_k}$$

Same because of chain rule.

$K : S^1 \longrightarrow \mathbb{R}^3$

$\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$\int_{S^1} K^* A$

Compare to $\int_{S^1} (\varphi \circ K)^* A$

$(\varphi \circ K)^* A = K^* \varphi^* A$

Changing $K \longrightarrow \varphi \circ K$

Same as changing

$A \longrightarrow \varphi^* A$

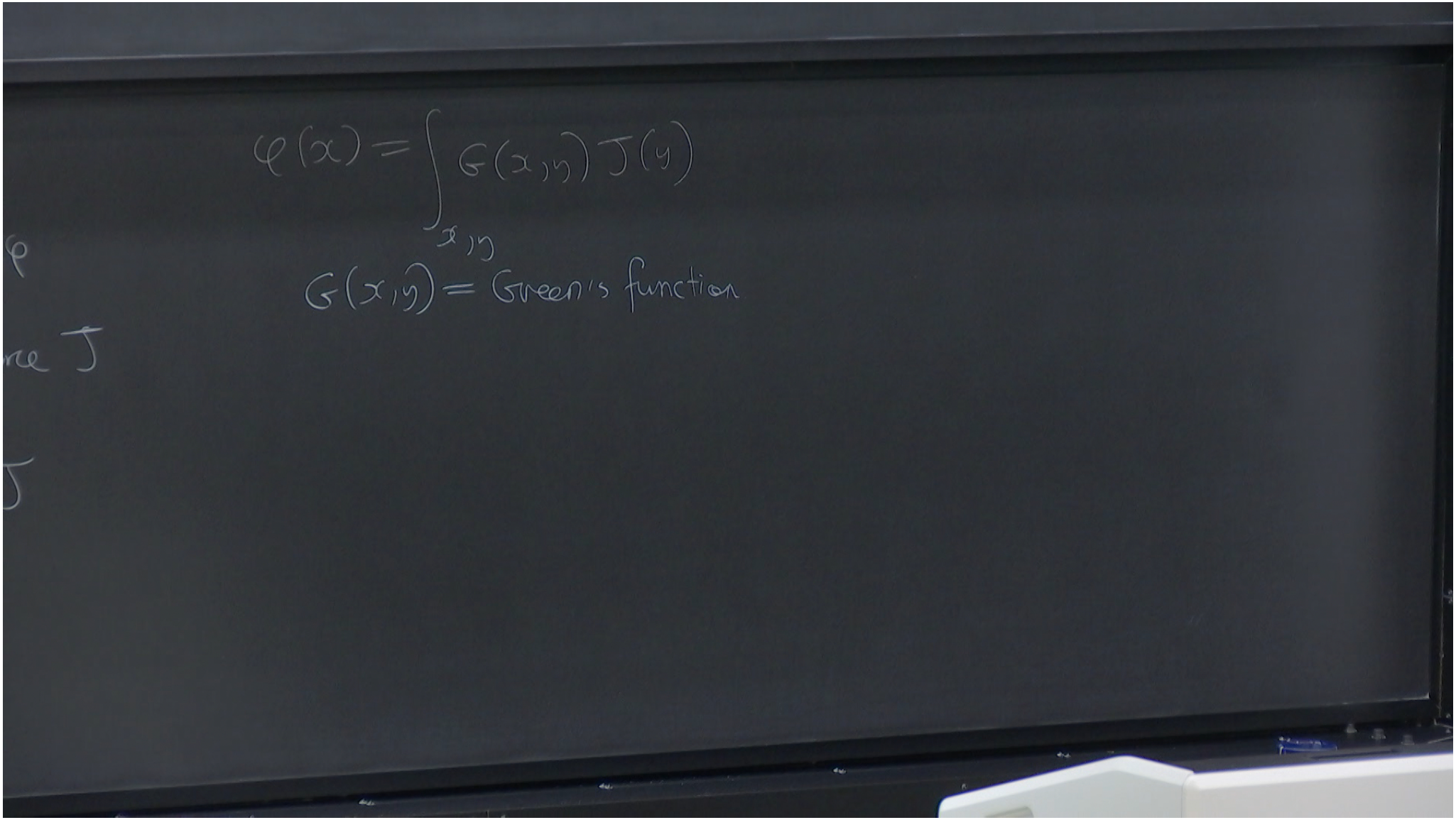
Propagators

Consider a free field, Lagrangian $\int \varphi \Delta \varphi = \Sigma \int \varphi \partial_{x_i}^2 \varphi$

Then, to compute the propagator we introduce a source J and solve EOM for Lagrangian

$$\int \varphi \Delta \varphi - \frac{1}{2} \varphi J \quad \underline{\text{Soln}} \quad \Delta \varphi = J$$

Solved by


$\varphi(x) = \int_{x,y} G(x,y)\, J(y)$
$G(x,y)$ = Green's function

Propagators

Consider a free field, Lagrangian $\int \varphi \Delta \varphi = \sum \int \varphi \partial_{x_i}^2 \varphi$

Then, to compute the propagator we introduce a source J and solve EOM for Lagrangian

$$\int \varphi^{(x)} \Delta \varphi^{(x)} - \frac{1}{2} \varphi^{(x)} J^{(x)} \quad \underline{\text{Soln}} \quad \Delta \varphi = J$$

Solved by

CS

$$\int A \cdot d A - \frac{1}{2} A \cdot J$$

then, J should be a 2-form.

If we want this to be gauge invariant,

under a small gauge trans.

$$\delta A = A + \varepsilon d\chi$$

mod ε^2

2-form.

gauge invariant,

ons.

For this to be invariant, need

$$d\tilde{J} = 0$$

The variational eq^ is

$$dA = J$$

[If we change $A \to A + d\chi$
the eq^ will still be solved]

a gauge.

We ask that

$$\sum \frac{\partial}{\partial x_i} A^i = 0$$

if $A = \sum A^i dx_i$

(Lorenz gauge)

To fix 1 sol^n, we need to choose a gauge.

We ask that

$$\sum \frac{\partial}{\partial x_i} A^i = 0$$

if $A = \sum A^i dx_i$

(Lorenz gauge)

We want A s.t.

1) $dA = J$

2) $\sum \frac{\partial}{\partial x_i} A_i = 0$

Want to write

$$A(x) = \int_y P(x,y) \wedge J(y)$$

so that these eqns hold.

$P(x,y)$ is a 2-form on $\mathbb{R}^3 \times \mathbb{R}^3$

so that $\int_y P(x,y) \wedge J(y)$ is a 1-form

$dA = J$ will hold if $dP(x,y) = \delta_{x=y} \, d(x_1-y_1) \wedge d(x_2-y_2) \wedge d(x_3-y_3)$

As,

$$d_x \int_y P(x,y) \wedge J(y) = \int_y d_x P(x,y) \wedge J(y) = \int_y (d_x + d_y)\left(P(x,y) \wedge J(y)\right)$$

In order that

A satisfies condition (2),

we need that, if we write

$$P = p^{i,j}(x,y)\,dx_i \wedge dy_j$$

then, P should satisfy

$$\sum \frac{\partial}{\partial x_i} p^{i,j} = 0$$

The unique soln to these equations is

$$\Phi = \varepsilon_{ijk} \frac{(x_i - y_i)\, d(x_j - y_j) \wedge d(x_k - y_k)}{\|x - y\|^3}$$

... still be solved]

ditwor (2),
we write
$(x,y)\,dx_i \wedge dy_j$
atisfy
0

The unique sol'n to these equations is

$$\mathcal{P} = \varepsilon_{ijk} \frac{(x_i - y_i)\,d(x_j - y_j) \wedge d(x_k - y_k)}{\|x-y\|^3}$$

Define a map $\alpha : \mathbb{R}^3 \times \mathbb{R}^3 \setminus \{x=y\} \longrightarrow S^2$

$$\alpha(x,y) = \frac{x-y}{\|x-y\|}$$

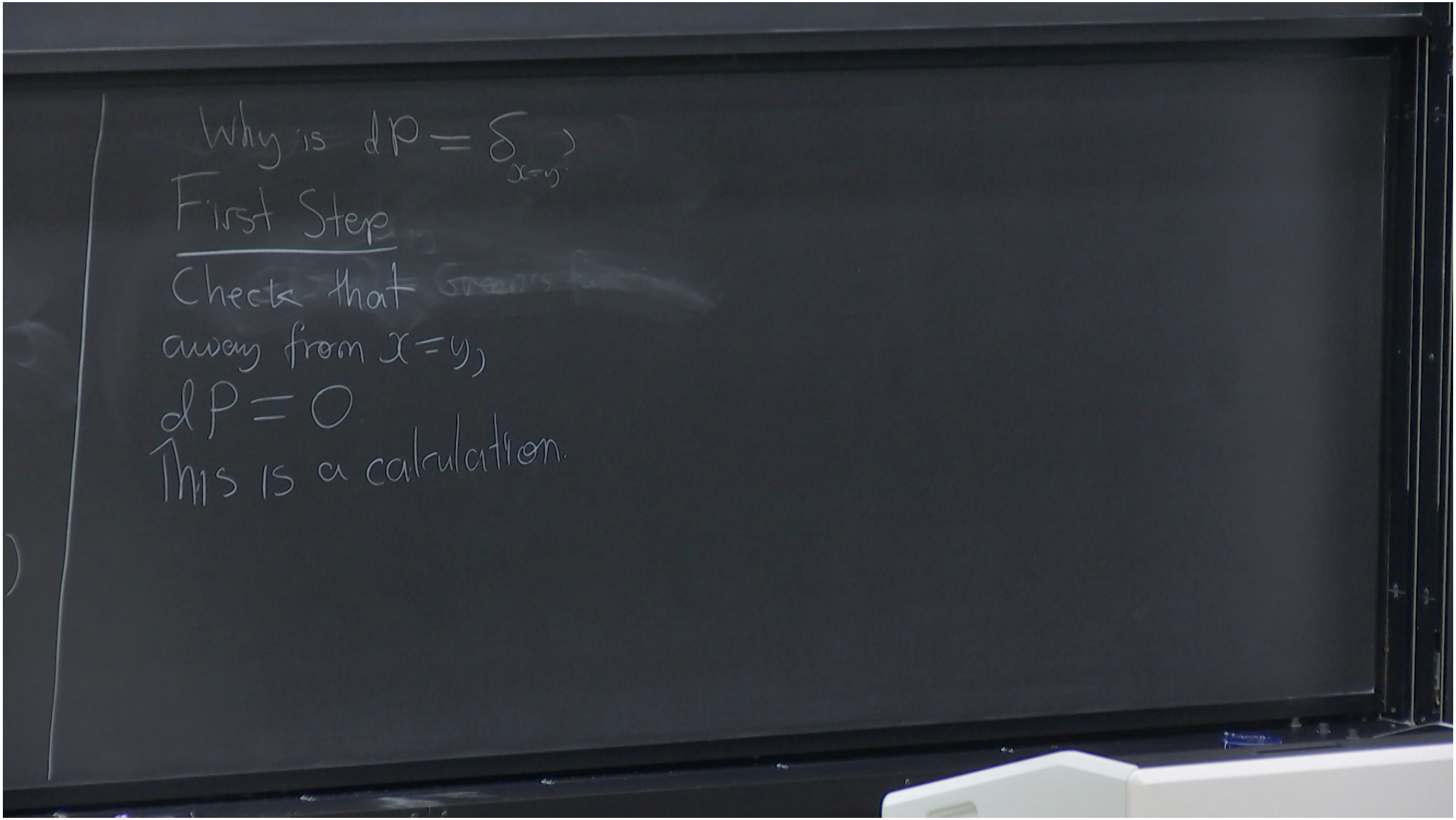
$\mathcal{P} = \alpha^* \omega$, where ω = the $SO(3)$ invariant volume form on S^2

To see that, if we embed $S^2 \subseteq \mathbb{R}^3$ coords u_i, the volume form on S^2 is $\omega = \varepsilon_{ijk} u_i \, du_j \, du_k$

$$\alpha^* \omega = \varepsilon_{ijk} \left(\frac{x_i - y_i}{\|x-y\|} \right) d\left(\frac{x_j - y_j}{\|x-y\|} \right) \wedge d\left(\frac{x_k - y_k}{\|x-y\|} \right)$$

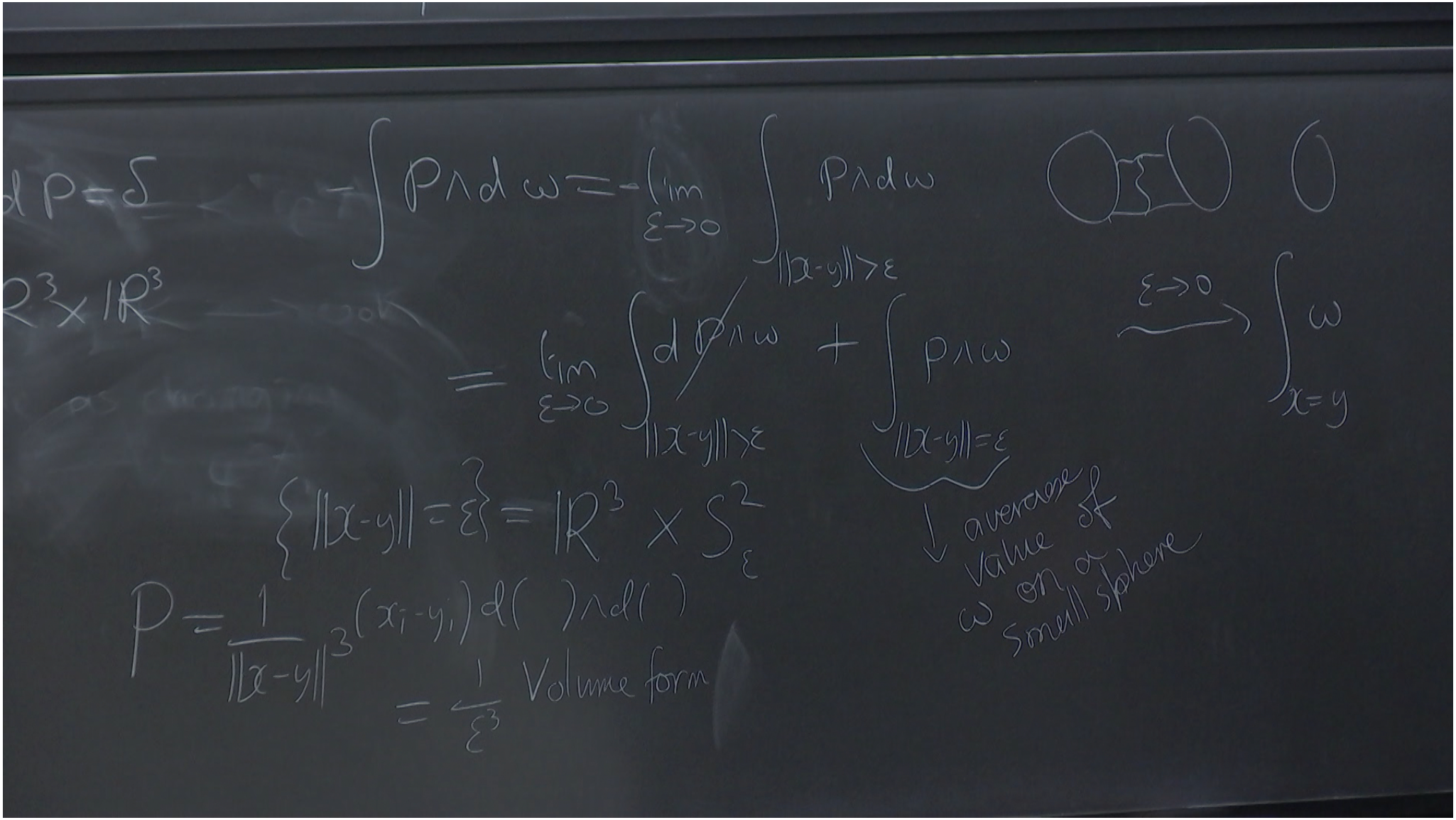
(a small calculation)

$$\frac{C}{\varepsilon_{ijk} \|x-y\|^3} (x_i - y_i) d(x_j - y_j) \wedge d(x_k - y_k)$$


Why is $dP = \delta_{x=y}$?
First Step
Check that
away from $x=y$,
$dP = 0$
This is a calculation.

In de Rham theorem's statement that $dP = \delta$ means, $\forall$ 3-forms ω on $\mathbb{R}^3 \times \mathbb{R}^3$

$$\int_{x,y} P \wedge d\omega = -\int_{x=y} \omega$$


$dP = \delta$
$-\int P \wedge d\omega = -\lim_{\varepsilon \to 0} \int_{\|x-y\| > \varepsilon} P \wedge d\omega$
$\mathbb{R}^3 \times \mathbb{R}^3$
$= \lim_{\varepsilon \to 0} \int_{\|x-y\| > \varepsilon} dP \wedge \omega + \int_{\|x-y\| = \varepsilon} P \wedge \omega$
$\varepsilon \to 0$
$\int_{x=y} \omega$
$\{\|x-y\| = \varepsilon\} = \mathbb{R}^3 \times S^2_\varepsilon$
$P = \frac{1}{\|x-y\|^3}(x_i - y_i)\, d(\,) \wedge d(\,)$
$= \frac{1}{\varepsilon^3}$ Volume form
average value of ω on a small sphere