Title: PSI 2019/2020 - Chern-Simons Theory Part 1 - Lecture 1

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Collection: PSI 2019/2020 - Chern-Simons Theory (Part 1)

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Chern-Simons Theory

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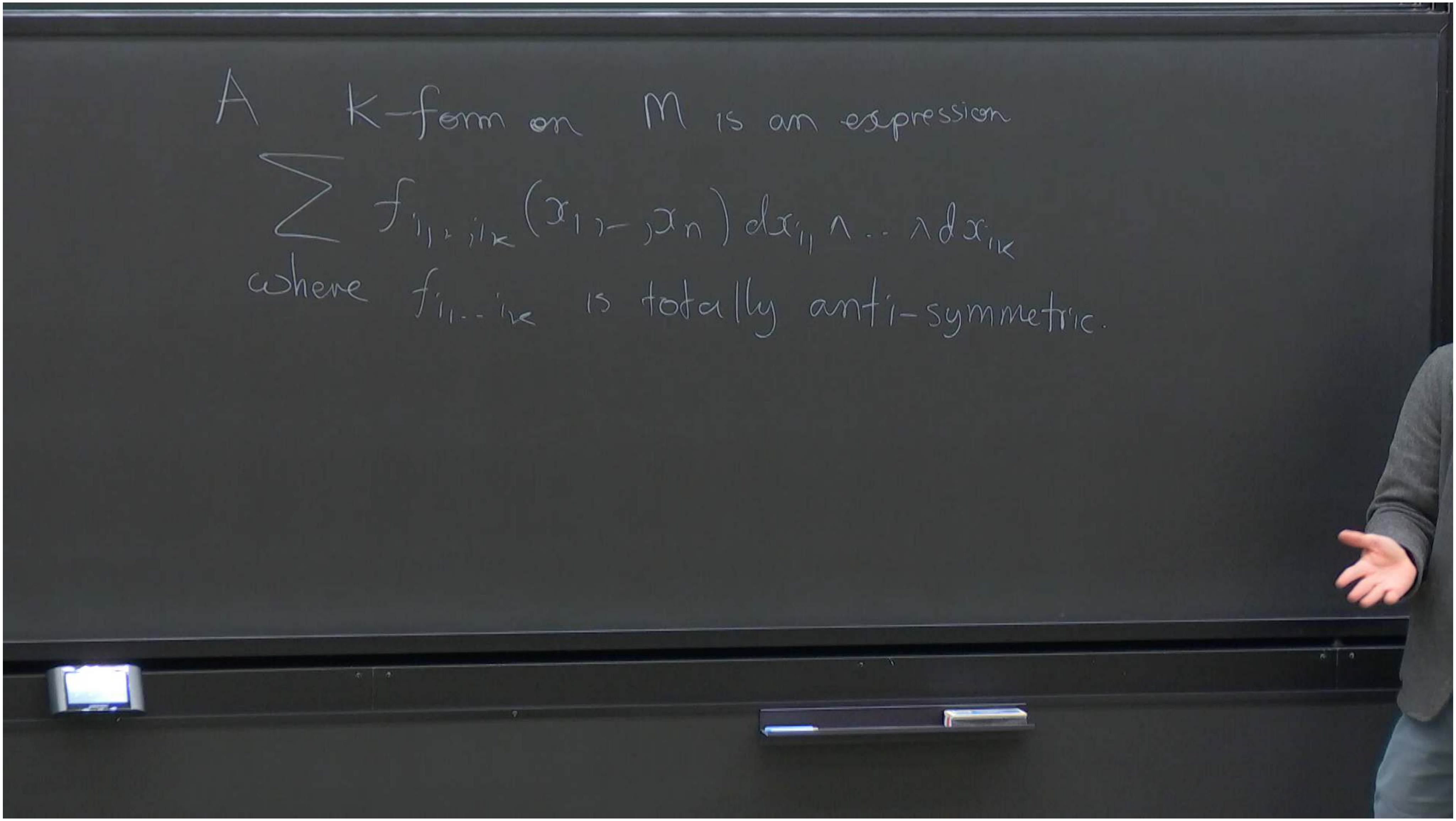
1) Differential Forms

If we have a manifold M of dim^n n

Local coordinates $x_1, \ldots, x_n$

A


A k-form on M is an expression
$\sum f_{i_1,\ldots,i_k}(x_1,\ldots,x_n)\, dx_{i_1}\wedge\ldots\wedge dx_{i_k}$
where $f_{i_1\ldots i_k}$ is totally anti-symmetric.

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RULE Treat the symbols $dx_1,\dots,dx_n$ as anti-commuting variables.

And, $\wedge$ means the product of anti-commuting variables.

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

$$dx_i \wedge dx_i = 0$$

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$$dx_i \wedge dx_i = 0$$

<u>Example</u>

If $\omega_1 = dx_2$

$$\omega_2 = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$$

What is $\omega_1 \wedge \omega_2$?

Answer:

$$dx_2 \wedge dx_1 \wedge dx_3$$

$$\cancel{+\, dx_2 \wedge dx_2 \wedge dx_3}$$

$$= -dx_1 \wedge dx_2 \wedge dx_3$$

$\omega_2 = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$

$= -dx_1 \wedge dx_2 \wedge dx_3$

de Rham operator

If $U \subseteq \mathbb{R}^n$ we let $\Omega^k(U)$ be space of k-forms on U

Define $d : \Omega^k(U) \longrightarrow \Omega^{k+1}(U)$

by $d\left(\sum f_{i_1 \dots i_k}(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_k}\right)$

$$= \sum \frac{\partial f_{i_1 \dots i_k}(x)}{\partial x_j}\, dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

Lemma If $\omega \in \Omega^k(U)$,

$d(d\omega) = 0$

Proof:

$$\omega = \sum f_{i_1 \ldots i_k}(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

$$d\omega = \sum \frac{\partial f_{i_1 \ldots i_k}}{\partial x_j}\, dx_j \wedge dx_{i_1} \ldots \wedge dx_{i_k}$$

$$d(d\omega) =$$

$$= \sum \underbrace{\frac{\partial^2 f_{i_1 \dots i_k}}{\partial x_{j_1} \partial x_{j_2}}}_{\text{symmetric}} \underbrace{dx_{j_1} \wedge dx_{j_2}}_{\text{anti-sym}} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

$$= 0$$

Definition

If $U \subseteq \mathbb{R}^n$

we define $H^k(U)$ to be

$$\{\omega \in \Omega^k(U),\ d\omega = 0\} \big/ \omega \sim \omega'$$

if $\omega - \omega' = d\alpha$

$\alpha \in \Omega^{k-1}(U)$

Examples:

$H^k(\mathbb{R}^n) = 0$ unless $k = 0$

$H^0 \mathbb{R}^n = \mathbb{R}$

$$H^k(\mathbb{R}^n \setminus 0) = \begin{cases} 0, & k \neq 0, n-1 \\ \mathbb{R} & k = 0, n-1 \end{cases}$$

ω'

$\omega' = d\alpha$

$^{k-1}(U)$

If $U \subseteq \mathbb{R}^n$

$V \subseteq \mathbb{R}^m$

and $f : U \longrightarrow V$

$g = (g_1, \dots, g_m) \qquad g_i : U \longrightarrow \mathbb{R}$

If

If $\omega = \sum f_{i_1,\dots,i_k}(y_1,\dots,y_m)\, dy_{i_1} \wedge \dots \wedge dy_{i_k}$

$\in \Omega^k(V)$

We define

$g^*\omega$ by the following rules

1) $g^* dy_1$

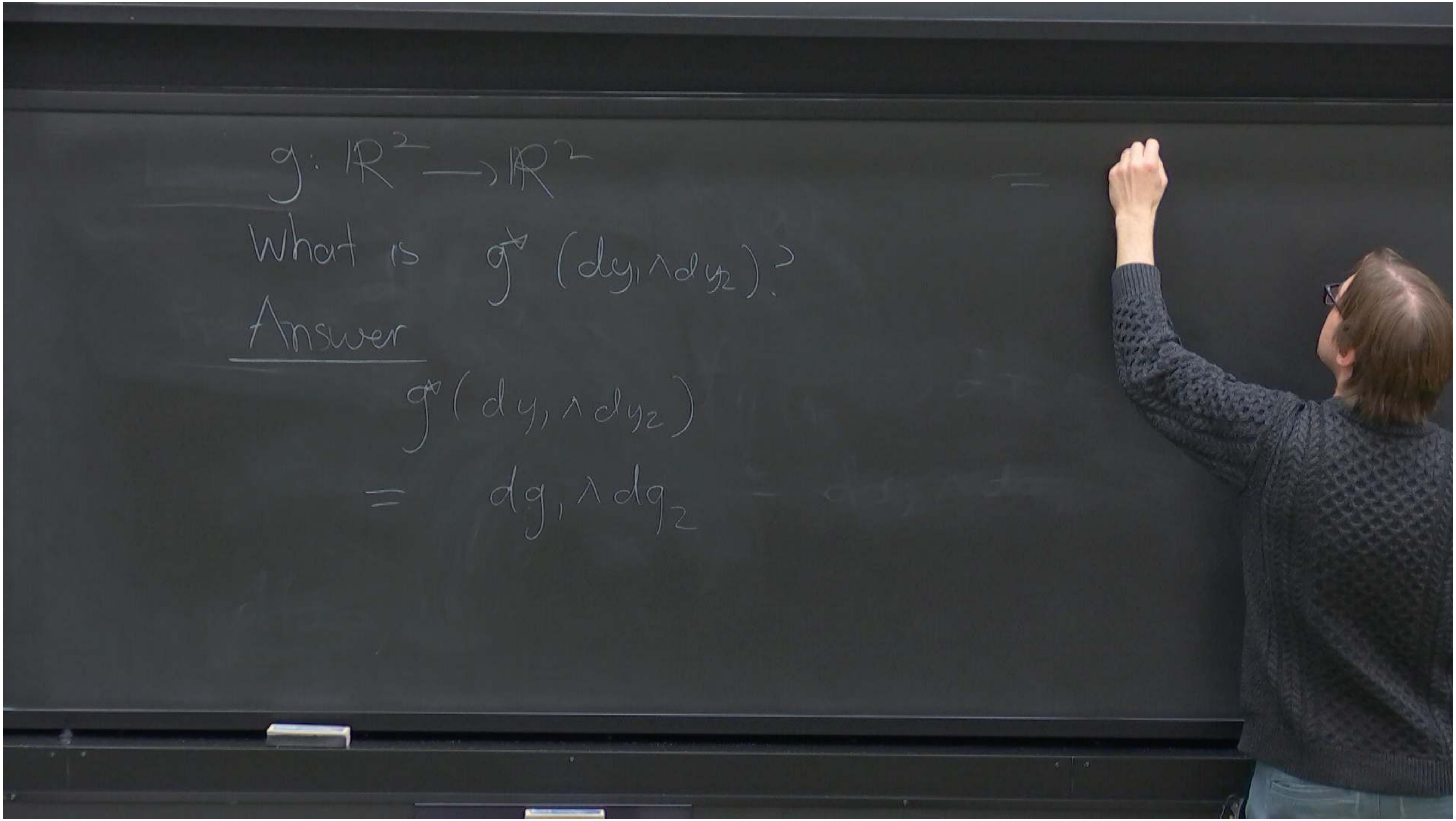
If $\omega = \sum f_{i_1 \ldots i_k}(y_1, \ldots, y_m)\, dy_{i_1} \wedge \ldots \wedge dy_{i_k}$

$\in \Omega^k(V)$

We define $g^*\omega$ by the following rules

1) $g^* dy_i = dg_i = \sum \frac{\partial g_i}{\partial x_\alpha} dx_\alpha$

2) $g^*\omega = \sum f_{i_1 \ldots i_k}(g_1(x), \ldots, g_m(x))\, dg_{i_1} \wedge \ldots \wedge dg_{i_k}$


$g: \mathbb{R}^2 \to \mathbb{R}^2$
What is $g^*(dy_1 \wedge dy_2)$?
Answer
$g^*(dy_1 \wedge dy_2)$
$= dg_1 \wedge dg_2$

$$= \left(\frac{\partial g_1}{\partial x_i} dx_i\right) \wedge \left(\frac{\partial g_2}{\partial x_j} dx_j\right)$$

$)?$

$$= \frac{\partial g_1}{\partial x_i} \frac{\partial g_2}{\partial x_j} \left(dx_i \wedge dx_j \right)$$

$$= \left(\frac{\partial g_1}{\partial x_1} \frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2} \frac{\partial g_2}{\partial x_1} \right) dx_1 \wedge dx_2$$

$$\ldots) \wedge \left(\frac{\partial g_2}{\partial x_j} dx_j \right)$$

$$\ldots \frac{\partial g_2}{\partial x_j} \left(dx_i \wedge dx_j \right)$$

$$\ldots - \frac{\partial g_1}{\partial x_2} \frac{\partial g_2}{\partial x_1} \Big) dx_1 \wedge dx_2$$

In general,

$g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$g^*(dy_1 \wedge \ldots \wedge dy_n) = \mathrm{Jac}(g)\, dx_1 \wedge \ldots \wedge dx_n$$

If we have a manifold M

what is $\Omega^k(M)$?

Answer:

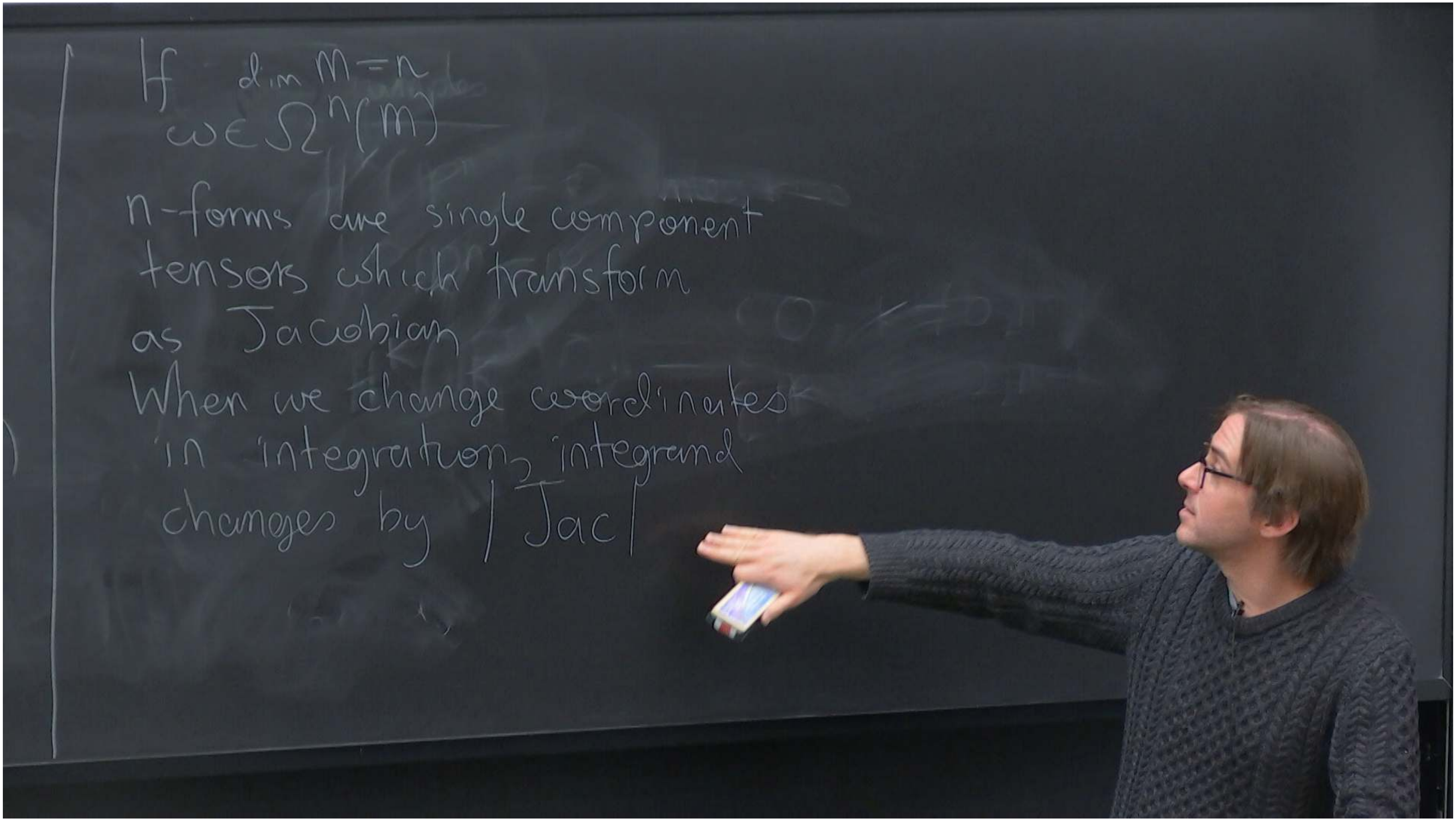
If $U_i \subseteq M$, $\varphi_i : U_i \to \mathbb{R}^n$ are coordinate charts

φ_{ij} transition fns

An element of $\Omega^k(M)$ is $\omega_i \in \Omega^k(U_i)$

such that on $U_i \cap U_j$

$(\varphi_{ij})^* \omega_j = \omega_i$


If $\dim M = n$
$\omega \in \Omega^n(M)$
n-forms are single component
tensors which transform
as Jacobian
When we change coordinates
in integration, integrand
changes by $|Jac|$