Title: PSI 2019/2020 - Chern-Simons Theory Part 1 - Lecture 1

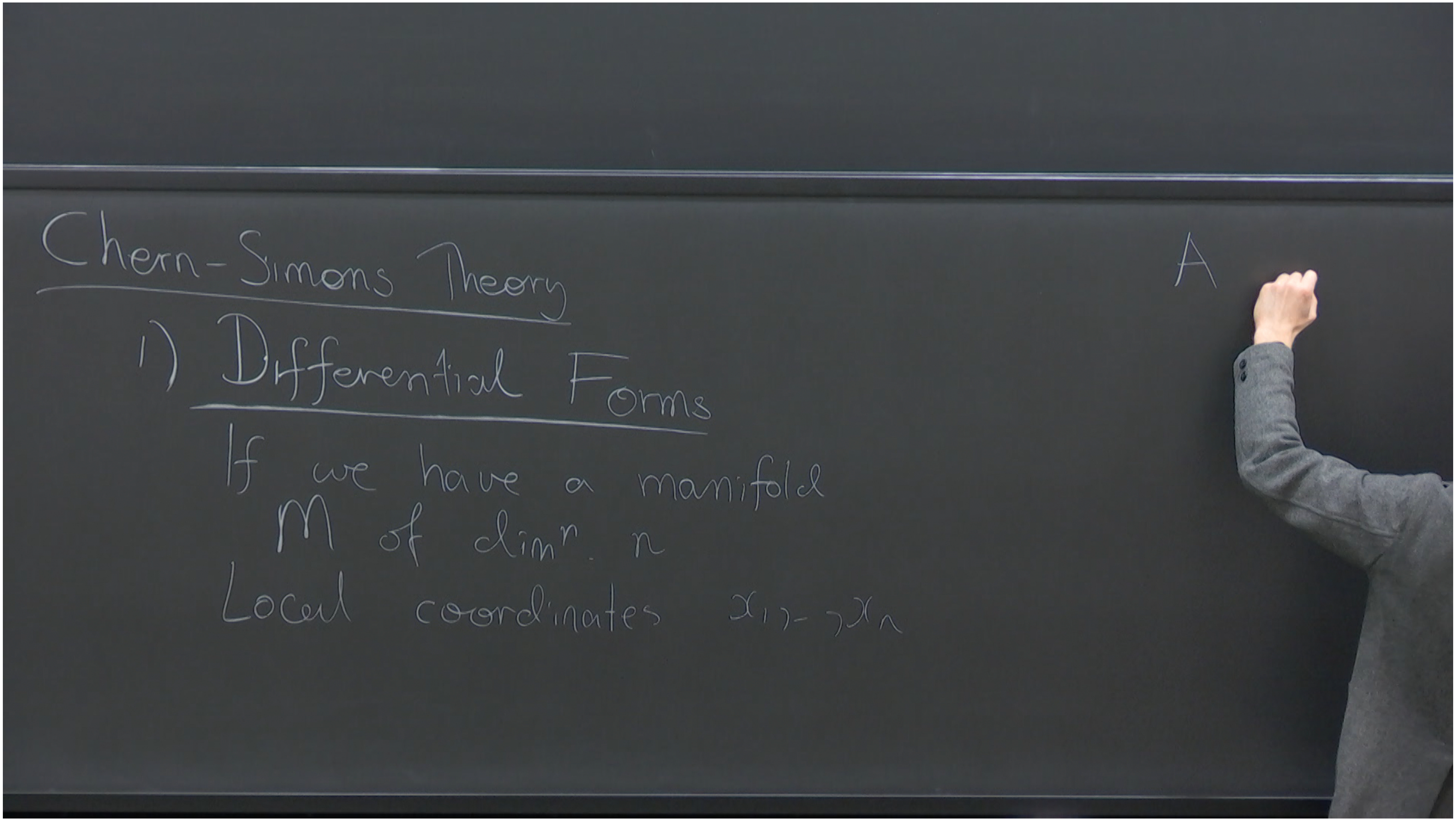
Speakers: Kevin Costello

Collection: PSI 2019/2020 - Chern-Simons Theory (Part 1)

Date: February 10, 2020 - 10:00 AM

URL: http://pirsa.org/20020024

Chern-Simons Theory


Chern-Simons Theory
1) Differential Forms
If we have a manifold
M of dimn. n
Local coordinates $x_1, \dots, x_n$
A

A k-form on M is an expression

$$\sum f_{i_1,\dots,i_k}(x_1,\dots,x_n)\, dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

where $f_{i_1\dots i_k}$ is totally anti-symmetric.

A k-form on M is an expression

$$\sum f_{i_1,\ldots,i_k}(x_1,\ldots,x_n)\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

where $f_{i_1\ldots i_k}$ is totally anti-symmetric.

RULE Treat the symbols $dx_1, \ldots, dx_n$ as anti-commuting variables.

And, $\wedge$ means the product of anti-commuting variables.

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

$$dx_i \wedge dx_i = 0$$

And, $\wedge$ means the product of anti-commuting variables.

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

$$dx_i \wedge dx_i = 0$$

<u>Example</u>

If $\omega_1 = dx_2$

$\omega_2 = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$

What is $\omega_1 \wedge \omega_2$?

Answer:

$$dx_2 \wedge dx_1 \wedge dx_3 + \cancel{dx_2 \wedge dx_2 \wedge dx_3}$$

$$= -dx_1 \wedge dx_2 \wedge dx_3$$

$\omega_2 = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$

$= -dx_1 \wedge dx_2 \wedge dx_3$

de Rham operator

If $U \subseteq \mathbb{R}^n$ we let $\Omega^k(U)$ be space of k-forms on U

Define $d : \Omega^k(U) \longrightarrow \Omega^{k+1}(U)$

by $d\left(\sum f_{i_1 \dots i_k}(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_k}\right)$

$$= \sum \frac{\partial f_{i_1 \dots i_k}(x)}{\partial x_j}\, dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

Lemma If $\omega \in \Omega^k(U)$,

$$d(d\omega) = 0$$

Proof:

$$\omega = \sum f_{i_1 \ldots i_k}(x)\, dx_{i_1} \wedge \quad \wedge dx_{i_k}$$

$$d\omega = \sum \frac{\partial f_{i_1 \ldots i_k}}{\partial x_j}\, dx_j \wedge dx_{i_1} \ldots \wedge dx_{i_k}$$

$$d(d\omega) =$$

$$= \sum \underbrace{\frac{\partial^2 f_{i_1 \dots i_k}}{\partial x_{j_1} \partial x_{j_2}}}_{\text{symmetric}} \underbrace{dx_{j_1} \wedge dx_{j_2}}_{\text{anti-sym}} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

$$= 0$$

Definition

If $U \subseteq \mathbb{R}^n$

we define $H^k(U)$ to be

$$\{\omega \in \Omega^k(U),\ d\omega = 0\} \Big/ \begin{array}{l} \omega \sim \omega' \\ \text{if } \omega - \omega' = d\alpha \\ \alpha \in \Omega^{k-1}(U) \end{array}$$

Examples:

$H^k(\mathbb{R}^n) = 0$ unless $k=0$

$H^0 \mathbb{R}^n = \mathbb{R}$

$$H^k(\mathbb{R}^n \setminus 0) = \begin{cases} 0, & k \neq 0, n-1 \\ \mathbb{R} & k = 0, n-1 \end{cases}$$

ω'

$\omega' = d\alpha$

$^{k-1}(U)$

If $U \subseteq \mathbb{R}^n$
$V \subseteq \mathbb{R}^m$
and $f : U \longrightarrow V$
$g = (g_1, \ldots, g_m)$ $g_i : U \longrightarrow \mathbb{R}$

If

If $\omega = \sum f_{i_1,\dots,i_k}(y_1,\dots,y_m)\, dy_{i_1} \wedge \dots \wedge dy_{i_k}$

$\in \Omega^k(V)$

We define

$g^*\omega$ by the following rules

1) $g^* dy_i$

If $\omega = \sum f_{i_1 \ldots i_k}(y_1, \ldots, y_m)\, dy_{i_1} \wedge \ldots \wedge dy_{i_k}$

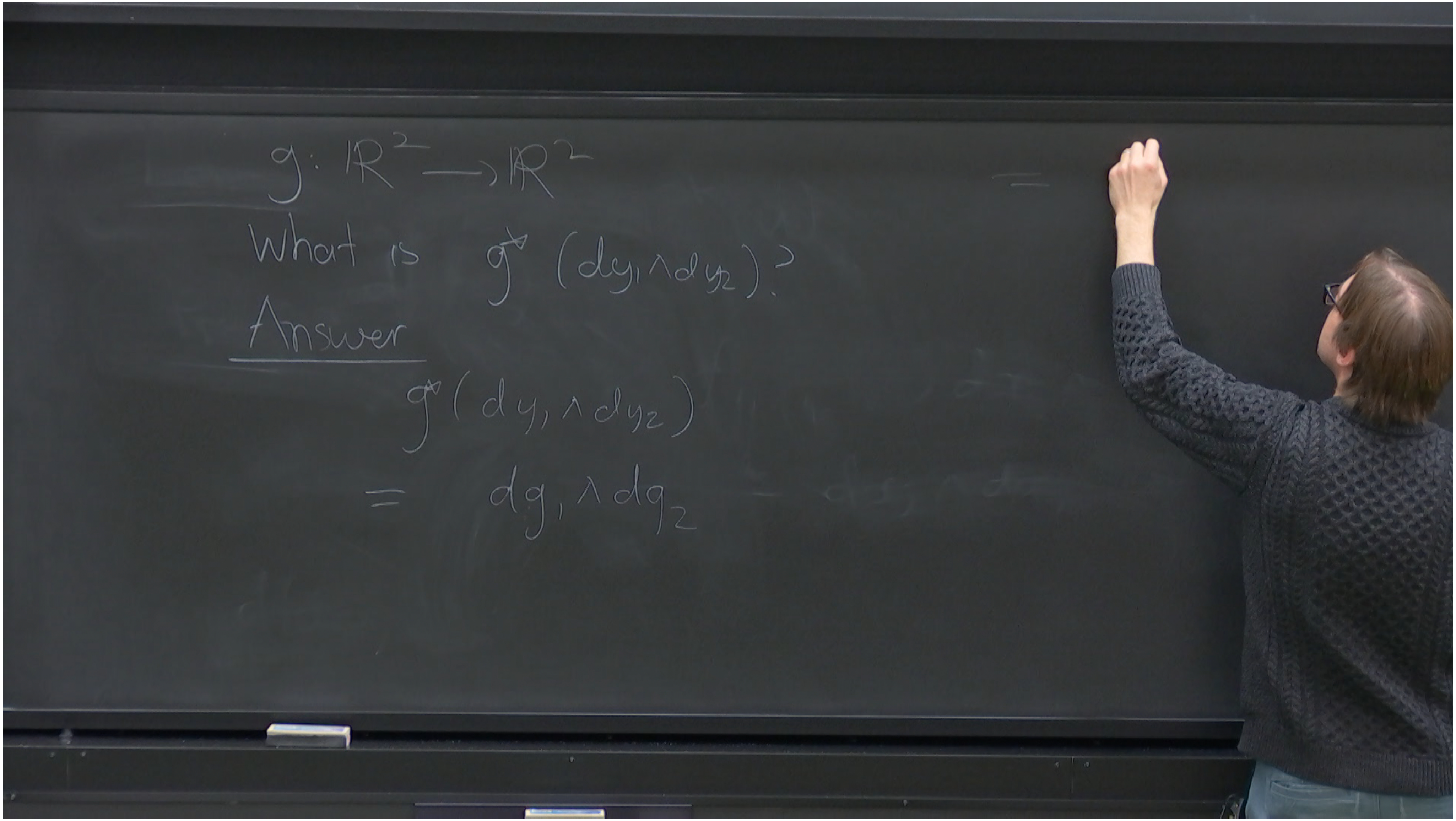
$\in \Omega^k(V)$

We define

$g^*\omega$ by the following rules

1) $g^* dy_i = dg_i = \sum \frac{\partial g_i}{\partial x_\alpha} dx_\alpha$

2) $g^*\omega = \sum f_{i_1 \ldots i_k}(g_1(x), \ldots, g_m(x))\, dg_{i_1} \wedge \ldots \wedge dg_{i_k}$

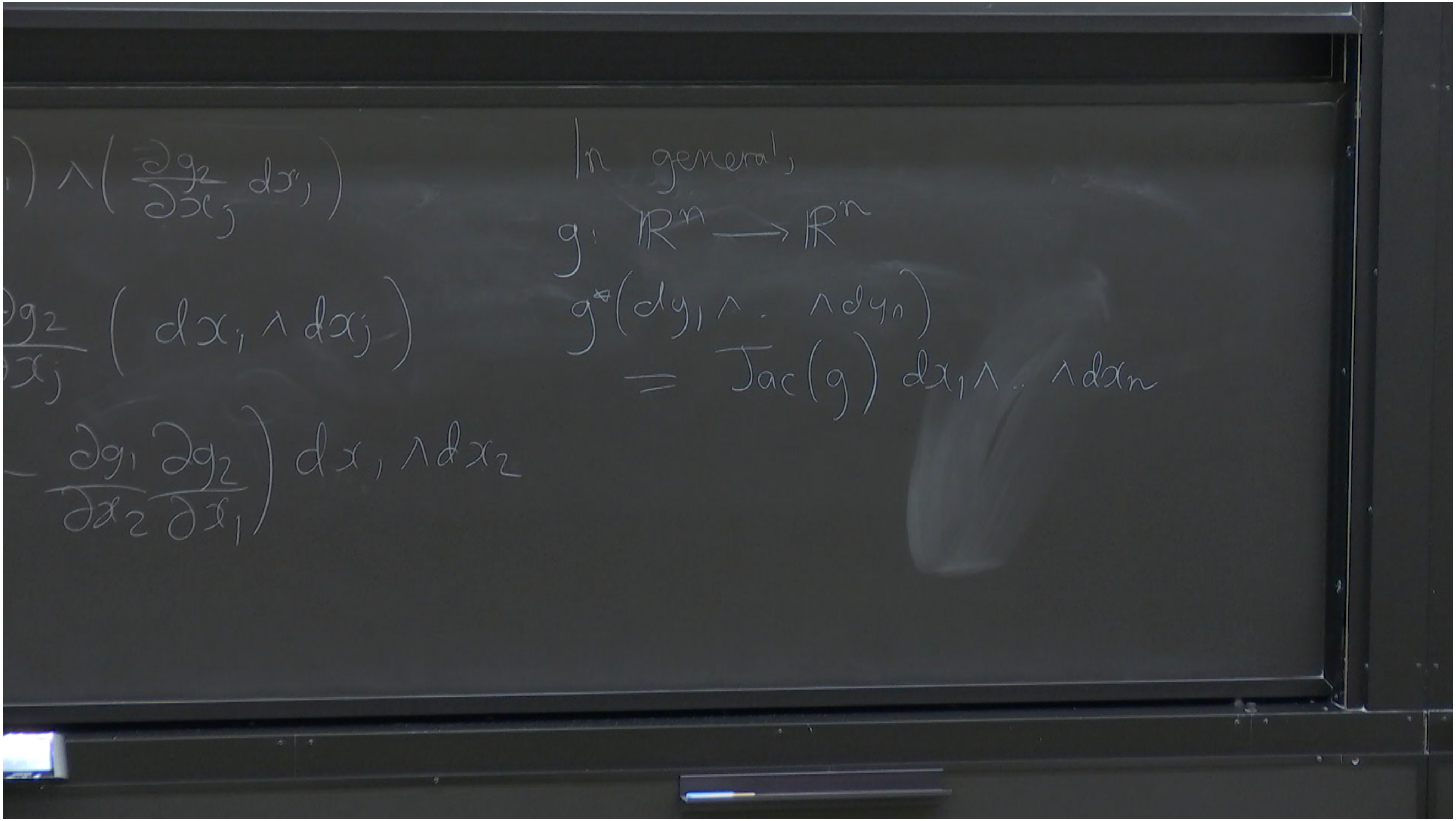

$g: \mathbb{R}^2 \to \mathbb{R}^2$
What is $g^*(dy_1 \wedge dy_2)$?
Answer
$g^*(dy_1 \wedge dy_2)$
$= dg_1 \wedge dg_2$

)?

$$= \left(\frac{\partial g_1}{\partial x_i}\, dx_i\right) \wedge \left(\frac{\partial g_2}{\partial x_j}\, dx_j\right)$$

$$= \frac{\partial g_1}{\partial x_i}\,\frac{\partial g_2}{\partial x_j}\left(dx_i \wedge dx_j\right)$$

$$= \left(\frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1}\right) dx_1 \wedge dx_2$$


$) \wedge \left(\frac{\partial g_2}{\partial x_j} dx_j \right)$
$\frac{\partial g_2}{\partial x_j} \left(dx_i \wedge dx_j \right)$
$\frac{\partial g_1}{\partial x_2} \frac{\partial g_2}{\partial x_1} \Big) dx_1 \wedge dx_2$
In general,
$g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$
$g^*(dy_1 \wedge \dots \wedge dy_n)$
$= \mathrm{Jac}(g)\, dx_1 \wedge \dots \wedge dx_n$

If we have a manifold M

what is $\Omega^k(M)$?

Answer

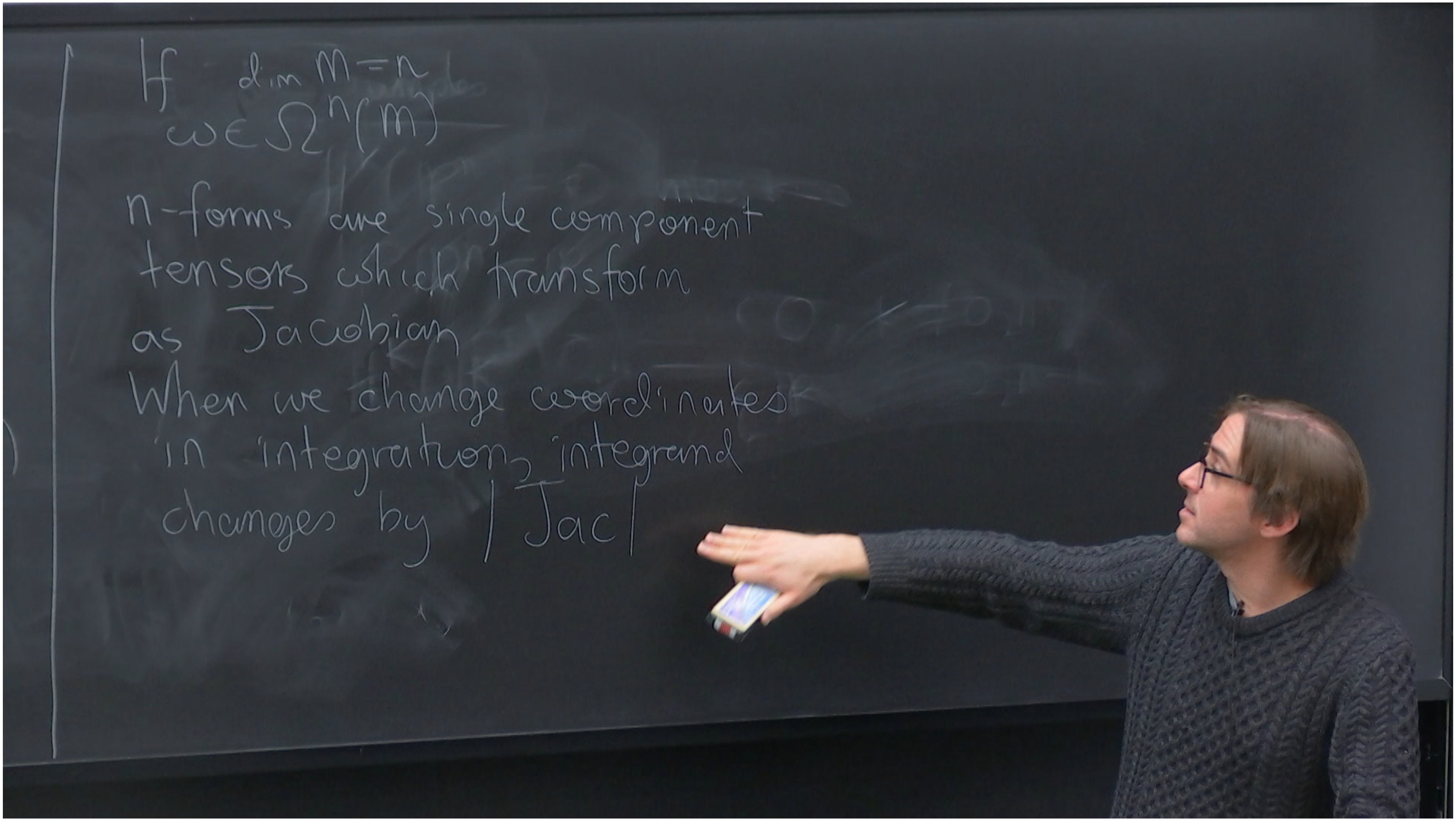
If $U_i \subseteq M \xrightarrow{\varphi_i} \mathbb{R}^n$ are coordinate charts

φ_{ij} transition fns

An element of $\Omega^k(M)$ is $\omega_i \in \Omega^k(U_i)$

such that on $U_i \cap U_j$

$\varphi_{ij}^* \omega_j = \omega_i$


If $\dim M = n$
$\omega \in \Omega^n(M)$
n-forms are single component
tensors which transform
as Jacobian
When we change coordinates
in integration, integrand
changes by $|Jac|$