

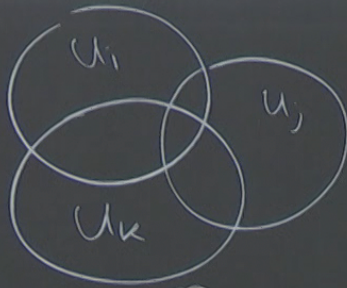
Title: PSI 2019/2020 - Chern-Simons Theory Part 1 - Lecture 3

Speakers: Kevin Costello

Collection: PSI 2019/2020 - Chern-Simons Theory (Part 1)

Date: February 14, 2020 - 10:00 AM

URL: <http://pirsa.org/20020026>

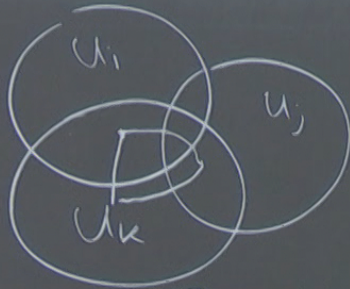


A_i
 A_j
 A_k

On $U_i \cap U_j$, A_i, A_j are related by X_{ij} , gauge transform.

$$A_i \xrightarrow{X_{ik}} A_k$$

$$A_j \xrightarrow{X_{jk}} A_k$$



A_i
 A_j
 A_k

On $U_i \cap U_j$, A_i, A_j are related by X_{ij} , gauge transform.
 Need $X_{ik} = X_{ij} X_{jk}$

$A_i \xrightarrow{X_{ik}} A_k$
 $A_j \xrightarrow{X_{jk}} A_k$

E=

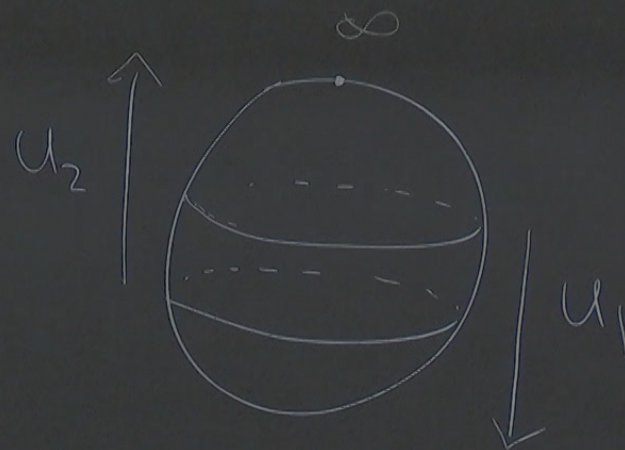
Example

$$M = S^2 = \mathbb{C} \cup \{\infty\}$$

$$U_1 = \{z, |z| < 2\}$$

$$U_2 = \{z, |z| > 1\}$$

$$U_1 \cap U_2 = \{z, 1 < |z| < 2\}$$



$$G = U(1)$$

A_i have no Lie alg. inside

$X_{ij} \in$ complex numbers of modulus 1

$$A_1 = 0$$

$$X_2(r, \theta) = e^{2\pi i n \theta}$$

$1 < r < 2$

On the region $1 < r < 2$

$$A_2 = \int_{12}^{-1} dx_{12} \\ = 2\pi i n d\theta$$

To define A_2 $r > 2$

choose $f(r)$, with

$$f(r) = 1 \quad \text{if } 1 < r < 2$$

$$f(r) = 0 \quad r \gg 0$$

$$A_2 = f(r) 2\pi i n d\theta \\ \text{for } r > 1$$

Question (In general)

Can we find a new gauge so that A is globally defined?

Answer ($G = U(1)$)

Consider the curvature $F(A_i)$

Under a gauge transformation, $F(A_i) \rightarrow X_{ji}^{-1} F(A_i) X_{ji}$

$$F(A_i) = F(A_j) \text{ on } U_i \cap U_j \quad = F(A_i) \text{ for } U(1)$$

Global two-form.

If there was a new gauge, where $\tilde{A}_i = \tilde{A}_i$ on $U_i \cap U_j$,
then $F(A) = F(\tilde{A}) = d\tilde{A}$

$$\Rightarrow [F(A)] \in H^2(M) = 0$$

$$\Leftrightarrow \int_{\Sigma} F(A) \text{ over every closed 2-cycle vanishes.}$$

In S^2 example,

$$A_2 = f(r) 2\pi i \, d\theta$$

$r > 1$

$$A_1 = 0 \quad r < 2$$

$$dA_1 = 0$$

$$F = dA_2 = -f'(r) 2\pi i \, dr \, d\theta$$

$$\int_{S^2} F = \left(\begin{matrix} = 0 & \text{if } r < 2 \end{matrix} \right) \left(\int_{\theta=0}^{2\pi} d\theta \right) \left(\int_{r=1}^{\infty} f'(r) \, dr \right)$$

$$dA_1 = 0$$

$$F = dA_2 = -f'(r) 2\pi i n dr$$

$$\int_{S^2} F = -2\pi i n \left(\int_{\theta=0}^{2\pi} d\theta \right) \left(\int_{r=2}^{\infty} f'(r) \right) = +2\pi i n$$

$$f=1 \text{ when } r=2, f=0 \text{ when } r=\infty$$

In general, if we have A_i on each patch
can we find a X_{ij} gauge trans.

Can we find X_i so that

$$\tilde{A}_i = X_i^{-1} dX_i + X_i^{-1} A_i X_i$$

where $\tilde{A}_i = \tilde{A}_j$ on $U_i \cap U_j$?

Answer

If and only if

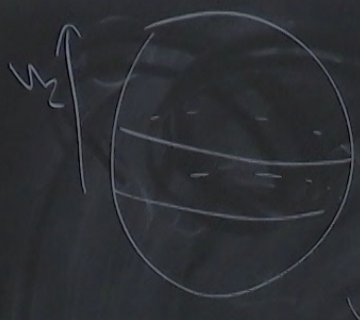
we can find X_i so that

$$X_{ij} = X_i X_j^{-1}$$

If this is true, then the equation

$$A_j = (X_j X_i^{-1}) d(X_i X_j^{-1}) + X_j X_i^{-1} A_i X_i X_j^{-1}$$

$$\Rightarrow X_j^{-1} dX_j + X_j^{-1} A_j X_j = X_i^{-1} dX_i + X_i^{-1} A_i X_i$$



$$X_{12}(r, \theta) = e^{2\pi i n \theta}$$

Can we find $X_1: U_1 \rightarrow S^1$

$$X_2: U_2 \rightarrow S^1$$

so $X_{12} = X_1 X_2^{-1} ?$

Intuition

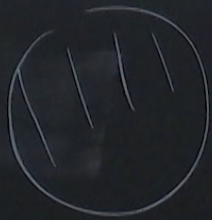
$$X_{12}(r, \theta)$$

θ ranges from 0 to 1
it winds n times around 0

Since $U_1 =$ a disc

Any $X_1: U_1 \rightarrow S^1$

X_1 can not wind around 0
at all!



Winding number

$$= \int_{\partial U_1} \frac{dx_1}{x_1} = \int_{U_1} d\left(\frac{dx_1}{x_1}\right) = 0$$

Chern-Simons Lagrangian

Take gauge group $U(n)$

If we have a gauge field,
then on each patch

have $F(A_i) = F(A_i)^a M_a$

a Lie alg index
 $M_a \in u(n)$

This is gauge covariant:

On $U_i \cap U_j$

$$F(A_j) = X_{ij}^{-1} F(A_i) X_{ij}$$

$$= \sum F(A_i)^a X_{ij}^{-1} m_a X_{ij}$$

We can build gauge INVARIANT quantities
by using traces.

$$\text{In particular, } \text{Tr}(F(A_i) \wedge F(A_i)) = \sum_{a,b} F(A_i)^a \wedge F(A_i)^b \text{Tr}(m_a m_b)$$

Easy to check, that
this is gauge invariant

If M is a 4-manifold
 $\text{Tr } F \wedge F$ (a 4-form)
is a Lagrangian,

can consider

$$\int_M \text{Tr} F \wedge F$$

(Gauge invariance $\Rightarrow$ the Lagrangian agrees on overlapping patches)

If M is a closed 4-manifold,

$\int_M \text{Tr} F \wedge F$ does not change if we vary the gauge field by small amounts

In the Abelian case,

$$\text{Vary } A \rightarrow A + \delta A$$

$$F(A) = dA$$

$$\delta dA \wedge dA = d((\delta A) \wedge dA)$$

= a total derivative

$$\int_M dx^1 dx^2 dx^3 dx^4 \text{Tr}(m_a m_b) F^{a_{12}} F^{b_{34}} \epsilon_{11'22'33'44'}$$

→ the Lagrangian agrees on overlapping patches

used 4-manifolds

not change if we vary the gauge field by small amounts

Further, $\int \text{Tr} F \wedge F$

is quantized.

It takes values in

$$4\pi^2 \mathbb{Z}$$

CLEVER way of writing CS action:

If N^3 is a 3-manifold

M^4 has boundary N^3

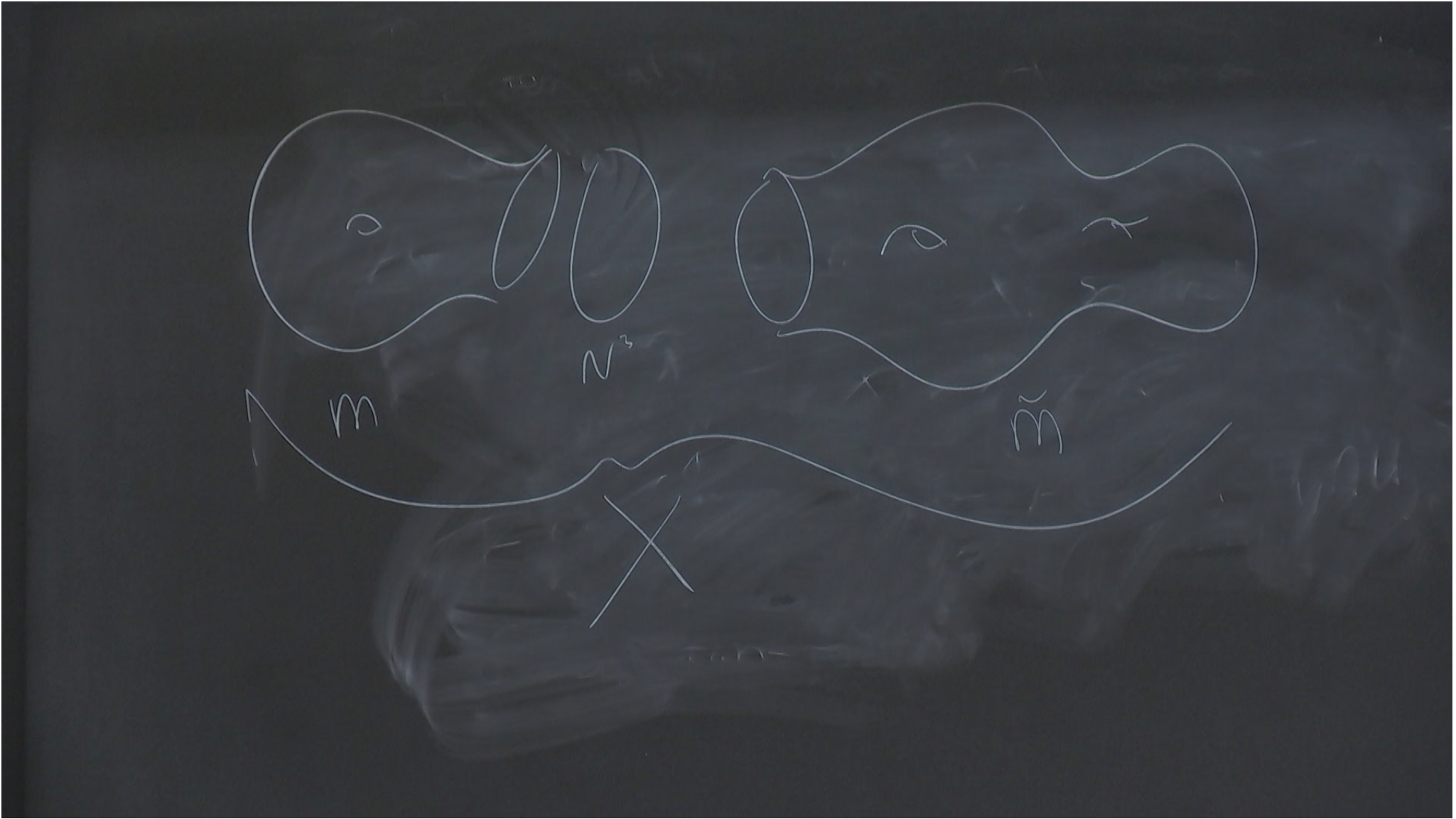
Choose a gauge field on N^3 , extend it to M^4

$$CS(A) = \int_{M^4} \text{Tr} F(A) \wedge F(A)$$

If $\tilde{M}^4$ also has boundary N^3

$$\tilde{M}^4 \cup_{N^3} M^4 = X$$

No boundary



Further, $\int_M \text{Tr} F \wedge F$

is quantized.

It takes values in

$$8\pi^2 \mathbb{Z}$$

LEVER way of writing CS action:

If N^3 is a 3-manifold

M^4 has boundary N^3

Choose a gauge field on
 N^3 , extend it to M^4

$$CS(A) = \int_{M^4} \text{Tr} F(A) \wedge F(A)$$

If $\tilde{M}^4$ also has boundary N^3

$$\tilde{M}^4 \cup_{N^3} M^4 = X$$

No boundary

$$\int_X \text{Tr} F(A) \wedge F(A)$$

$$= \int_{M^4} \text{Tr} F(A) \wedge F(A) - \int_{\tilde{M}^4} \text{Tr} F(A) \wedge F(A)$$

$$\int_X \text{Tr} F \wedge F \text{ is quantized.}$$

$$e^{iK/4\pi \int_{M^4} \text{Tr} F \wedge F} \text{ is independent of choice of } M^4.$$

Conclude

CS Lagrangian defined as $\int_{M^4} \text{Tr} F \wedge F$

makes sense, if we study $\frac{iK}{4\pi} \text{CS}$

$$\int_X e^{\frac{iK}{4\pi} \text{CS}}$$

where $K \in \mathbb{Z}$
 $K = \text{level}$