Title: PSI 2019/2020 - Chern-Simons Theory Part 1 - Lecture 2

Speakers: Kevin Costello

Collection: PSI 2019/2020 - Chern-Simons Theory (Part 1)

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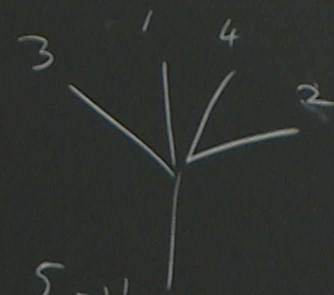
URL: http://pirsa.org/20020025

$\star_\hbar$

$a \star_\hbar b$

$= ab + \hbar\{a,b\}$

$+ \hbar^2 a \circ_2 b + \dots$



A is a assoc. alg over $\mathbb{C}[[\hbar]]$

mod $\hbar$ is commutative

Operad over $\mathbb{C}[[\hbar]]$

Gen. by $\mu(a,b)$ μ^{00}

$\lim_{\hbar \to 0} \frac{\mu - \mu^{00}}{\hbar}$ is divisible by $\hbar$

$\mu_5 : 5 \to 1$

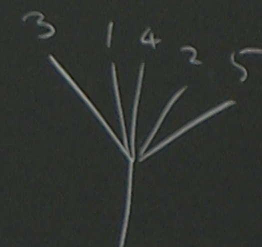
$\mu_3 : 3 \to 1$

$\mu_3 \circ_2 \mu_5$ $\quad i = 1 .. 5$

$\mu_5(a_1, \mu_2(a_2, a_3, a_4), \dots a_5)$

$\mathrm{Ass}(n,1)$

$= S_n$



A, $a_1, \dots, a_5 \in A$

$a_3 a_1 a_4 a_2 a_5$

Integration

$\omega \in \Omega^n(M)$

M an n-dim manifold

Locally, $\omega = f(x)\,dx_1 \wedge \ldots \wedge dx_n$

In other coordinates,

$\tilde{x}$,

$$\omega = \underline{\mathrm{Jac}(\quad)}\, f(\tilde{x})\, d\tilde{x}_1 \wedge \ldots \wedge d\tilde{x}_n$$

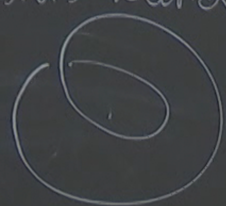
$$\det\left(\frac{\partial x_j}{\partial \tilde{x}_i}\right)$$

Def^n

An orientation on a manifold is a collection of local coordinate systems so that $\det \mathrm{Jac} > 0$ when we change from one coordinate to another.

Example

Möbius band



does not admit an orientation

If we have a frame $e_{1,2}$

If $f_i = A_i^j e_i$

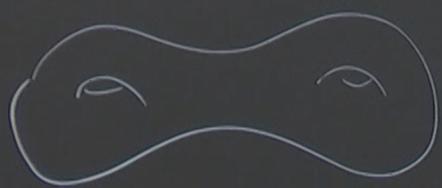
If we have a frame $e_1, e_2, \ldots, e_n$ locally, then locally it describes an orientation,

If $f_i = A_i^j e_j$ is another frame, it gives same orientation if $\det A > 0$.

In dim^n 2,

frames e_1, e_2 and e_2, e_1 give opposite orientations

1) Cut M into n-d

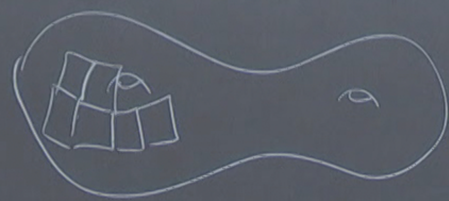


Let M be an oriented manifold of dimension n.

$\omega \in \Omega^n(M)$

Then, we define $\int_M \omega$ by:

1) Cut M into n-d cubes



iented manifold

n.

ω by.

2) On each cube choose coordinates

x_i, range from 0,1

so that on interfaces between cubes,

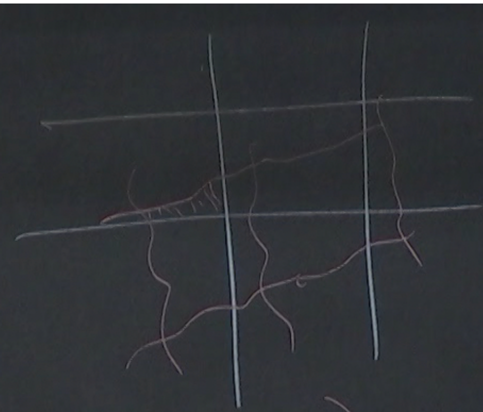
$\det \mathrm{Jac} > 0$

$$\int_M \omega = \sum_{\text{cubes}} \int\int\cdots\int_{x_i=0} \omega$$

here, $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$

Key Point This is independent of how we chop M into cubes.

or



of
o cubes.

$$\left.\begin{aligned}&\sum \int_{\text{cubes}} \\ &\sum \int_{\text{cubes}}\end{aligned}\right\} = \sum \int_{\substack{\text{smaller cubes} \\ \text{which fit in both} \\ \text{white and pink}}}$$

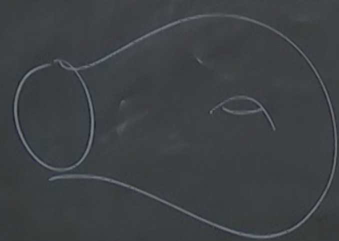
$$= \sum \int_{\substack{\text{smaller}\\ \text{cubes}\\ \text{which fit in both}\\ \text{white and pink}}}$$

For each cube, answer is independent of coordinates, so

$$\sum \int_{\text{cubes}} \omega = \sum \int_{\text{cubes}} \omega$$

Stokes Theorem

If M is an oriented n-d manifold with boundary



i.e. some coordinate patches $x_1, \dots, x_n$ are restricted by $x_1 \geq 0$

If $\omega \in \Omega^{n-1} M$

Then,

$$\int_M d\omega = \int_{\partial M} \iota^* \omega$$

≥ 0 where $\iota : \partial M \hookrightarrow M$ is inclusion.

How is ∂M oriented?

we can choose a frame

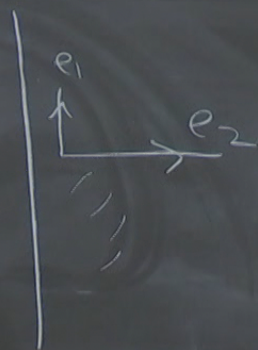
$e_1, \ldots, e_n$ near the boundary

so that

$e_1, \ldots, e_{n-1}$ are parallel

e_n is inward-pointing

he boundary

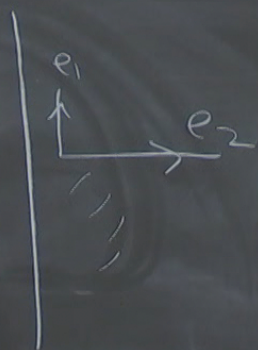


pointing

e
the boundary

del

-pointing

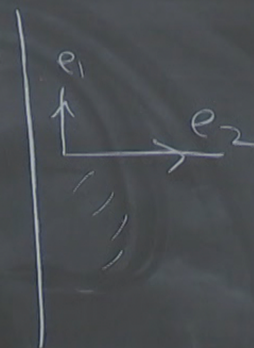


If $e_1, \ldots, e_n$ is an oriented frame

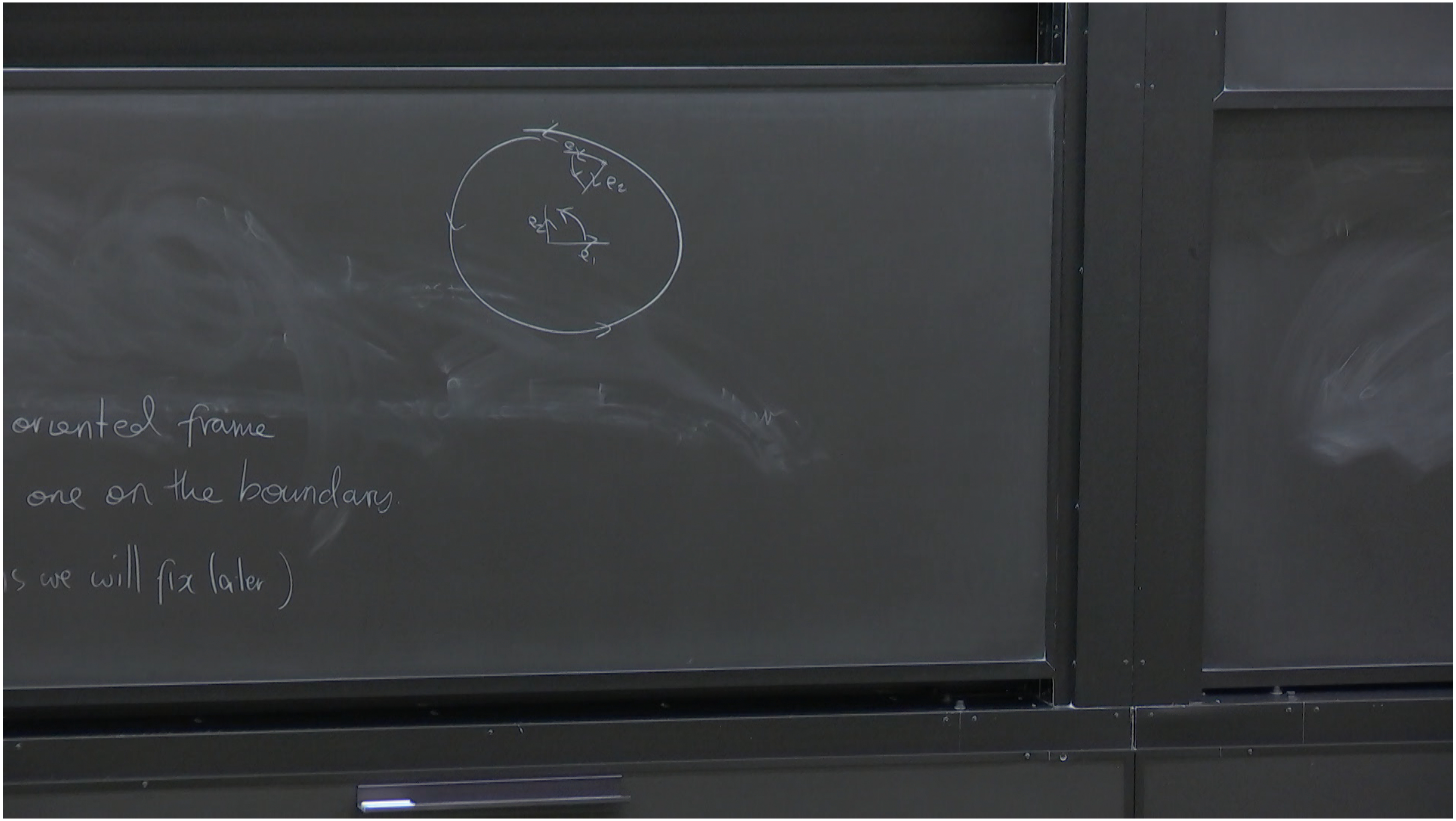
Then $e_1, \ldots, e_{n-1}$ is one on the boundary.

hoose a frame
near the boundary

are parallel
is inward-pointing



If $e_1, \dots, e_n$ is an oriented frame
Then $e_1, \dots, e_{n-1}$ is one on the boundary.
(Some annoying signs we will fix later)


e_2
e_2
e_1
oriented frame
one on the boundary.
s we will fix later)

Proof of Stokes Theorem

- Divide M into cubes.

$$\int_M d\omega = \sum_{\text{cubes}} \int_{x_1 \ldots x_n = 0}^{1} d\omega$$

On

$\omega =$

On each cube,

$$\omega = \sum f_{i_1 \ldots i_{n-1}} \, dx_{i_1} \wedge \ldots \wedge dx_{i_n}$$

$$d\omega = \sum \frac{\partial f_{i_1 \ldots i_{n-1}}}{\partial x_j} \, dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_n}$$

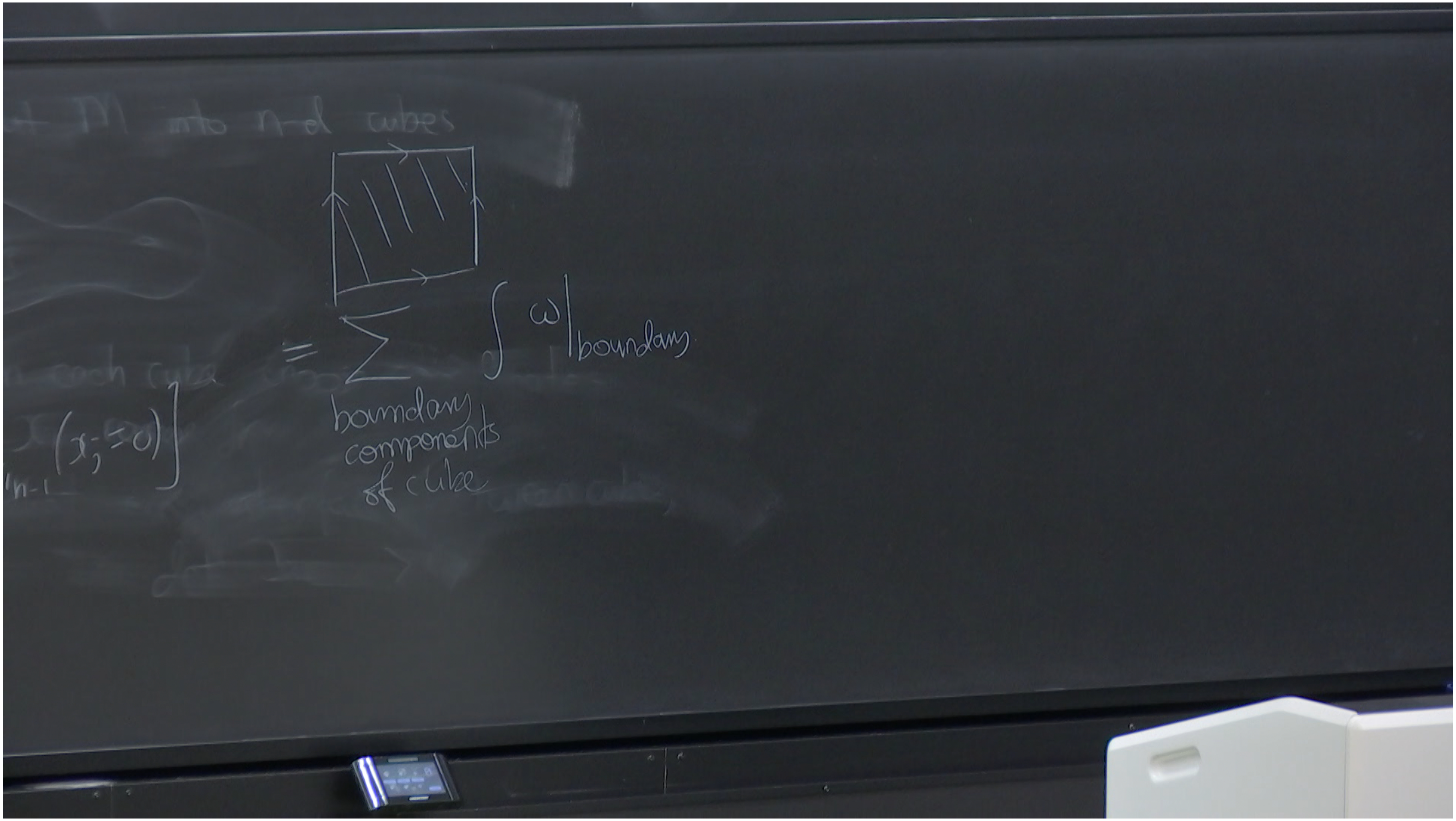
$$= \sum \varepsilon_{j i_1 \ldots i_{n-1}} \frac{\partial f_{i_1 \ldots i_{n-1}}}{\partial x_j} \, dx_1 \wedge \ldots \wedge dx_n$$

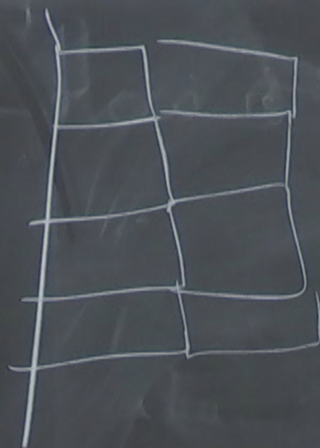
1) Cut M into n-d cubes

$$\int_{x_1,\ldots,x_n=0}^{1} \varepsilon_{j i_1 \ldots i_{n-1}} \frac{\partial f_{i_1 \ldots i_{n-1}}}{\partial x_j} dx_1 \ldots dx_n$$

Integrate over x_j variable

Gets $\sum_j \int_{x_i, i \neq j} \varepsilon_{j i_1 \ldots i_{n-1}} \left[f_{i_1 \ldots i_{n-1}}(x_j = 1) - f_{i_1 \ldots i_{n-1}}(x_j = 0) \right]$


into n-d cubes
$= \sum \int \omega|_{boundary}$
boundary components of cube

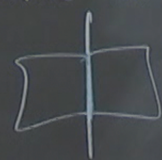


$$\sum_{\text{cubes}} \int d\omega = \sum \int \omega$$

contributions from boundaries of each cube

Key point

Interface between two cubes



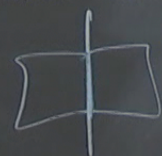
appears twice
with, opposite
sign!

ω

ons
lanes
ube

Key point

Interface between two cubes



appears twice
with opposite
sign!

Except for those which live on ∂M

Conclude, $\displaystyle\int_M d\omega = \int_{\partial M} \omega$, If M is a manifold

A gauge field (locally on M)

for some Lie algebra $\mathfrak{g}$

is an expression

$$A = \sum A^a t_a$$

t_a is a basis of $\mathfrak{g}$

$A^a \in \Omega^1(M)$

$A \in \Omega^1(M) \otimes \mathfrak{g}$

m)

of g

$A \in \Omega^1(m) \otimes g$

$F(A) \in \Omega^2(m) \otimes g$

is

$$dA^a + \frac{1}{2} f^a_{bc} A^b \wedge A^c$$

f^a_{bc} are structure constants of g

Even for a non-trivial bundle, $F(A)$ makes sense

$$F(A) \in \Omega^2(M) \otimes \mathfrak{g}$$

<u>Bianchi Identity</u>

$$dF(A)^a + f^a_{\ bc} A^b \wedge F(A)^c = 0$$

We can form a 4-form

$$K_{ab} F(A)^a \wedge F(A)^b \quad (K = \text{Killing form})$$

$$\in \Omega^4(M)$$

If $X : M \longrightarrow G$

$$\begin{pmatrix} G = SU(n) \\ \mathfrak{g} = su(n) \end{pmatrix}$$

$A \in \Omega^1(M) \otimes su(n)$

Then, a gauge transformation

sends

$$A \longrightarrow X^{-1} dX + X^{-1} A X$$

In coordinates,

$$A = \sum dx_i \, t_a \, A^{ia}$$

$$t_a \in su(n)$$

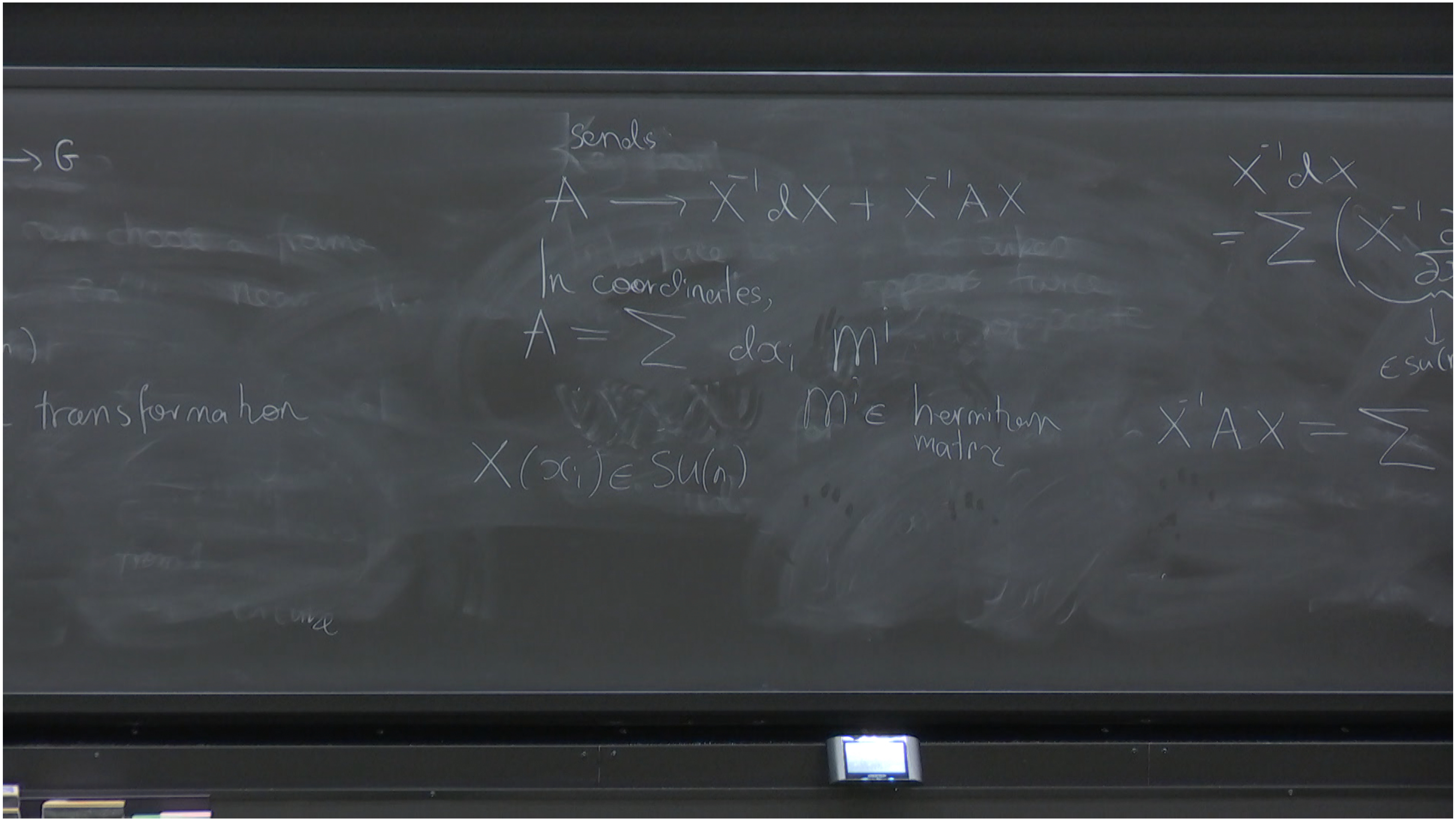
$$X(x_i) \in SU(n)$$

ansformation

$+ X^{-1}AX$

$$X^{-1}dX = \sum \Big(\underbrace{X^{-1}\frac{\partial}{\partial x_i}X}_{\in su(n)} \Big) dx_i$$

$t_a A^{ia}$


→ G
sends
$A \longrightarrow X^{-1} dX + X^{-1} A X$
In coordinates,
$A = \sum dx_i M^i$
$M^i \in$ hermitian matrix
$X(x_i) \in SU(n)$
transformation
$X^{-1} dX$
$= \sum (X^{-1}$
$\in su(n$
$X^{-1} A X = \sum$

$$\ldots X + X^{-1} A X$$

$$X^{-1} dX = \sum \underbrace{\left(X^{-1} \frac{\partial}{\partial x_i} X\right)}_{\in su(n)} dx_i$$

$i\, M^i$

$M^i \in$ hermitian matrix

$$X^{-1} A X = \sum dx_i\, X^{-1} M_i X$$

Gauge Field with non-trivial bundle

— M, manifold

U_i coordinate patches

$A_i \in \Omega^1(U_i) \otimes su(n)$

— If $U_i \cap U_j$ is not empty

on $U_i \cap U_j$ we have

$X_{ij} : U_i \cap U_j \longrightarrow SU(n)$

a gauge transformation

so that

$$A_j = X_{ij}^{-1} A_i X_{ij} + X_{ij}^{-1} d X_{ij}$$

- If U_i, U_j, U_k intersect

$$A_k = X_{ik}^{-1} (A_i + d) X_{ik}$$

$$= X_{jk}^{-1} (A_j + d) X_{jk} = X_{jk}^{-1} X_{ij}^{-1} (A_i + d) X_{ij} X_{jk}$$

- trivial bundle

d manifold

patches

su(n)

mpty

e

$$X_{ij} : U_i \cap U_j \longrightarrow su(n)$$

a gauge transformation

so that

$$A_j = X_{ij}^{-1} A_i X_{ij} + X_{ij}^{-1} d X_{ij}$$

- If U_i, U_j, U_k intersect

$$A_k = X_{ik}^{-1} (A_i + d) X_{ik}$$

$$= X_{jk}^{-1} (A_j + d) X_{jk} = X_{jk}^{-1} X_{ij}^{-1} (A_i + d) X_{ij} X$$

$\longrightarrow SU(n)$

unsformation

$A_i X_{ij} + X_{ij}^{-1} d X_{ij}$

intersect

$A_i + d) X_{ik}$

$+d) X_{jk} = X_{jk}^{-1} X_{ij}^{-1} (A_i + d) X_{ij} X_{jk}$

Need the

"cocycle condition"

$X_{ik} = X_{ij} X_{jk}$

$$\nabla_A^i = \partial_{x_i} + A^{ia} t_a$$

acts on n-component field

$$X : M \to SU(n)$$

permute the n components by a unitary matrix

A^X = A after gauge transformation

$$\nabla_{A^X} = X^{-1} \nabla_A X$$

Lecture 12:

$$\mathcal{L}_{\nu SM} = -\frac{1}{4} G^a_{\mu\nu} G_a^{\mu\nu} - \frac{1}{4} W^a_{\mu\nu} W_a^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

$$- (D_\mu h)^\dagger (D^\mu h) + m^2 (h^\dagger h) - \frac{\lambda}{4} (h^\dagger h)^2$$

$$+ i \left[\bar{q}_L \not{D} q_L + \bar{u}_R \not{D} u_R + \bar{d}_R \not{D} d_R + \bar{l}_L \not{D} l_L + \bar{\nu}_R \not{D} \nu_R + \bar{e}_R \not{D} e_R \right]$$

$$- \left[\bar{q}_L Y_d h d_R + \bar{q}_L Y_u \tilde{h} u_R + \bar{l}_L Y_e h e_R + \bar{l}_L Y_\nu \tilde{h} \nu_R + h.c. \right]$$

$$- \frac{1}{2} \left[\bar{\nu}_{R,c} M_m \nu_R + h.c. \right]$$

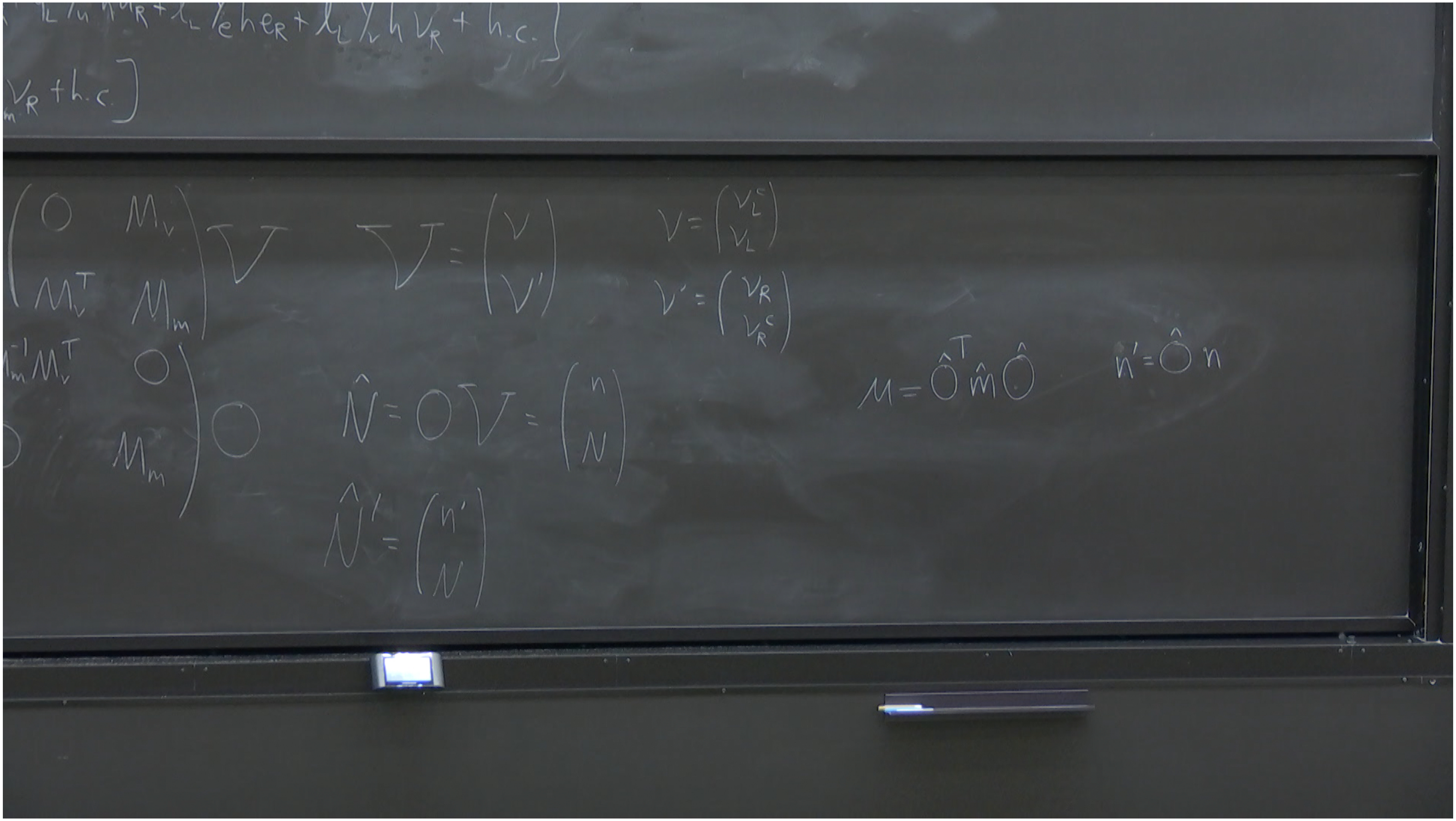
$$[\bar{q}_L Y_d h d_R + \bar{q}_L Y_u \tilde{h} u_R + \bar{l}_L Y_e h e_R + \bar{l}_L Y_\nu \tilde{h} \nu_R + h.c.]$$

$$-\frac{1}{2}\left[\bar{\nu}_{R,c} M_m \nu_R + h.c.\right]$$

$$L_{\nu SM} = \ldots \frac{1}{2} \bar{\mathcal{V}} \not{\partial} \mathcal{V} - \frac{1}{2} \bar{\mathcal{V}} \begin{pmatrix} 0 & M_\nu \\ M_\nu^T & M_m \end{pmatrix} \mathcal{V} \qquad \mathcal{V} = \begin{pmatrix} \nu \\ \nu' \end{pmatrix} \qquad \nu = \begin{pmatrix} \nu_L^c \\ \nu_L \end{pmatrix}$$

$$\nu' = \begin{pmatrix} \nu_R \\ \nu_R^c \end{pmatrix}$$

$$\begin{pmatrix} 0 & M_\nu \\ M_\nu^T & M_m \end{pmatrix} = O^T \begin{pmatrix} -M_\nu M_m^{-1} M_\nu^T & 0 \\ 0 & M_m \end{pmatrix} O \qquad \hat{N} = O \mathcal{V} = \begin{pmatrix} n \\ N \end{pmatrix}$$


$\mathcal{V} = \begin{pmatrix} \nu \\ \nu' \end{pmatrix}$
$\nu = \begin{pmatrix} \nu_L^c \\ \nu_L \end{pmatrix}$
$\nu' = \begin{pmatrix} \nu_R \\ \nu_R^c \end{pmatrix}$
$\hat{N} = O\,\mathcal{V} = \begin{pmatrix} n \\ N \end{pmatrix}$
$\hat{N}' = \begin{pmatrix} n' \\ N \end{pmatrix}$
$M = \hat{O}^T \hat{m} \hat{O}$
$n' = \hat{O}\, n$

$$-\frac{1}{2}\left[\bar{\nu}_{R,c} M_m \nu_R + h.c.\right]$$

$$L_{\nu SM} = \dots \frac{i}{2}\bar{\mathcal{V}}\not\partial\mathcal{V} - \frac{1}{2}\bar{\mathcal{V}}\begin{pmatrix} 0 & M_\nu \\ M_\nu^T & M_m \end{pmatrix}\mathcal{V} \qquad \mathcal{V} = \begin{pmatrix} \nu \\ \nu' \end{pmatrix} \qquad \nu = (\qquad \nu' = ($$

$$\begin{pmatrix} 0 & M_\nu \\ M_\nu^T & M_m \end{pmatrix} = O^T \begin{pmatrix} -M_\nu M_m^{-1} M_\nu^T & 0 \\ 0 & M_m \end{pmatrix} O \qquad \hat{N} = O\mathcal{V} = \begin{pmatrix} n \\ N \end{pmatrix}$$

$$\rightarrow -\frac{i}{2}\bar{\hat{N}}'\not\partial\hat{N}' - \frac{1}{2}\bar{\hat{N}}'\hat{M}'\hat{N}' \qquad \begin{pmatrix} \hat{m} & \\ & M_m \end{pmatrix} \qquad \hat{N}' = \begin{pmatrix} n' \\ N \end{pmatrix}$$

$$\mathcal{V} = \begin{pmatrix} \nu \\ \nu' \end{pmatrix}$$

$$\nu = \begin{pmatrix} \nu_L^c \\ \nu_L \end{pmatrix}$$

$$\nu' = \begin{pmatrix} \nu_R \\ \nu_R^c \end{pmatrix}$$

$$M = \hat{O}^T \hat{m} \hat{O} \qquad n' = \hat{O}\, n$$

$$\hat{N} = O\mathcal{V} = \begin{pmatrix} n \\ N \end{pmatrix}$$

$$\hat{N}' = \begin{pmatrix} n' \\ N \end{pmatrix}$$

$$\frac{M_\nu^2}{M_m} \qquad M_m$$

$$-\frac{1}{2}\left[\bar{\nu}_{R,c} M_m \nu_R + h.c.\right]$$

DM + leptogenesis

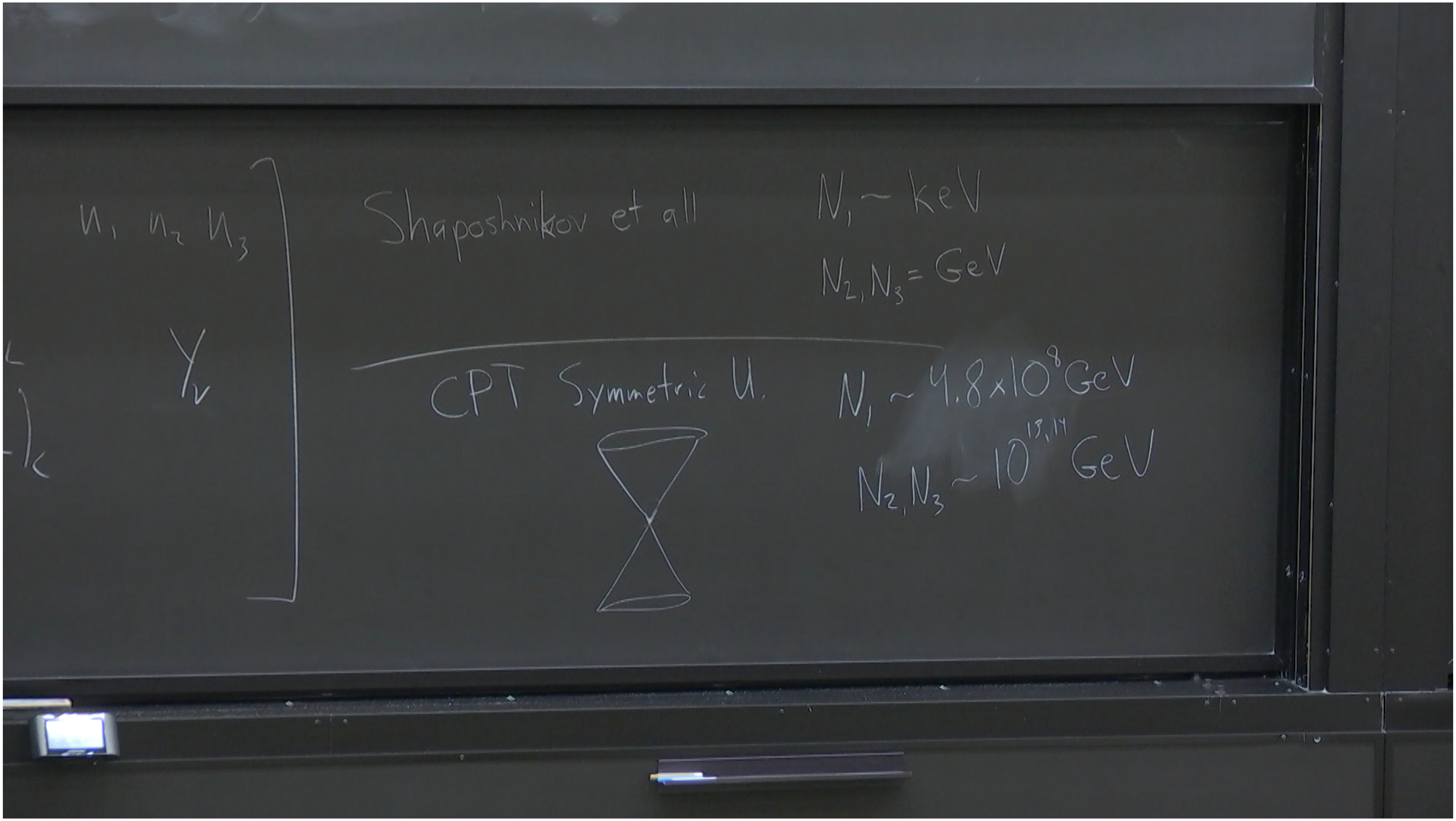
N_1 $\quad$ n_1 $(m_{n_1} = 0)$ $\quad$ $n_1 \; n_2 \; n_3$

N_2, N_3

$$N_{2,3} \to \begin{cases} h + \ell_L \\ (h + \ell_L)_c \end{cases}$$

Y_ν

T_{EW}

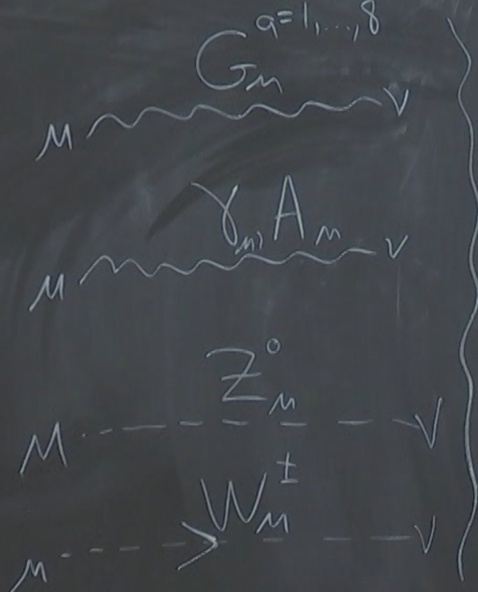

n_1 n_2 n_3
Shaposhnikov et all
$N_1 \sim$ keV
$N_2, N_3 =$ GeV
Y_ν
CPT Symmetric U.
$N_1 \sim 4.8 \times 10^8$ GeV
$N_2, N_3 \sim 10^{13,14}$ GeV

SM Feynman diagrams:

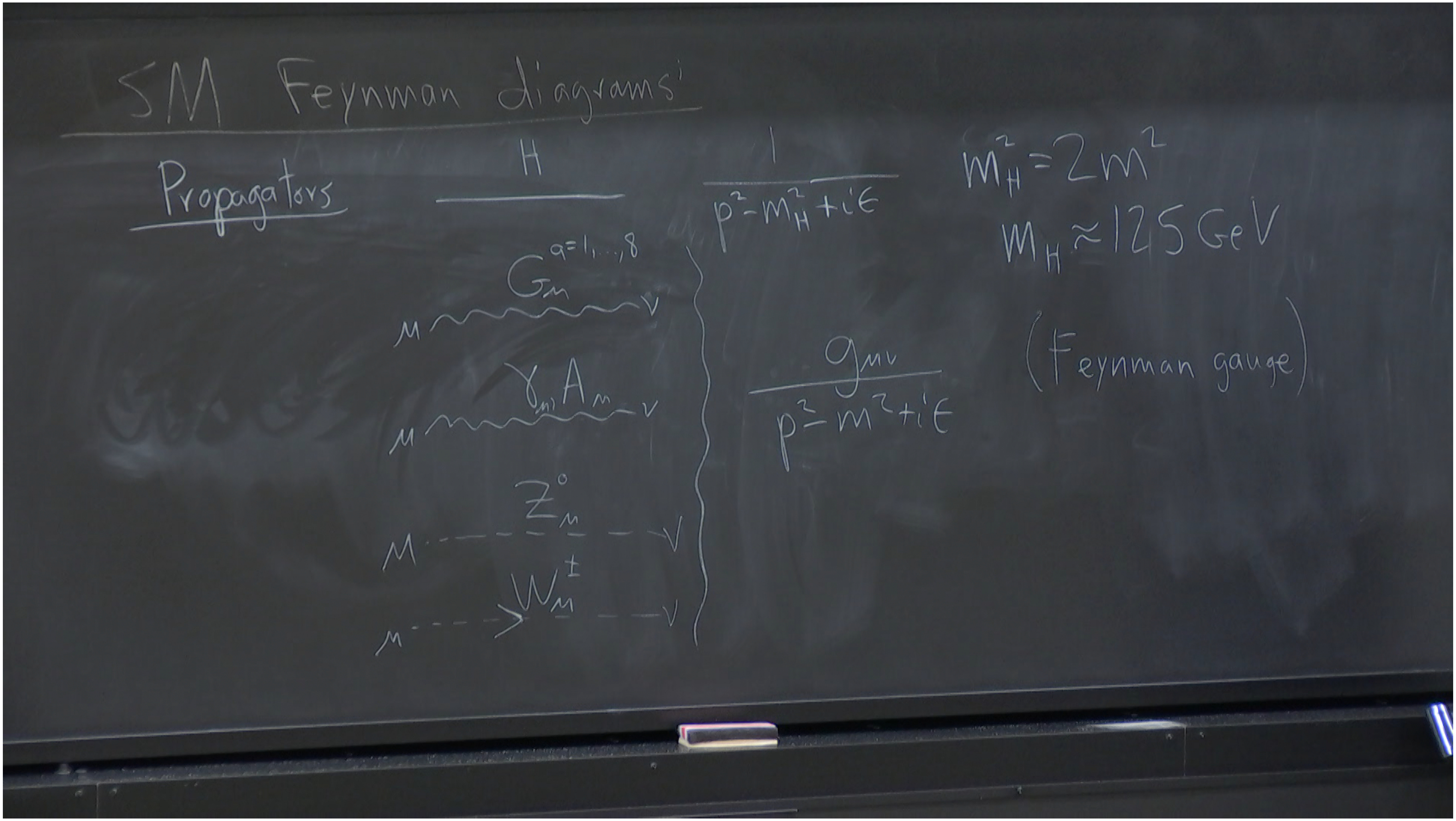
Propagators

H

$$\frac{1}{p^2 - m_H^2 + i\epsilon} \qquad m_H^2 = 2m^2$$



$$\frac{g_{\mu\nu}}{p^2 - m^2 + i\epsilon} \qquad \text{(Feynman gauge)}$$


SM Feynman diagrams:
Propagators
H
$\frac{1}{p^2 - m_H^2 + i\epsilon}$
$m_H^2 = 2m^2$
$m_H \approx 125$ GeV
$G_\mu^{a=1,\dots,8}$
μ ν
γ_μ, A_μ
$\frac{g_{\mu\nu}}{p^2 - m^2 + i\epsilon}$
(Feynman gauge)
Z^0_μ
$W^\pm_\mu$

SM Feynman diagrams:

Propagators

H : $\dfrac{1}{p^2 - m_H^2 + i\epsilon}$

$m_H^2 = 2m^2$

$m_H \approx 125\ \text{GeV}$

$G_\mu^{a=1,\dots,8}$: $\mu \quad \nu$

γ_μ, A_μ : $\mu \quad \nu$

$\dfrac{g_{\mu\nu}}{p^2 - m^2 + i\epsilon}$ (Feynman gauge)

Z^0_μ : $\mu \quad \nu$

$W^\pm_\mu$: $\mu \quad \nu$

$m_Z \simeq 91\ \text{GeV}$

$m_W \simeq 80\ \text{GeV}$

$$\xrightarrow[f]{} \qquad \frac{1}{\not{p} - m + i\epsilon}$$

$$d = d, s, b \quad m \simeq \{5,\ 95,\ 4200\}\ \text{MeV}$$

$$u = u, c, t \quad m \simeq \{2,\ 1275,\ 170\,000\}\ \text{MeV}$$

$$e = e, \mu, \tau \quad m = \{.5,\ 105,\ 1800\}\ \text{MeV}$$

$$\nu = \{n_1\ n_2\ n_3\} << \underline{\underline{\text{eV}}}$$