Title: PSI 2019/2020 - Chern-Simons Theory Part 1 - Lecture 5

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Collection: PSI 2019/2020 - Chern-Simons Theory (Part 1)

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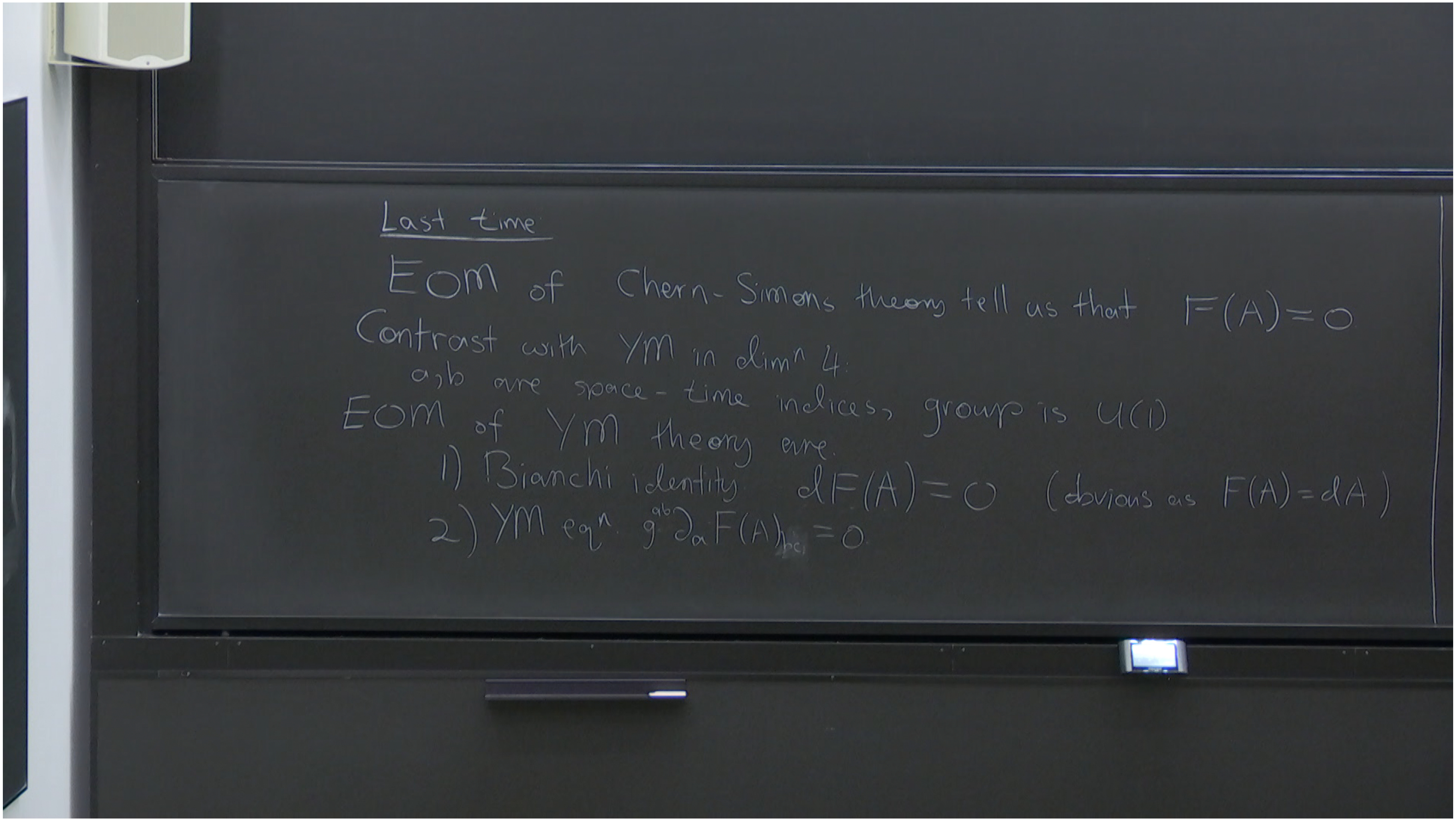

Last time
EOM of Chern-Simons theory tell us that $F(A)=0$
Contrast with YM in dim^n 4:
a, b are space-time indices, group is $U(1)$
EOM of YM theory are
1) Bianchi identity $dF(A)=0$ (obvious as $F(A)=dA$)
2) YM eq^n. $g^{ab}\partial_a F(A)_{bc}=0$

$A) = 0$

$F(A) = dA\,)$

2) can be rewritten as follows
Define a new 2-form
$\ast F$ by

$$(\ast F)_{ab} = g_{ac}\, g_{bd}\, \varepsilon^{cdef} F_{cd}$$

(we can use $g_{ab} = \delta_{ab}$ to simplify)

Eq^n 2 is:

$$d \ast F = 0$$

of YM theory are:

1) Bianchi identity $dF(A)=0$ (obvious as F

2) YM eq^n. $g^{ab}\partial_a F(A)_{bc} = 0$.

YM

There are local operators!

For $G=U(1)$, the value of $F(A)_{ab}$ at a point of space-time is a gauge invariant measurement.

$G=U(n)$, e.g. $\mathrm{Tr}(F\wedge F)$ at a point is gauge invariant.

NO

Lo

(obvious as $F(A) = dA$)

$d * F = 0$

point

CS

NO local measurements!

Locally, $F(A) = 0 \Rightarrow$ we can use a gauge transformation to set $A = 0$.

Eg. $G = U(1)$, $A \in \Omega^1(\mathbb{R}^3)$

If $dA = 0$, since $H^1(\mathbb{R}^3) = 0$, $A = d\chi$ for some χ.

Then, $\dot{A} = e^{-\chi} d e^{\chi}$ so A is gauge trivial.

Wilson Loops

Now suppose that we have a manifold M and $K \subseteq M$ is a circle in M

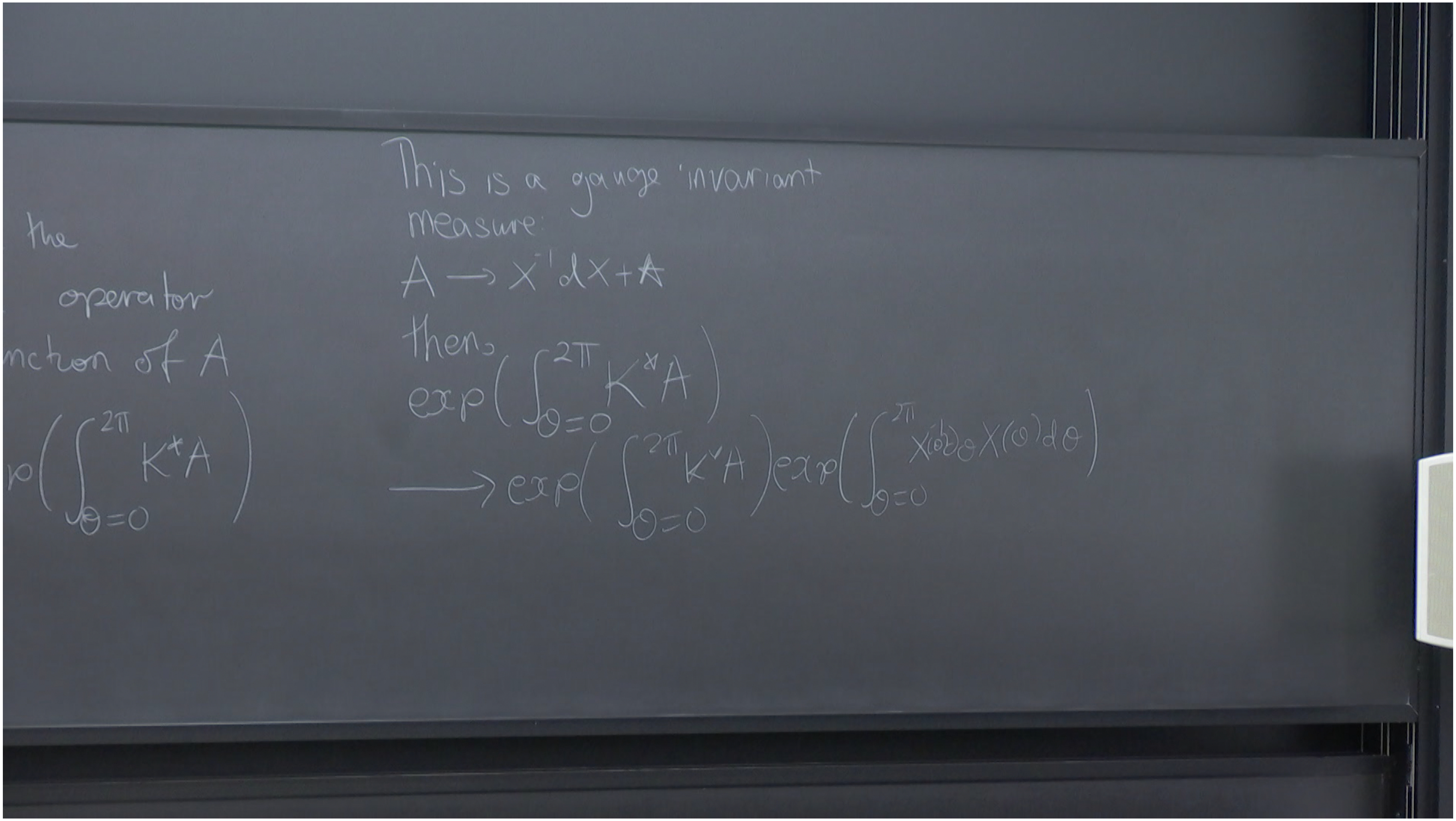
$$K(\theta) \in M$$

$$\theta \in [0, 2\pi]$$

If $G = U(1)$

Then, define the <u>Wilson loop</u> operator to be the function of A which sends

$$A \longrightarrow \exp\left(\int_{\theta=0}^{2\pi} K^{*}A \right)$$


the
operator
nction of A
$\exp\left(\int_{\theta=0}^{2\pi} K^{*}A\right)$
This is a gauge invariant measure:
$A \longrightarrow X^{-1}dX + A$
then,
$\exp\left(\int_{\theta=0}^{2\pi} K^{\gamma}A\right)$
$\longrightarrow \exp\left(\int_{\theta=0}^{2\pi} K^{\gamma}A\right)\exp\left(\int_{\theta=0}^{2\pi} X(\theta)^{-1}\partial_\theta X(\theta)\,d\theta\right)$

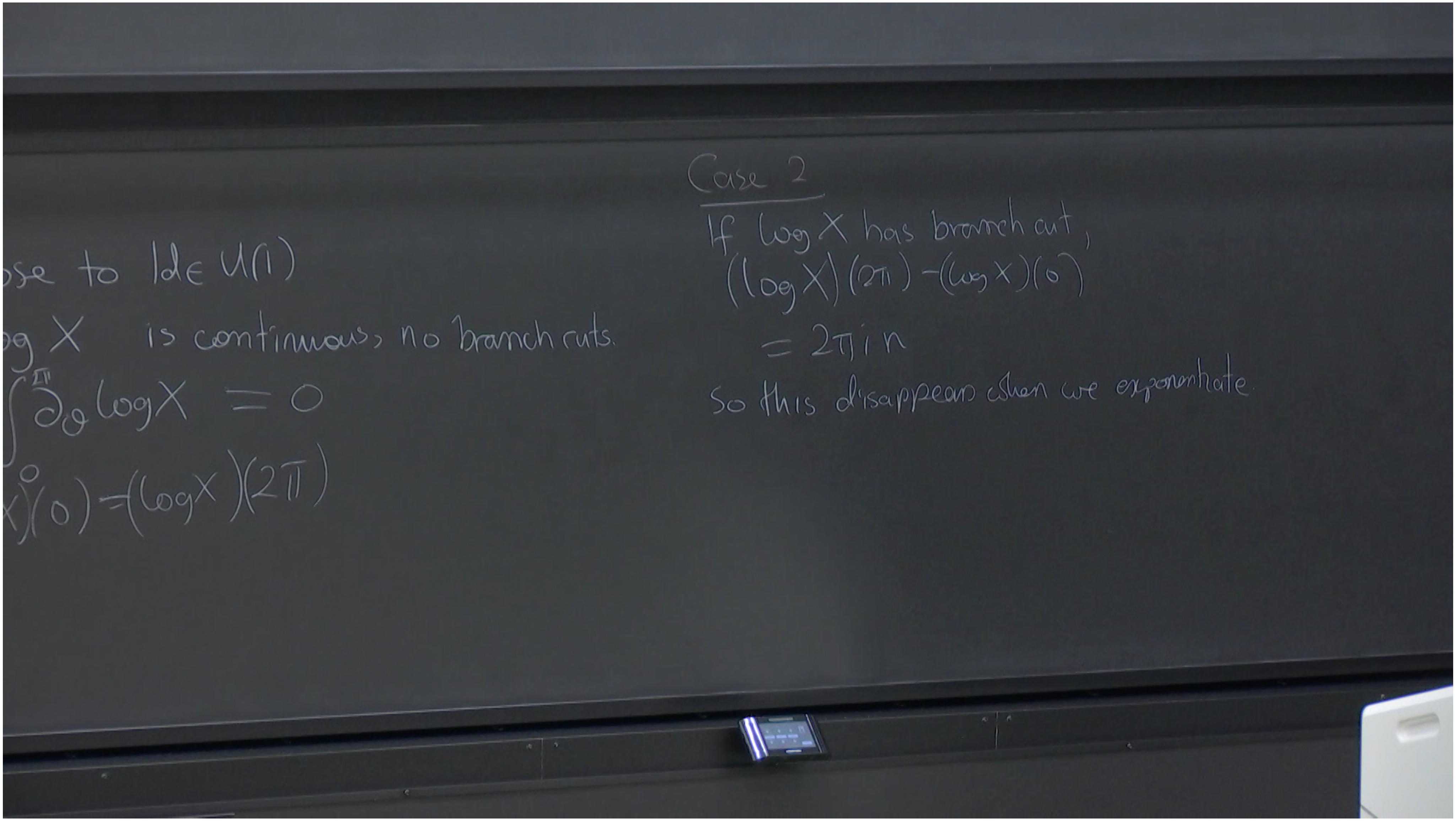
$X(\theta) \in U(1)$

Case 1 X close to $\mathrm{Id} \in U(1)$

Then, $\log X$ is continuous, no branch cuts.

$$\int X^{-1} \partial_\theta X = \int_0^{2\pi} \partial_\theta \log X = 0$$

as $(\log X)(0) = (\log X)(2\pi)$


ose to Id ∈ U(1)
og X is continuous, no branch cuts.
∫ ∂θ log X = 0
X)(0) = (log X)(2π)
Case 2
If log X has branch cut,
(log X)(2π) − (log X)(0)
= 2πin
So this disappears when we exponentiate.

Alternative

Choose $\Sigma \subseteq M$ of dim^n 2, with

$\partial \Sigma = K$

Then, the Wilson line is $\exp\left(\int_\Sigma F(A)\right)$

(special to $U(1)$)

As, $\int_{\Sigma} dA = \int_{\partial\Sigma} A$

If we vary Σ a little, to Σ', with same boundary, then

Σ:

Σ':

$$\int_{\Sigma} F - \int_{\tilde{\Sigma}} F = \int_{\Sigma \cup \tilde{\Sigma}} F$$

$\Sigma \cup \tilde{\Sigma}$ = boundary of a 3d region in M,

$dF = 0$,

$$\int_{\Sigma \cup \tilde{\Sigma}} F = \int_{3d\ \text{region}} dF = 0$$

For YM, there are <u>two</u> natural loop operators!

If $K \subseteq M^4$,

$K = \partial \Sigma$,

- Wilson loop $= \exp\left(\int_\Sigma F(A)\right)$
- t'Hooft loop $= \exp\left(\int_\Sigma * F(A)\right)$

Because $dF(A) = 0$

$d * F(A) = 0$

both of these measuremen

do not change if we

e <u>two</u> natural loop operators!

M^4,

$$\ldots p\left(\int_{\Sigma} F(A)\right)$$

$$\ldots p\left(\int_{\Sigma} {*F(A)}\right)$$

Because $dF(A) = 0$

$d{*F(A)} = 0$

both of these measurements do not change if we vary Σ.

If $G = U(1)$

Then, define the <u>Wilson loop</u> operator to be the function of A

This is a gauge invariant measure:

$A \rightarrow X^{-1} dX + A$

then, $\left(\int^{2\pi} \ldots A\right)$

If $K_1, K_2 \subseteq \mathbb{R}^3$

$G = \mathbb{R}$ (the same as $U(1)$, but only allow gauge transf. near Id)

$\int_{K_1} A$ is gauge invariant.

Qn If $K_1, K_2 \subseteq \mathbb{R}^3$ are two knots which don't touch, what is

t only allow gauge
t)

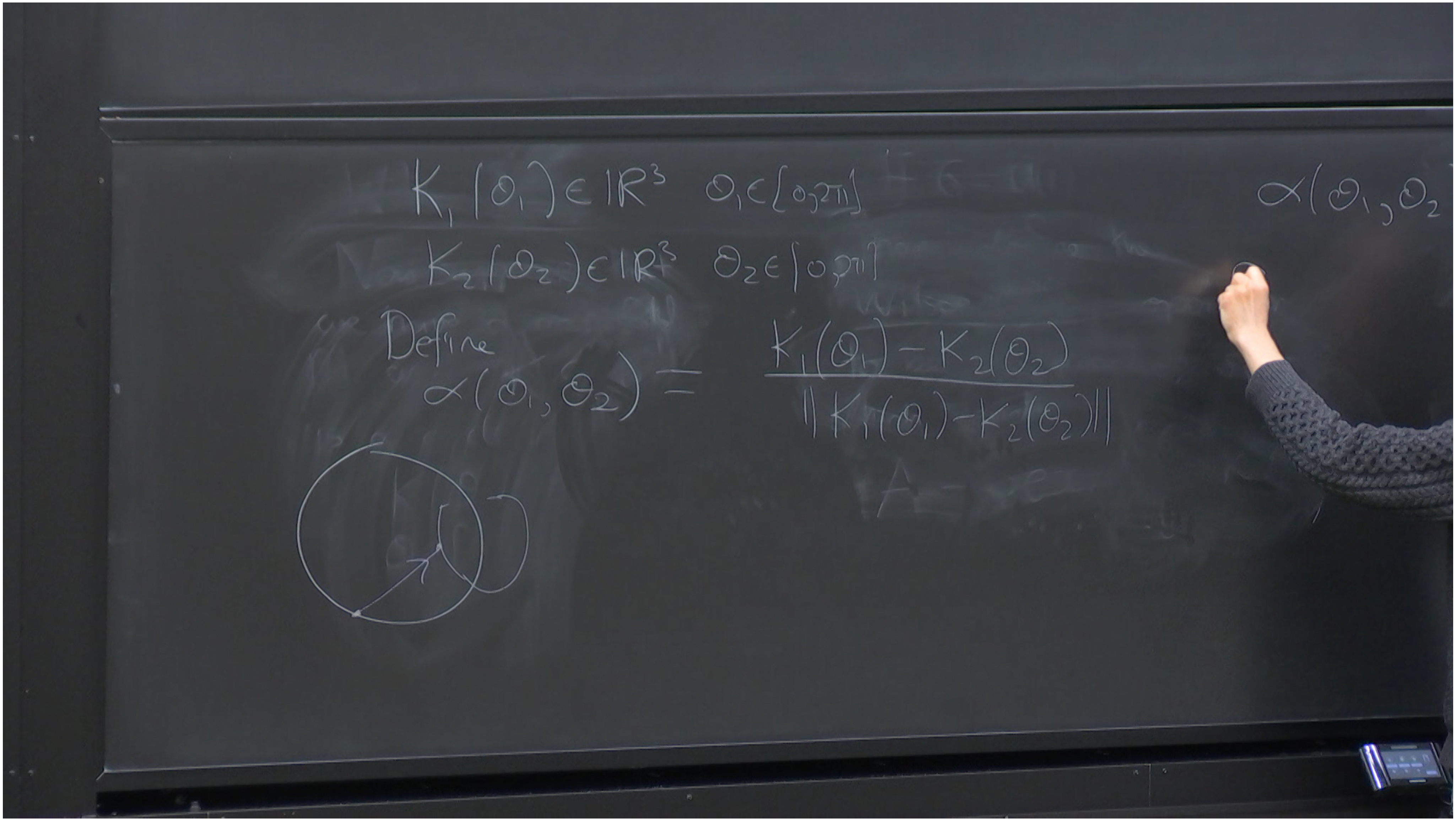
ots which

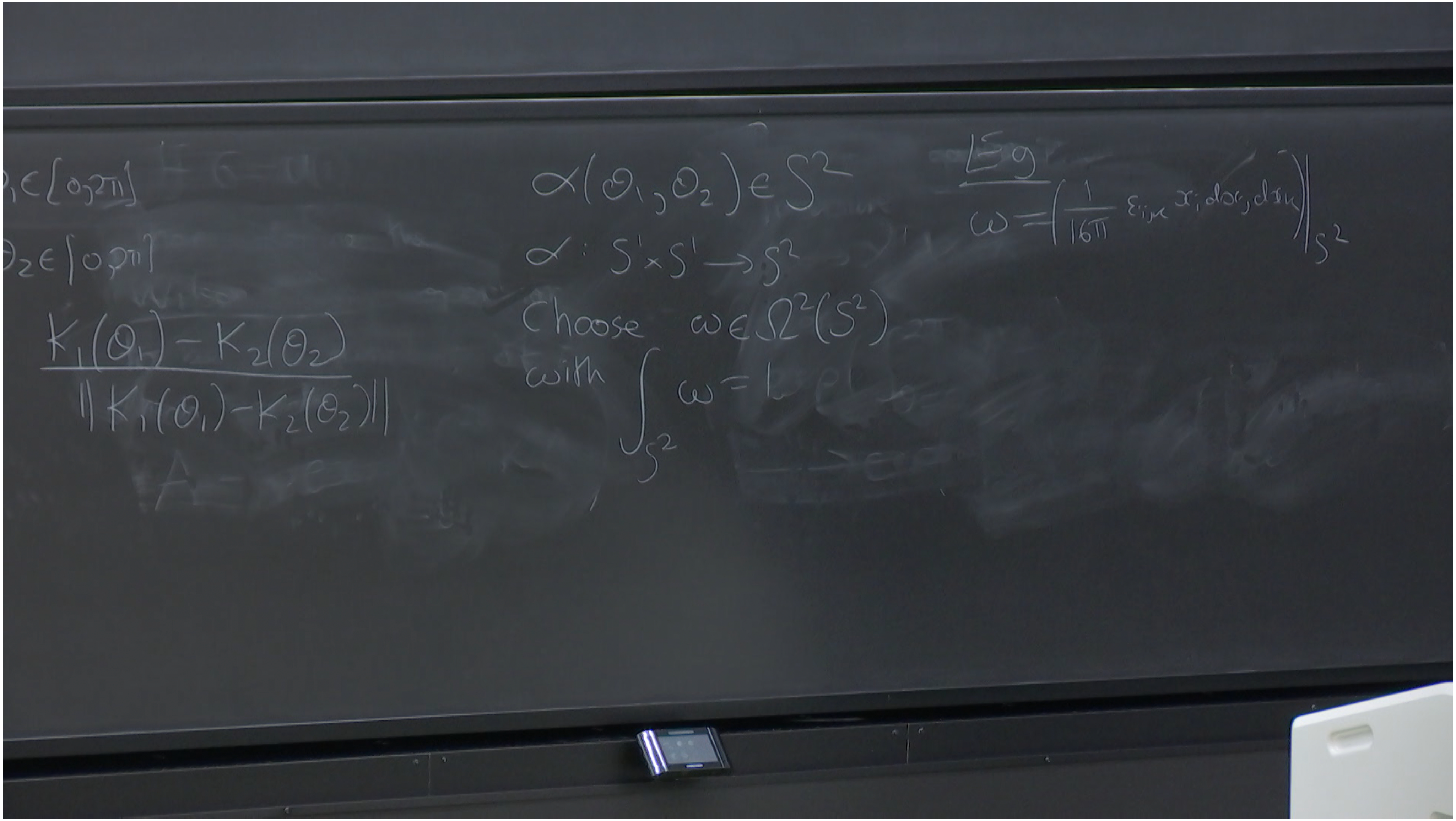
$$\left\langle \int_{K_1} A , \int_{K_2} A \right\rangle = \int_{A} e^{\int CS(A)} \int_{K_1} A \int_{K_2} A$$

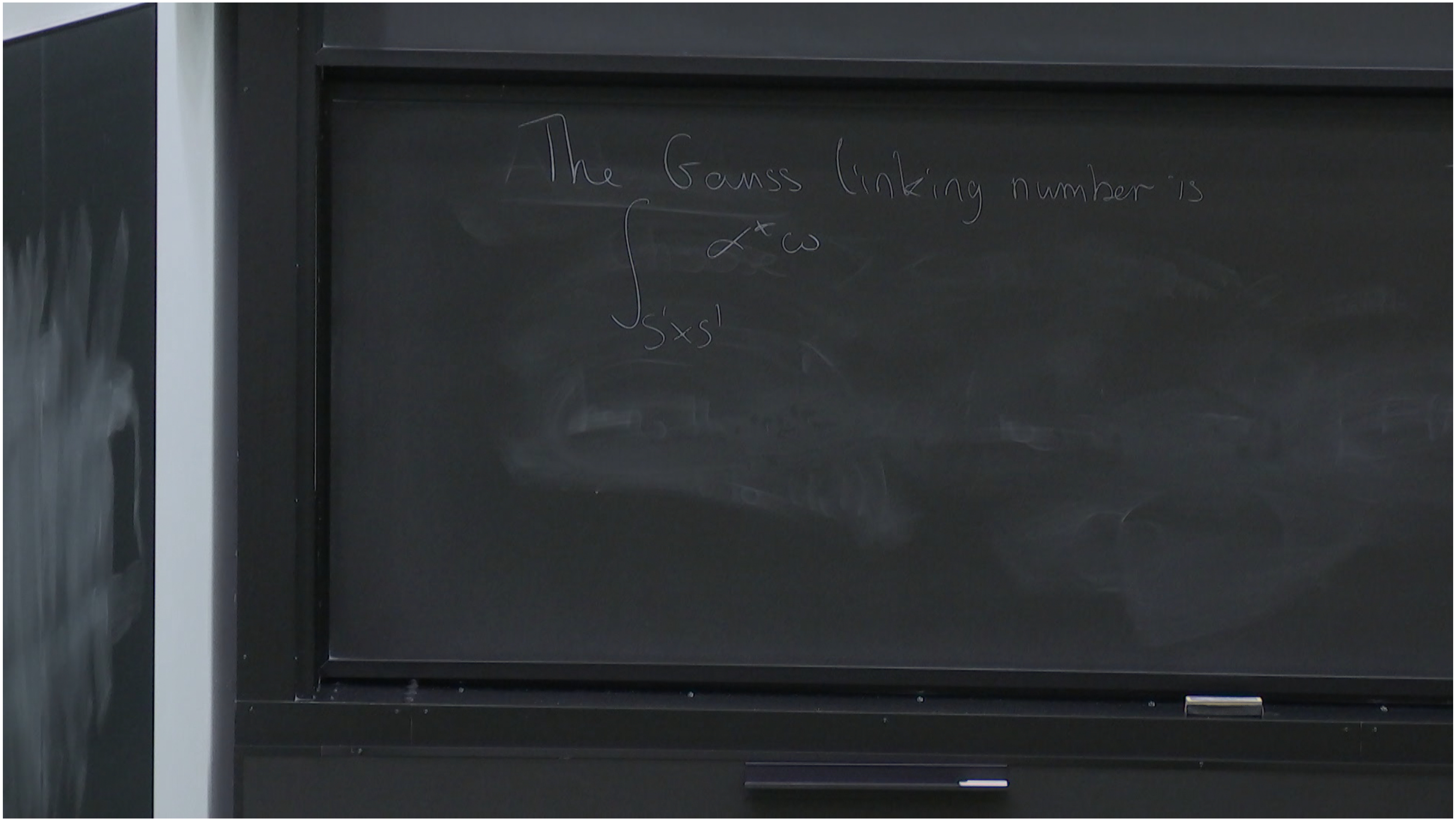
Theorem

This is the Gauss linking number

This is a gauge invariant


$K_1(\theta_1) \in \mathbb{R}^3 \quad \theta_1 \in [0, 2\pi]$
$K_2(\theta_2) \in \mathbb{R}^3 \quad \theta_2 \in [0, 2\pi]$
Define
$\alpha(\theta_1, \theta_2) = \dfrac{K_1(\theta_1) - K_2(\theta_2)}{\|K_1(\theta_1) - K_2(\theta_2)\|}$
$\alpha(\theta_1, \theta_2$


$\theta_1 \in [0, 2\pi]$
$\theta_2 \in [0, 2\pi]$
$\frac{K_1(\theta_1) - K_2(\theta_2)}{\|K_1(\theta_1) - K_2(\theta_2)\|}$
$\alpha(\theta_1, \theta_2) \in S^2$
$\alpha : S^1 \times S^1 \to S^2$
Choose $\omega \in \Omega^2(S^2)$
with $\int_{S^2} \omega = 1$
E.g.
$\omega = \left(\frac{1}{16\pi} \varepsilon_{ijk} x_i dx_j dx_k \right) \Big|_{S^2}$


The Gauss linking number is
$\int_{S^1 \times S^1} \alpha^* \omega$

Key points

1) This does not change if $\omega \longmapsto \omega + d\chi$

$\chi \in \Omega^1(S^2)$

$$\int_{S^1\times S^1} \alpha \wedge d\chi = \int_{S^1\times S^1} d\alpha \wedge \chi = 0$$ as $S^1\times S^1$ has no boundary

CS

NO local measurements!

2) If we vary the knots,

say $K_1(t,\theta_1)$ t ranges $0 \to 1$

$K_1(t,\theta_1)$ never touches K_2,

then the linking number does not change.

$$\alpha(t,\theta_1,\theta_2) : [0,1] \times S^1 \times S^1 \longrightarrow S^2$$

$$t \quad \theta_1 \quad \theta_2$$

$$\int_{[0,1]\times S^1 \times S^1} d\alpha^* \omega = \int_{[0,1]\times S^1\times S^1} \alpha^* d\omega = 0$$

So,

$$\int_{0\times S^1\times S^1} \alpha^* \omega = \int_{1\times S^1\times S^1} \alpha^* \omega$$

So that the linking number at $t=0$ and $t=1$ is the same.

Theorem

('t Hooft)

$K_1, K_2 \in \mathbb{R}^3$

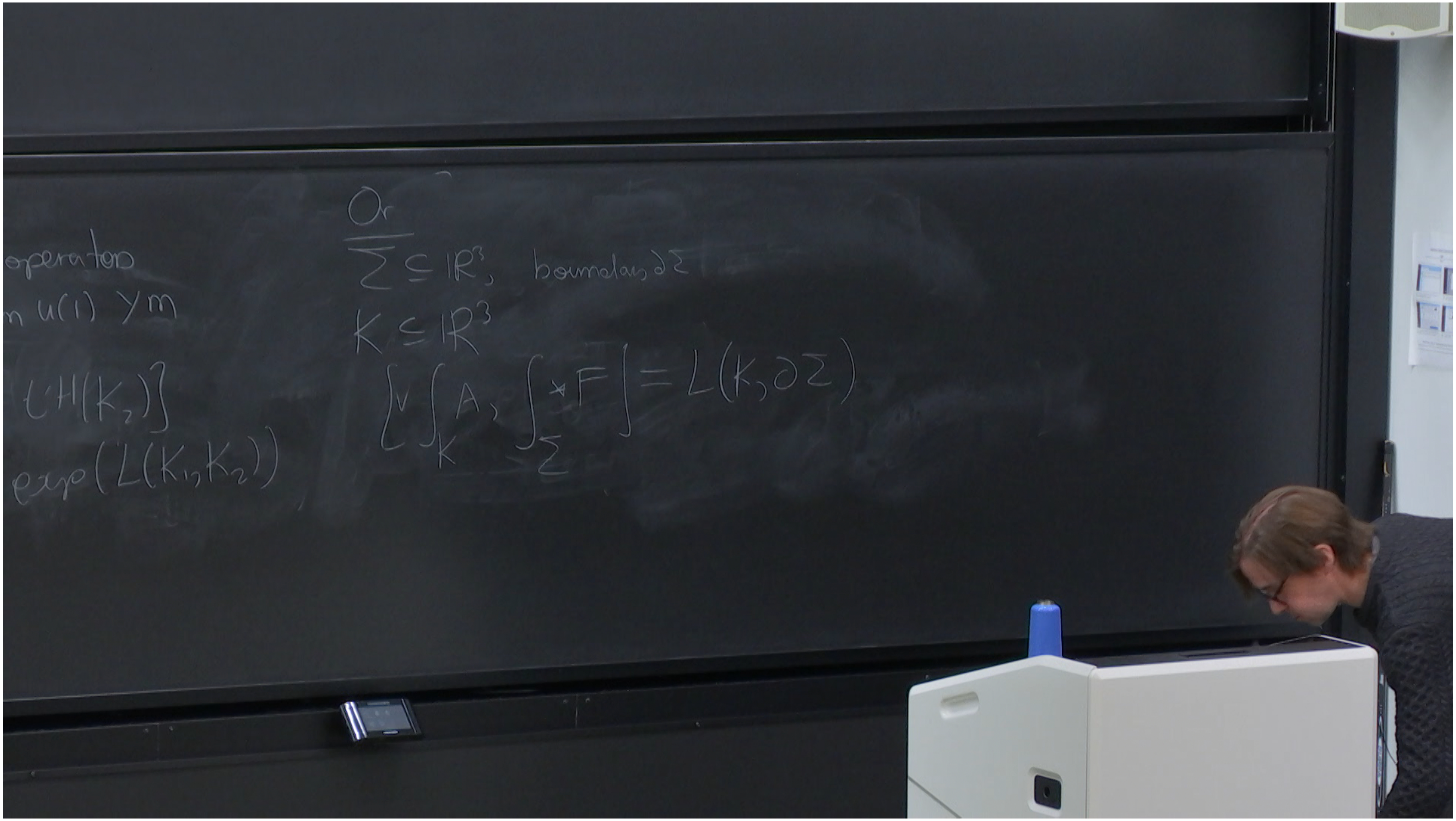
$W(K_1)$

$t'H(K_2)$

Viewed as operators in quantum $U(1)$ YM

$$[W(K_1), t'H(K_2)] = \exp(L(K_1, K_2))$$

Or


operators
u(1) ym
Or
$\Sigma \subseteq \mathbb{R}^3$, boundary $\partial\Sigma$
$K \subseteq \mathbb{R}^3$
$\left[\int_K A, \int_\Sigma *F\right] = L(K, \partial\Sigma)$
$\exp(L(K_1, K_2))$