Title: Noncommutative Zhu algebra and quantum field theory in four and three dimensions

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Series: Quantum Fields and Strings

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Abstract: For any vertex operator algebra V, Y. Zhu constructed an associative algebra Zhu(V) that captures its representation theory (more generally, given a finite order automorphism g of V, there exists an algebra Zhu_g(V) that captures g-twisted representation theory of V).

To a 4d N=2 superconformal theory T, one assigns a vertex algebra V[T] by the construction of Beem et al. We explain one role of Zhu algebra in this context. Namely, we show that a certain quotient of the Zhu algebra describes what happens to the Schur sector of the theory T under the dimensional reduction on S^1. This connects the VOA construction in 4d N=2 SCFT to the topological quantum mechanics construction in 3d N=4 SCFT, with the latter being given by the aforementioned quotient of the Zhu algebra. In the process, we will discuss how to reformulate the VOA construction on an S^3 x S^1 geometry.

Noncommutative Zhu algebra and SCFT

(based on M.D. '19, M.D.-Y. Wang '19 & M.D.-M. Fluder '19)

Intro Two algebraic structures

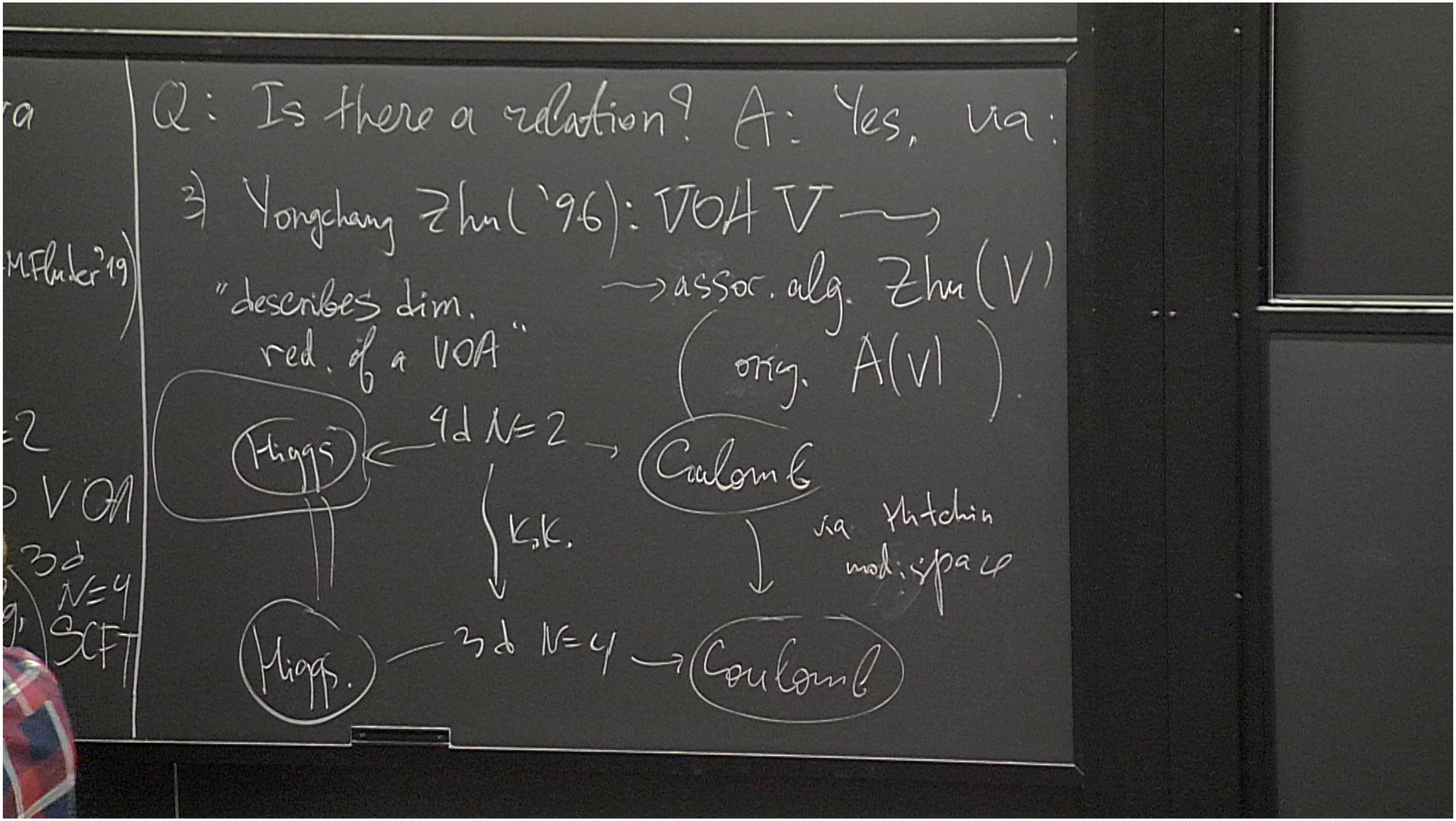
1) BLLPRvR ('13) 4d $\mathcal{N}=2$ SCFT $\supset$ VOA

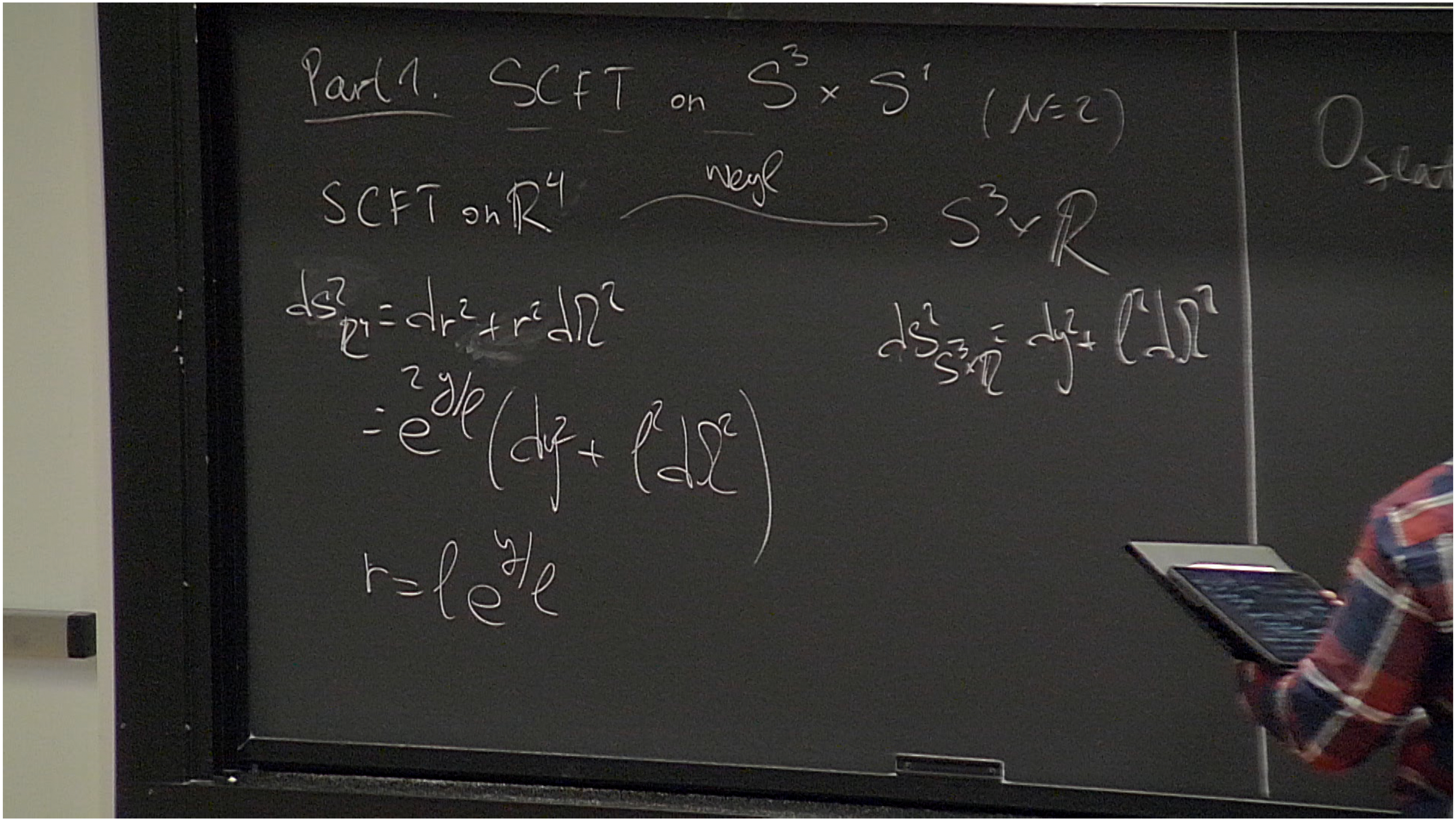
2) { CPY ('19) ; BPR ('16) } 1 TQM $\subset$ 3d $\mathcal{N}=4$ SCFT

(i.e. assoc. alg. + trace)

Q: Is ...

3) Yongch...


Q: Is there a relation? A: Yes, via:
3) Yongchang Zhu ('96): VOA V →
→ assoc. alg. Zhu(V)
"describes dim. red. of a VOA"
(orig. A(V))
Higgs
4d N=2
Coulomb
K.K.
via Hitchin mod. space
Higgs.
3d N=4
Coulomb
M.Fluder'19)
V OA
3d N=4 SCFT


Part 1. SCFT on $S^3 \times S^1$ $(\mathcal{N}=2)$
SCFT on $\mathbb{R}^4$ $\xrightarrow{\text{weyl}}$ $S^3 \times \mathbb{R}$
$ds^2_{\mathbb{R}^4} = dr^2 + r^2 d\Omega^2$
$= e^{2y/\ell}(dy^2 + \ell^2 d\Omega^2)$
$r = \ell e^{y/\ell}$
$ds^2_{S^3\times\mathbb{R}} = dy^2 + \ell^2 d\Omega^2$

$N=2)$

$\times \mathbb{R}$

$dy^2 + \ell^2 d\Omega^2$

$$\mathcal{O}_{flat} = \left(e^{y/\ell}\right)^{-E} \mathcal{O}_{cyl}$$

E: conf. dim.

Conf. Kill. sp.: $\xi = \epsilon + i \not{x} \eta$

$\bar{\xi} = \bar{\epsilon} + i \not{x} \bar{\eta}$

8 of them $(\epsilon, \bar{\epsilon}) \sim e^{-y/2\ell}$

8 of them $(\eta, \bar{\eta}) \sim e^{y/2\ell}$

(In Lagr. case: see [MD, ..., Fluder '19])

Let's "close $\mathbb{R}$":

$$y \sim y + \beta \ell$$

Because of (*): this breaks SUSY.

<u>Solution</u>: for each variable $\mathcal{F}$

$$\mathcal{F}(y+\beta\ell) = e^{-\beta R}\,\mathcal{F}(y)$$

R - gener. of $U(1)_R \subset SU(2)_R$

Preserve

...reaks SUSY.

...riable $\mathcal{F}$

$\mathcal{F}(y)$

...ener. of $U(1)_R \subset SU(2)_R$

Preserves 8 superch., commuting w/ $\mathcal{H} - R$.

dilat. gen.

$\mathfrak{s} \subset \mathfrak{su}(2,2|2)$

$\mathfrak{s} = \widetilde{\mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1)}$ centr. ext. by $\mathcal{H} - R$.

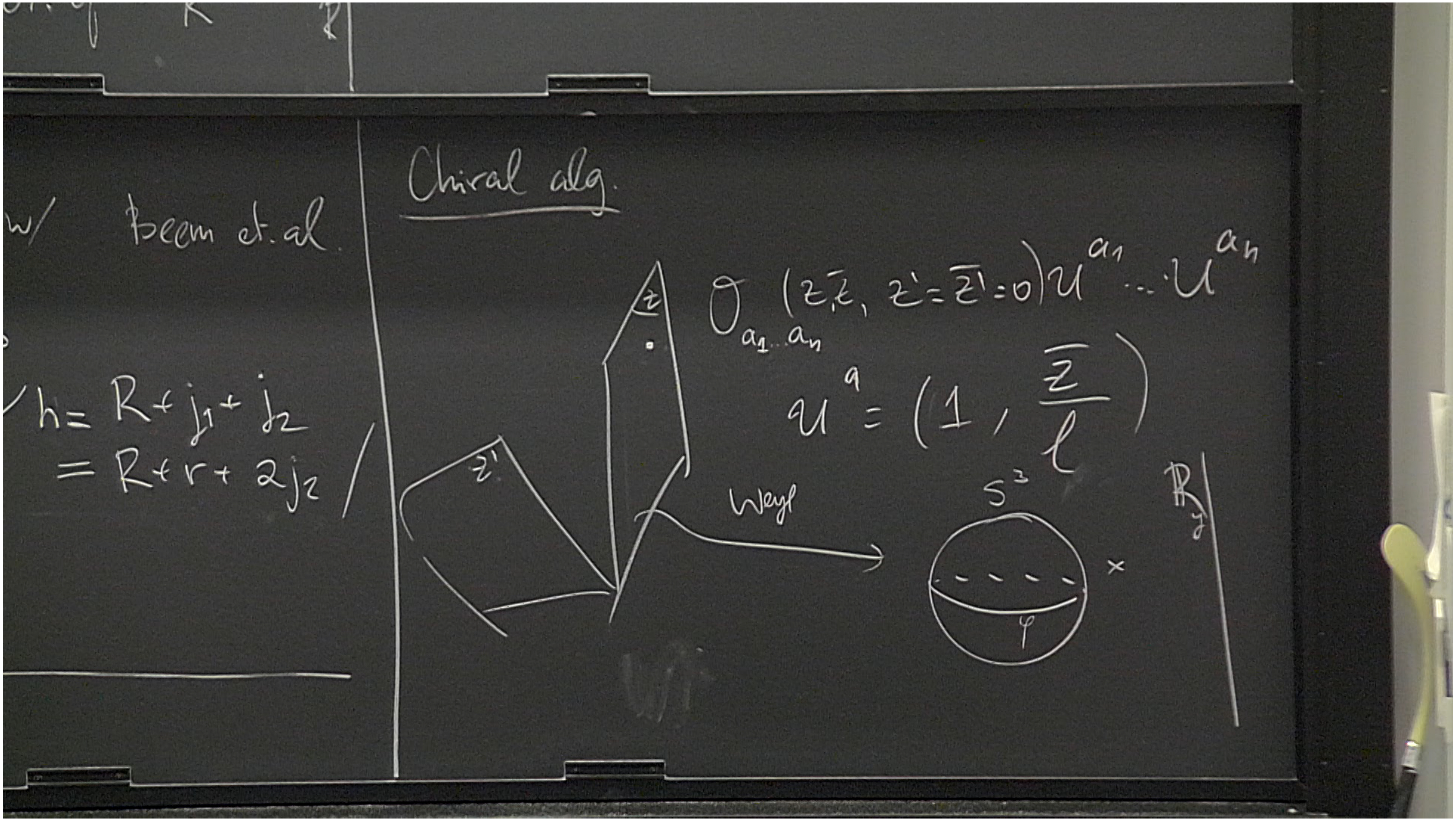
<u>Rk:</u> This is the b.g. that computes the Schur index.

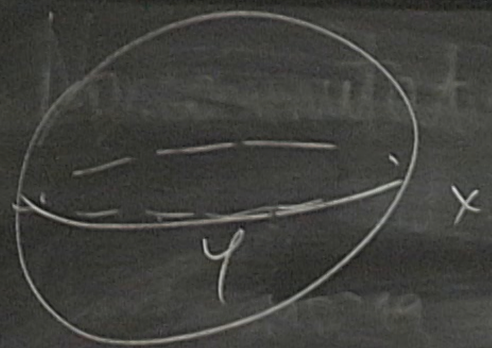
Rk: This is compatible w/ Beem et. al

i.e., $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathfrak{s}$

Rk: $(-1)^{2(R+r)}$ preserves $\mathbb{Q}_1$ & $\mathbb{Q}_2$

$$h = R + j_1 + j_2 = R + r + 2j_2$$


Chiral alg.
w/ Beem et. al
$h = R + j_1 + j_2$
$= R + r + 2j_2$
$\mathcal{O}_{a_1 \dots a_n}(z, \bar{z}, z' = \bar{z}' = 0)\, u^{a_1} \cdots u^{a_n}$
$u^a = \left(1, \frac{\bar{z}}{\ell}\right)$
z
z'
Weyl
$S^3 \times \mathbb{R}_y$
φ



$\times \; \mathbb{R}_y$

z plane maps to $S^1_\varphi \times \mathbb{R}_y$

$$w = \ell\varphi - iy$$

$$\mathcal{O}^{cyl}(w,\bar{w}) = \mathcal{O}_{a_1 \dots a_n}(w, w', \theta = \tfrac{\pi}{2})\, u^{a_1} \dots u^{a_n}$$

$$u = \left(e^{i\bar{w}/2\ell}, e^{-i\bar{w}/2\ell} \right)$$

close $\mathbb{R}$

$\mathcal{O}^{cyl}_{(w,\bar{w}}$

S^1_y

tr-re.

S^1_y

$$O_{flat} = \left(e^{y/\ell}\right)^{-E} \qquad O_{cyl}$$

E = conf. dim.

Conf. Kill. sp.: $\xi = \epsilon + i \not{x}\, \eta$

$\bar\xi = \bar\epsilon + i \not{x}\, \eta$

(*) 8 of them $(\epsilon, \bar\epsilon) \sim e^{-y/2\ell}$

8 of them $(\eta, \bar\eta) \sim e^{y/2\ell}$

In Lagr. case: see [MD, ..., Fluder '19]

$\Rightarrow$ VOA on $\mathbb{T}^2 = S^1_\varphi \times S^1_y$

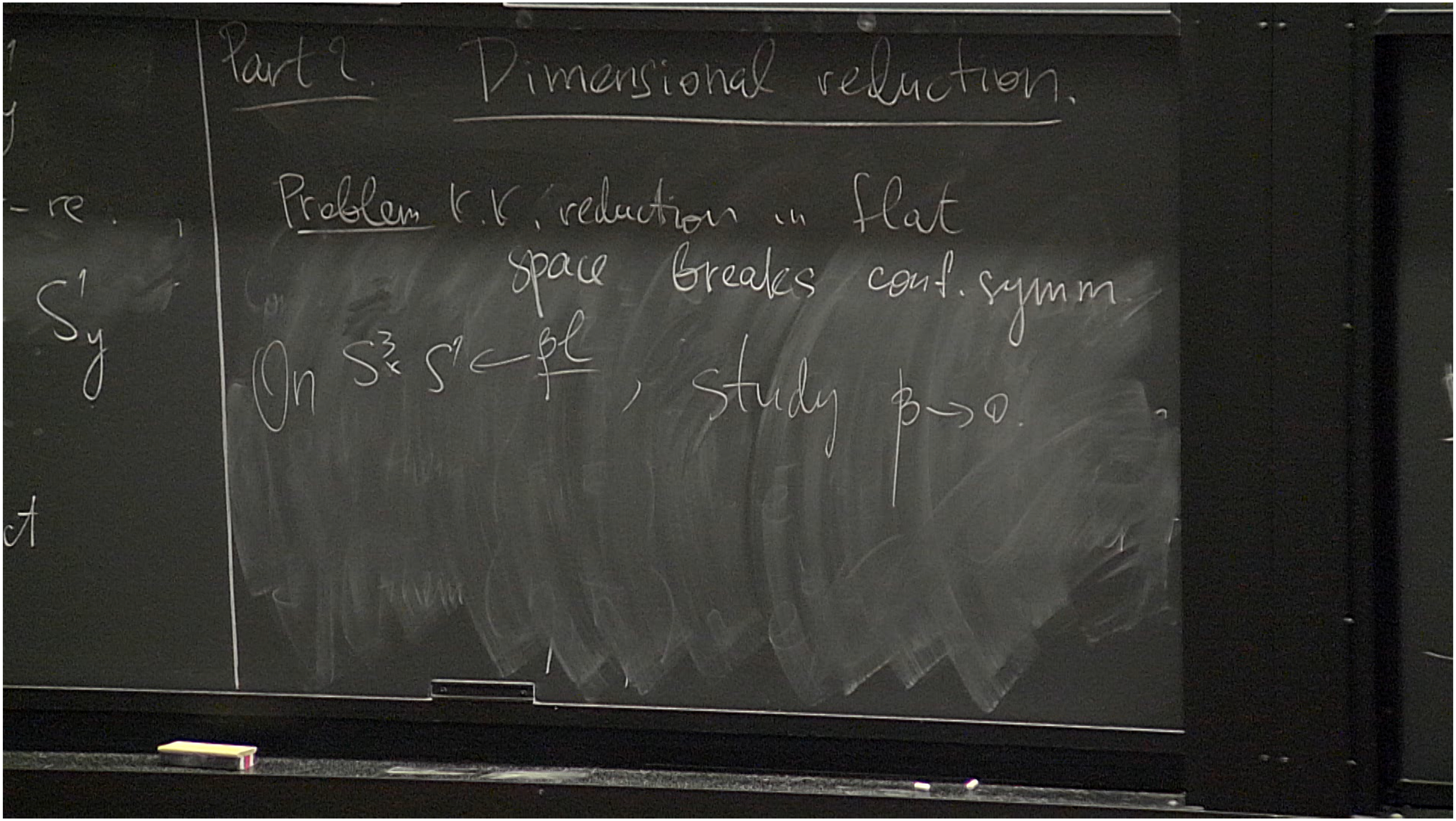
w/ NS-R spin str-re.

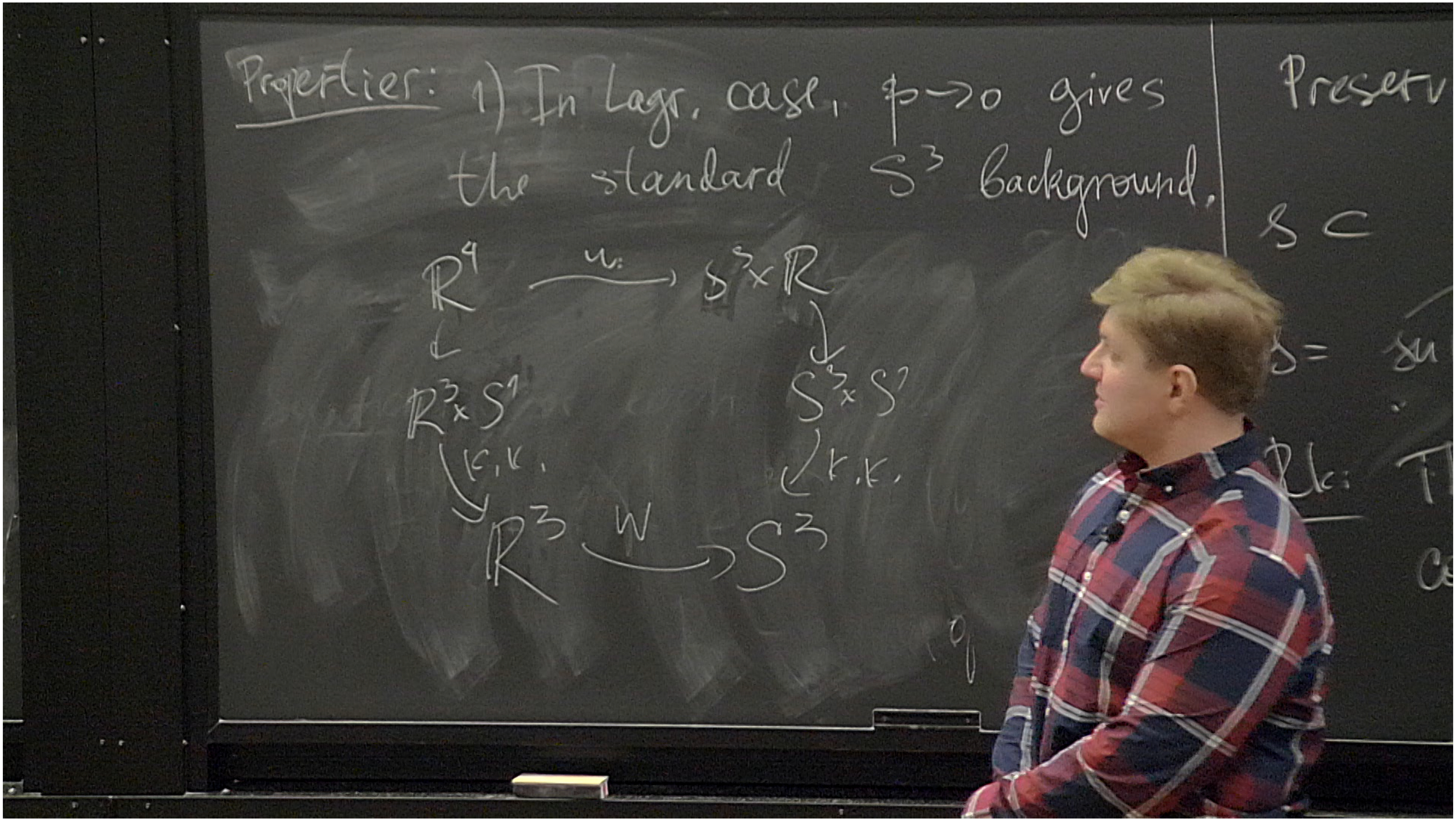
<u>Rk</u>: 1) Turn on $(-1)^{2(R+r)}$ along S^1_y

$\Rightarrow$ flips R $\to$ NS.

2) Can add $(-1)^{2(R+r)}$ defect

$\Rightarrow$ flips NS $\to$ R


Part 2. Dimensional reduction.
Problem K.K. reduction in flat space breaks conf. symm.
On $S^3 \times S^1 \leftarrow \beta\ell$, study $\beta \to 0$.
S^1_y


Properties: 1) In Lagr. case, ħ→0 gives
the standard S³ background.
R⁴
u:
S³×R
R³×S¹
S³×S¹
K.K.
K.K.
R³
W
S³
Preserv
S ⊂
S = su
Rk: Th
c

gives

kground.

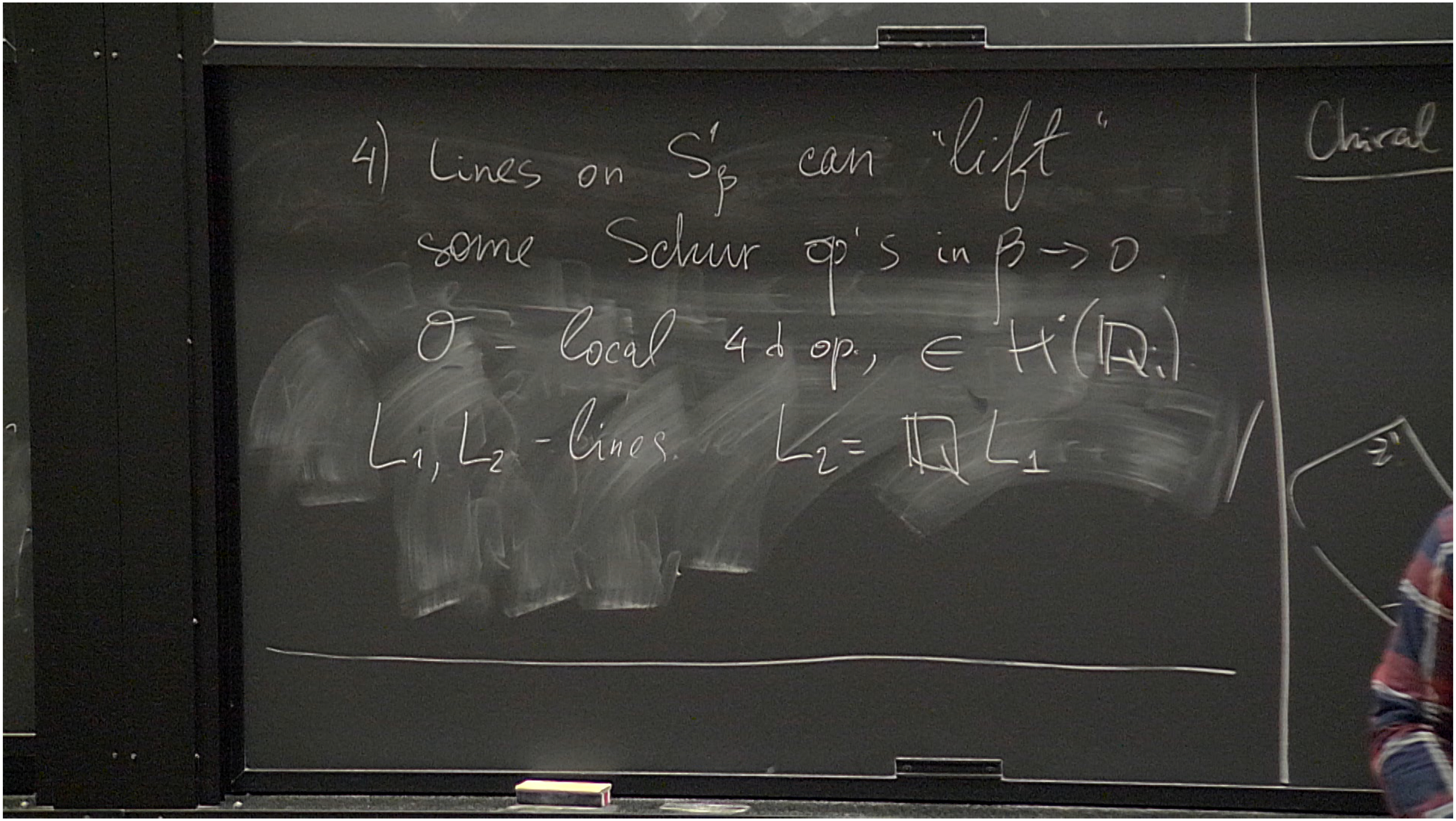
2) $\beta \to 0$ is divergent. Cardy behavior:

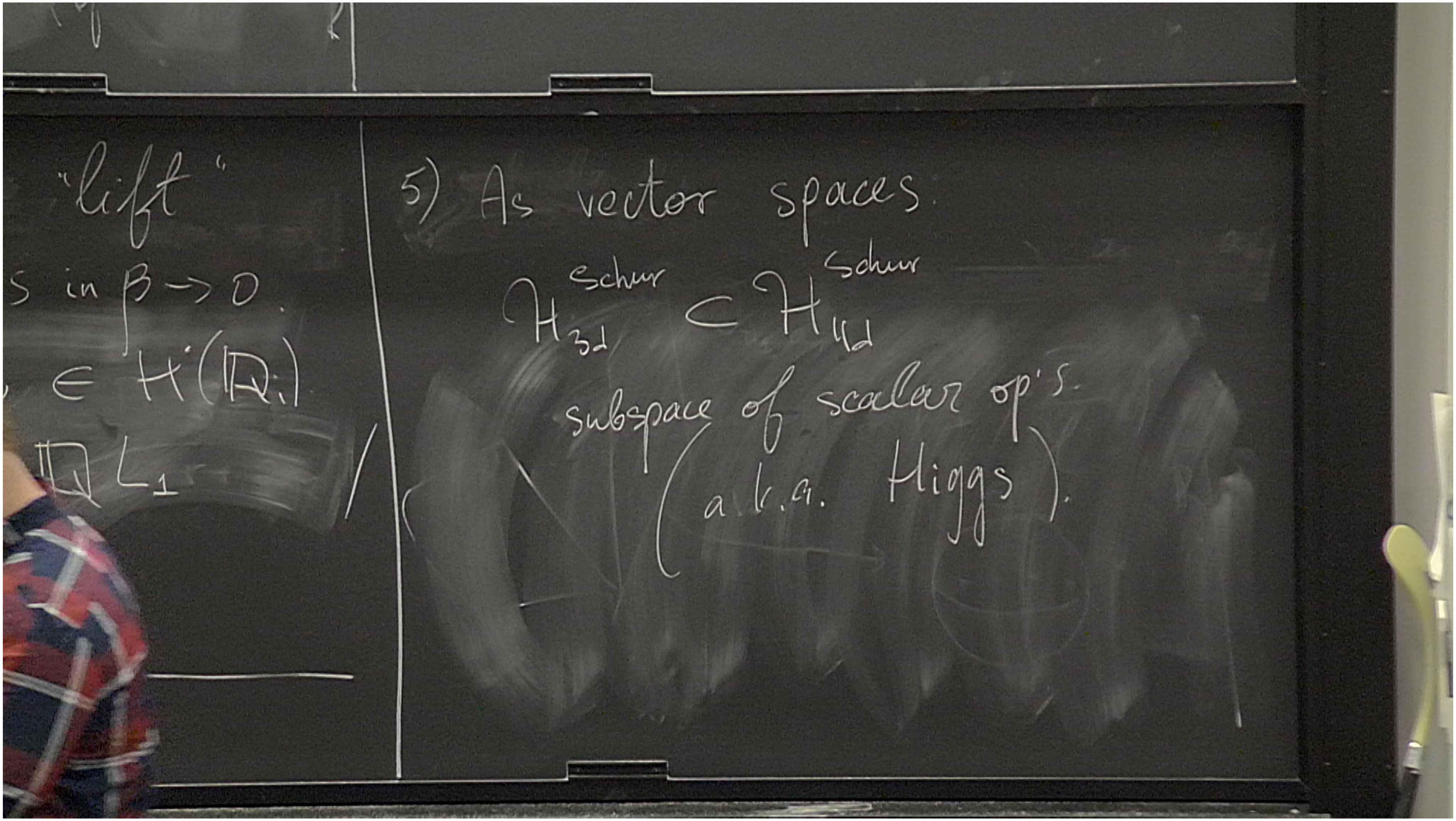
$$\log Z(S^3 \times S^1) \sim \frac{8\pi^2}{\beta}(c_{4d} - a_{4d}), \quad c_{4d} > a_{4d}$$

This can be removed by the E.-H. counterterm.

3) Problematic if $c_{4d} \le a_{4d}$.

Only consider $c_{4d} > a_{4d}$.


4) Lines on S¹_β can "lift"
some Schur op's in β → 0.
O - local 4d op, ∈ H(Q)
L₁, L₂ - lines. L₂ = Q L₁
Chiral

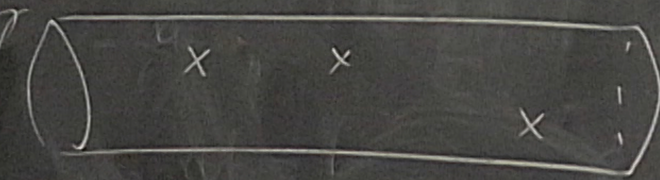

"lift"
s in $\beta \to 0$.
$\in H^{\cdot}(\mathbb{Q})$
$\mathbb{Q} L_1$
5) As vector spaces:
$\mathcal{H}^{\text{Schur}}_{3d} \subset \mathcal{H}^{\text{Schur}}_{4d}$
subspace of scalar op's.
(a.k.a. Higgs).

Algebraic structures?

Captured by correlators!

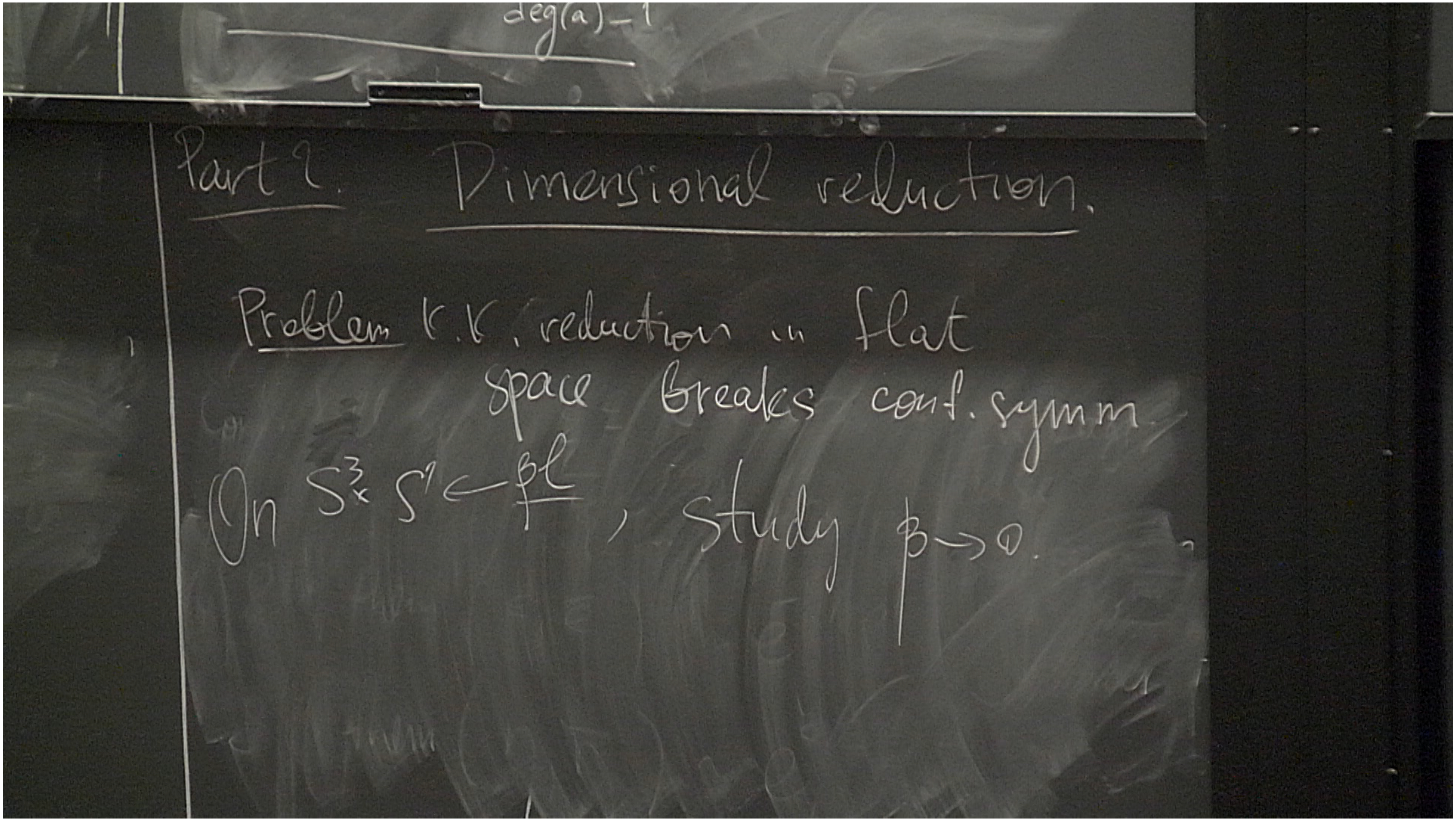
Part 3. Zhu algebra

Intuitively: VOA V on:

β  $\beta \to 0$

Close $\mathbb{R}$

$\mathcal{O}^{cyl}_{(w,\bar{w})}$

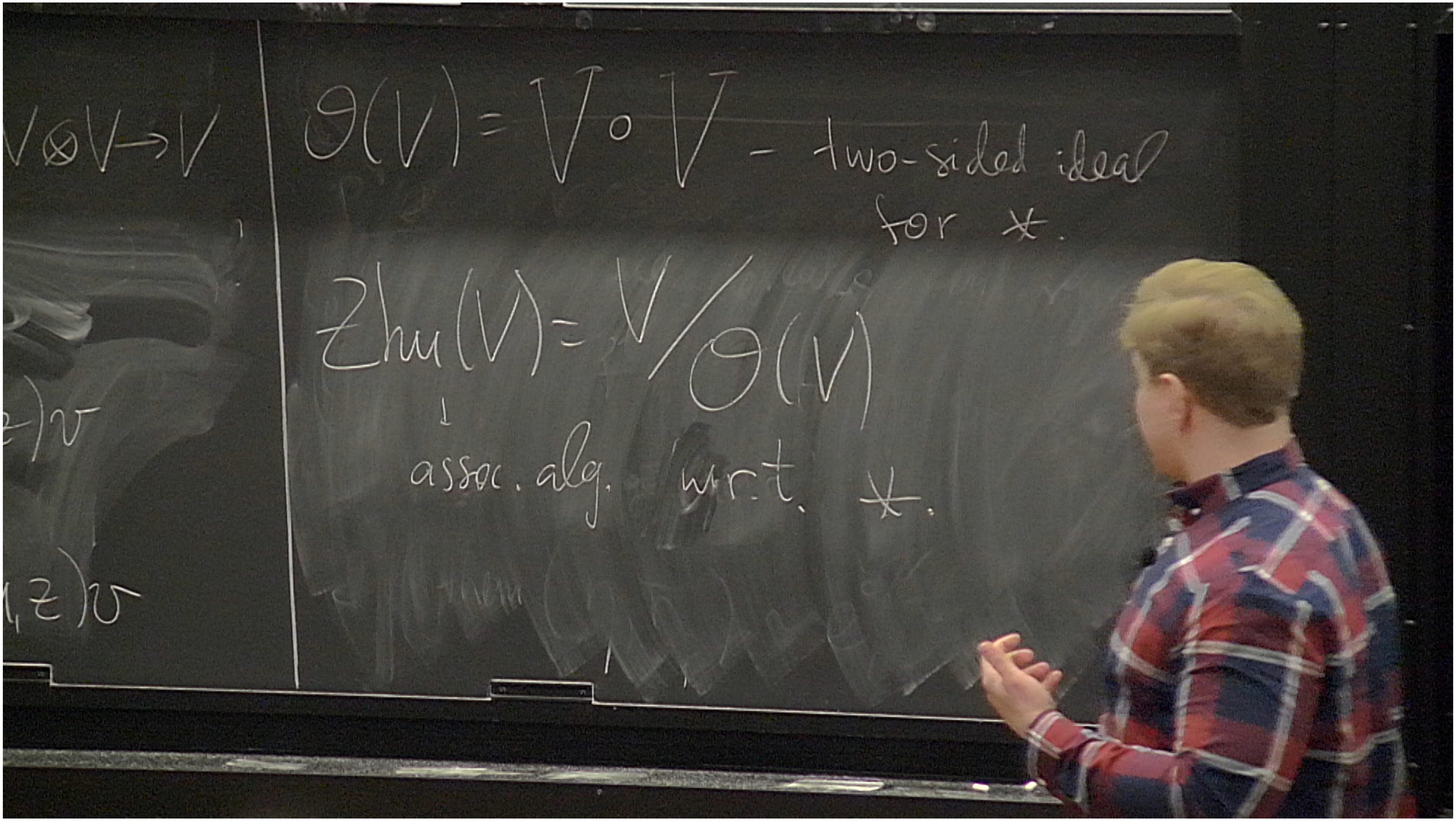

deg(a) - 1
Part 2. Dimensional reduction.
Problem K.K. reduction in flat space breaks conf. symm.
On $S^3_x \times S^1 \leftarrow \beta\ell$, study $\beta \to 0$.

$$o(a) \circ o(a) = o(a * b), \quad * : V \otimes V \to V \qquad \theta(V) =$$

$$u, v \in V,$$

$$u \circ v = \mathrm{Res}_z \frac{(1+\hbar z)^{\deg(u)}}{z^2} Y(u,z)v$$

$$u * v = \mathrm{Res}_z \frac{(1+\hbar z)^{\deg(u)}}{z} Y(u,z)v$$


V ⊗ V → V
O(V) = V ∘ V - two-sided ideal
for *.
Zhu(V) = V/O(V)
assoc. alg. w.r.t. *.

$V \otimes V \to V$

$z)v$

$1, z)v$

$O(V) = V \circ V$ – two-sided ideal for $*$.

$\mathrm{Zhu}(V) = V / O(V)$

assoc. alg. w.r.t. $*$.

V-modules $\xleftrightarrow{1-1}$ $\mathrm{Zhu}(V)$-modules.

Part 4 "$\tau \to 0$" (high-temperature) limit of torus correlators.

$$\langle O_1(w_1)\dots O_n(w_n)\rangle = \frac{\mathrm{Tr}_M\, q^{L_0 - c/24}\, O_1(w_1)\dots O(w_n)}{\mathrm{Tr}_M\, q^{L_0 - c/24}}$$

$M = V$ vac. module.

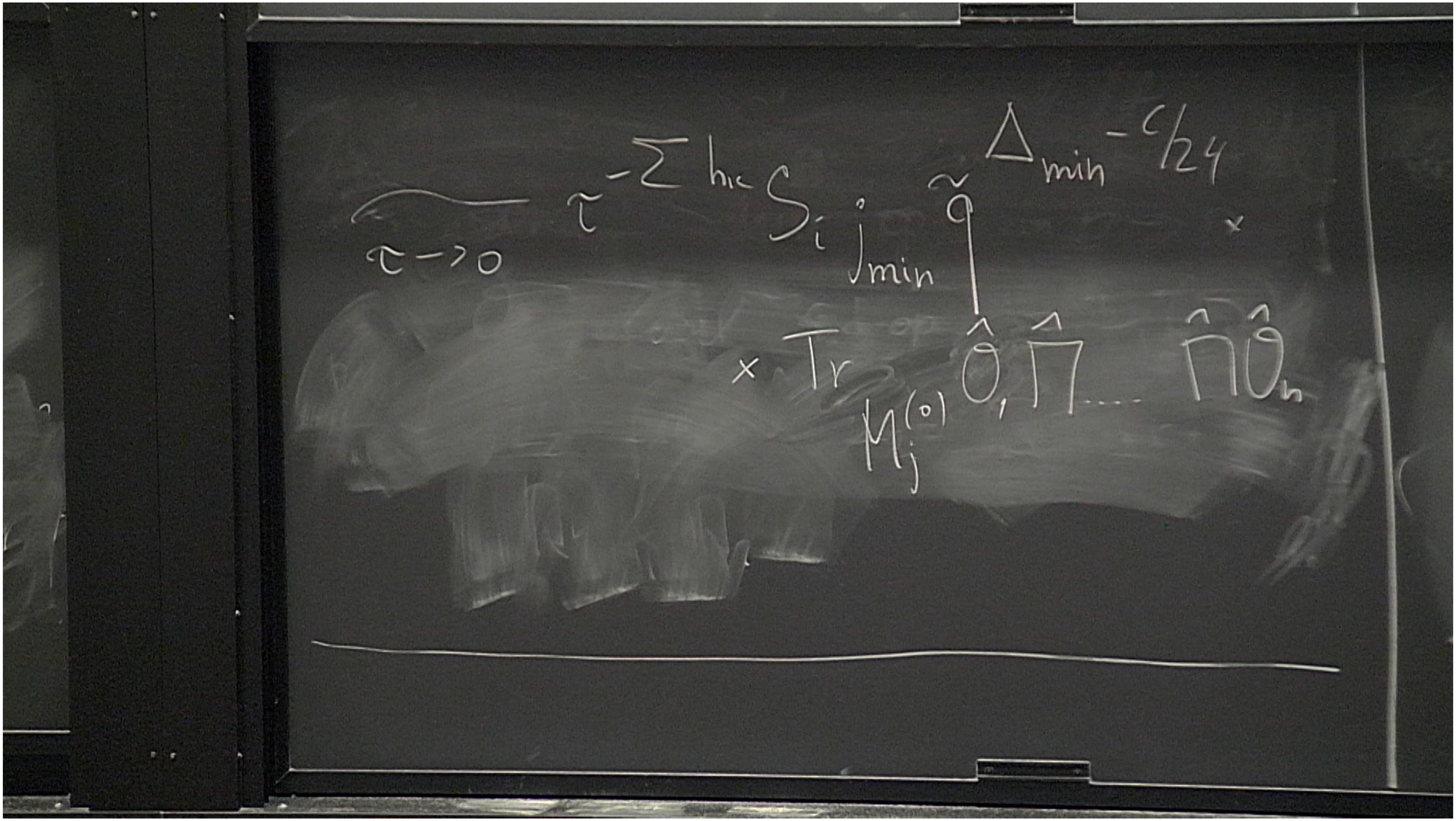
$$q = e^{2\pi i \tau}$$

5) As vector spaces:

$$\mathcal{H}^{\text{schur}}_{3d} \subset \mathcal{H}^{\text{schur}}_{4d}$$

subspace of scalar op's.

(a.k.a. Higgs).


$\xrightarrow[\tau\to 0]{} \tau^{-\sum h_k} S_{i\, j_{min}} \tilde{q}^{\Delta_{min} - c/24}$
$\times \mathrm{Tr}_{M_j^{(0)}} \hat{O}_1 \hat{\Pi} \cdots \hat{\Pi} \hat{O}_n$

$M = V$ var. module. $\mathrm{Tr}_M\, q^{L_0 - \dots}$

$q = e^{2\pi i \tau}$

$\tilde{\tau} = -\frac{1}{\tau}$, $w_i = \frac{\dots}{\tau}$

$\tilde{q} = e^{2\pi i \tilde{\tau}}$

$\tau \to 0$ $\quad j_{min}$

$\times \, \mathrm{Tr}_{M^{(0)}_{j_{min}}} \hat{O}_1 \hat{\Pi} \dots \hat{\Pi} \hat{O}_n$

$M^{(0)}_{j_{min}}$ — lowest L_0 eigensp. of $M_{j_{min}}$

$a \in V$, $\quad [a] \in \mathrm{Zhu}(V)$

(homogen. el.)

$$[a] \longmapsto \hbar^{\deg(a)} \frac{\mathrm{Tr}_{M^{(0)}_{min}}\, o(a)}{\mathrm{Tr}_{M^{(0)}_{min}}\, 1}$$

$$\boxed{T : \mathrm{Zhu}(V) \to \mathbb{C}[\hbar]}$$

$$T(\alpha\beta) =: B_T(\alpha, \beta)$$

$$\mathcal{N} = \ker B_T$$

$$\Downarrow$$

$$[t] - \# \cdot t_h^2$$

$$\boxed{\mathcal{A}_H = \mathrm{Zhu}(V) / \mathcal{N}}$$

$$j_1 = j_2 = \frac{1}{2}$$

$$h = \underset{1}{\underset{\shortparallel}{R}} + j_1 + j_1 = 2$$

$$(\mathcal{A}_H, T_s)$$

zero modes

with mult

$a \in V$

$=$

$o(a$

$\frac{1}{2}$

$j_1 = 2$

T_5

Example affine VOA $V_k(\mathfrak{g})$.

$$j_A = [J_A(z)].$$

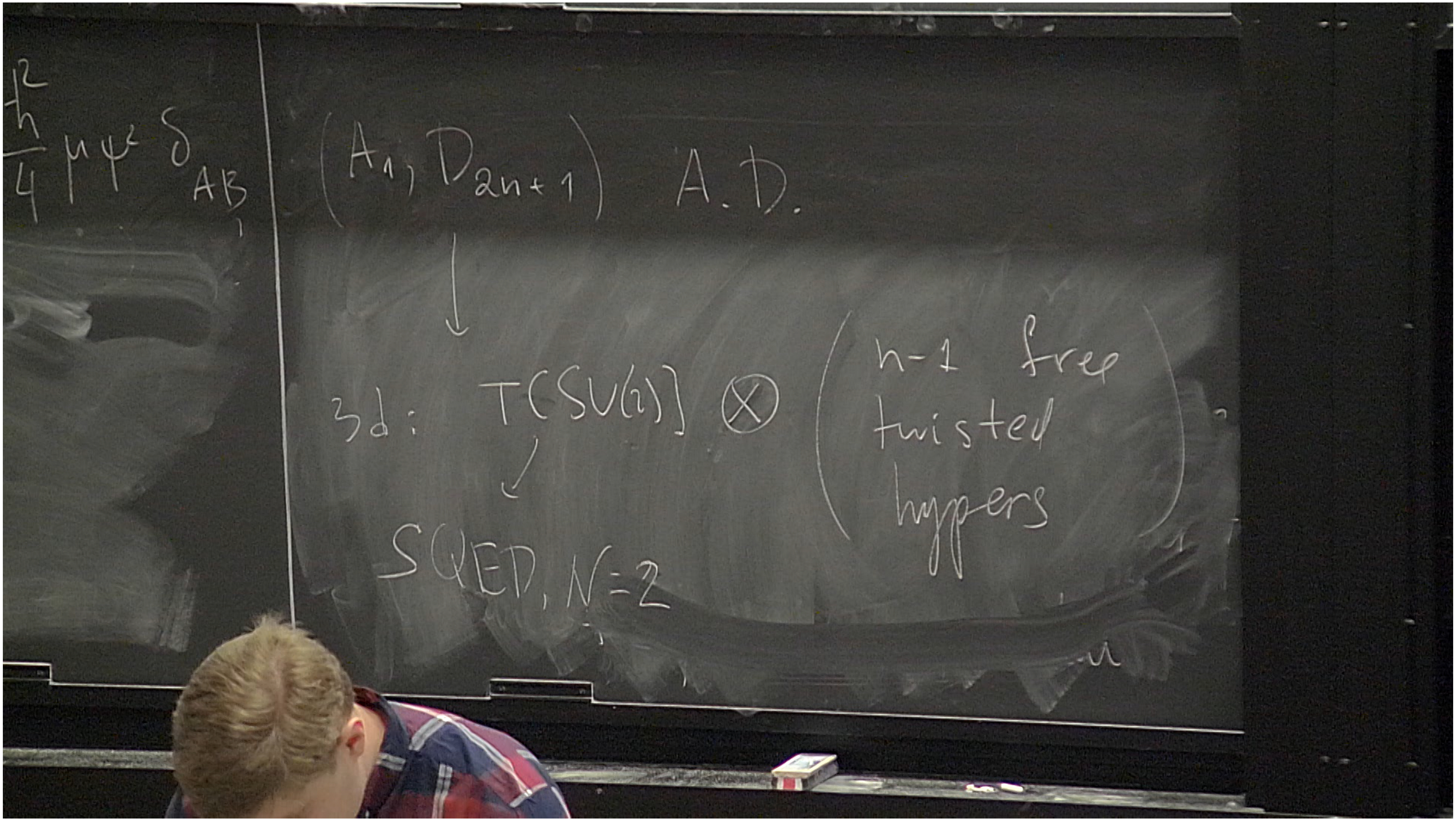
$$j_A * j_B = (j_A j_B) + i\hbar f_{AB}{}^{C} j_C$$

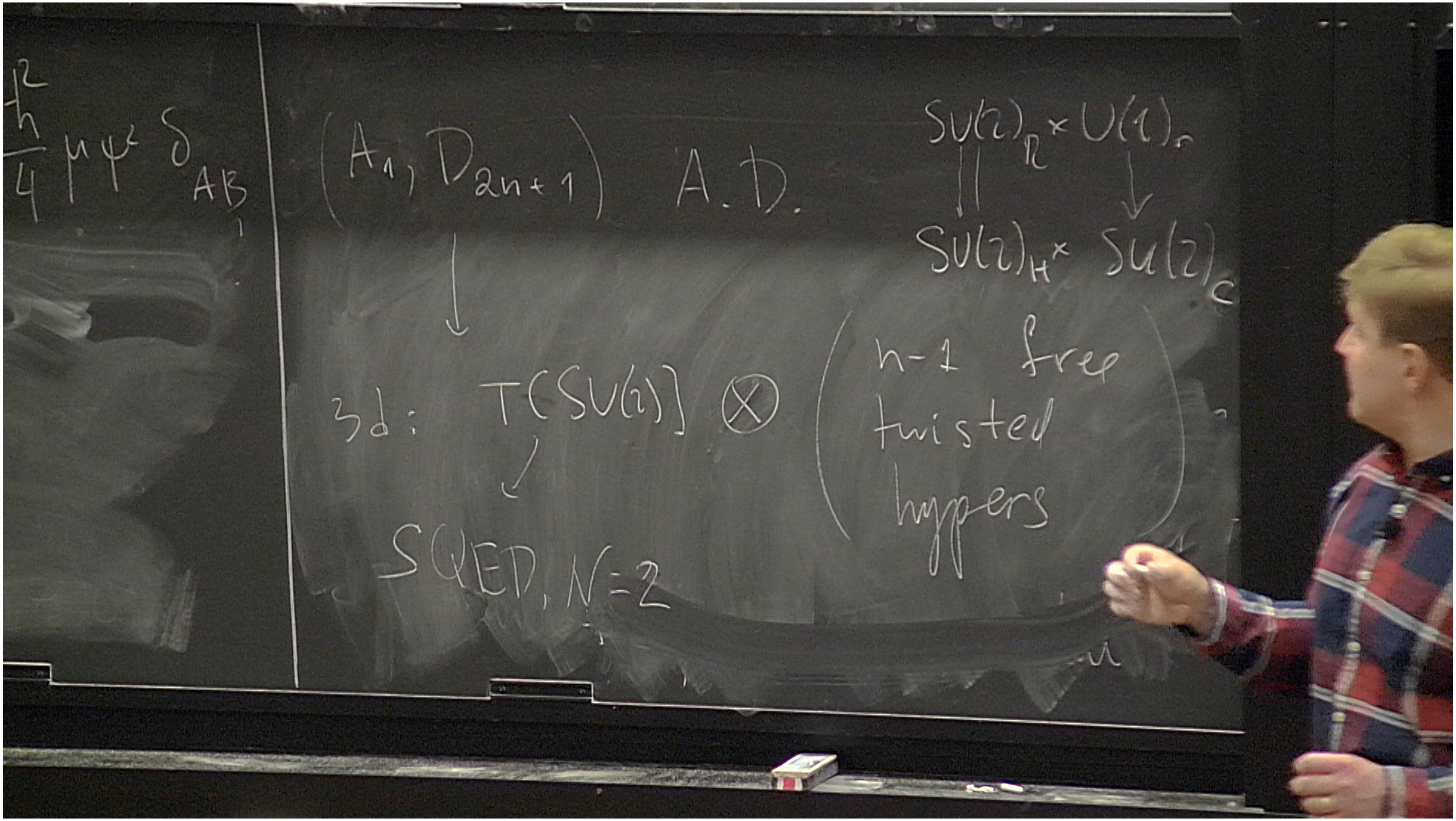
$$t = \frac{1}{\psi^2 (k + h^\vee)} \sum_A (j_A j_A)_{\mathfrak{g}}$$

$$T(t) = \hbar^2 \cdot \Delta_{min}, \quad \underline{t - \hbar^2 \Delta_{min} \in \mathbb{N}}$$

$$j_A \star j_B = :j_A j_B: + \frac{i\hbar}{2} f_{AB}{}^{C}\, j_C + \frac{\hbar^2}{4}\mu\psi^2\delta_{AB}$$

$$\mu = \frac{4(k+h^\vee)\Delta_{min}}{\dim \mathfrak{g}}$$


$\mu \psi^2 \delta_{AB}$
(A_1, D_{2n+1}) A.D.
3d: $T[SU(2)] \otimes$
n-1 free twisted hypers
SQED, N=2


(A1, D2n+1) A.D.
SU(2)R × U(1)r
SU(2)H × SU(2)C
3d: T[SU(2)] ⊗ (n-1 free twisted hypers)
SQED, N=2

$\frac{h^2}{4}\mu\psi^2\delta_{AB}$

(A_1, D_{2n+1}) A.D.

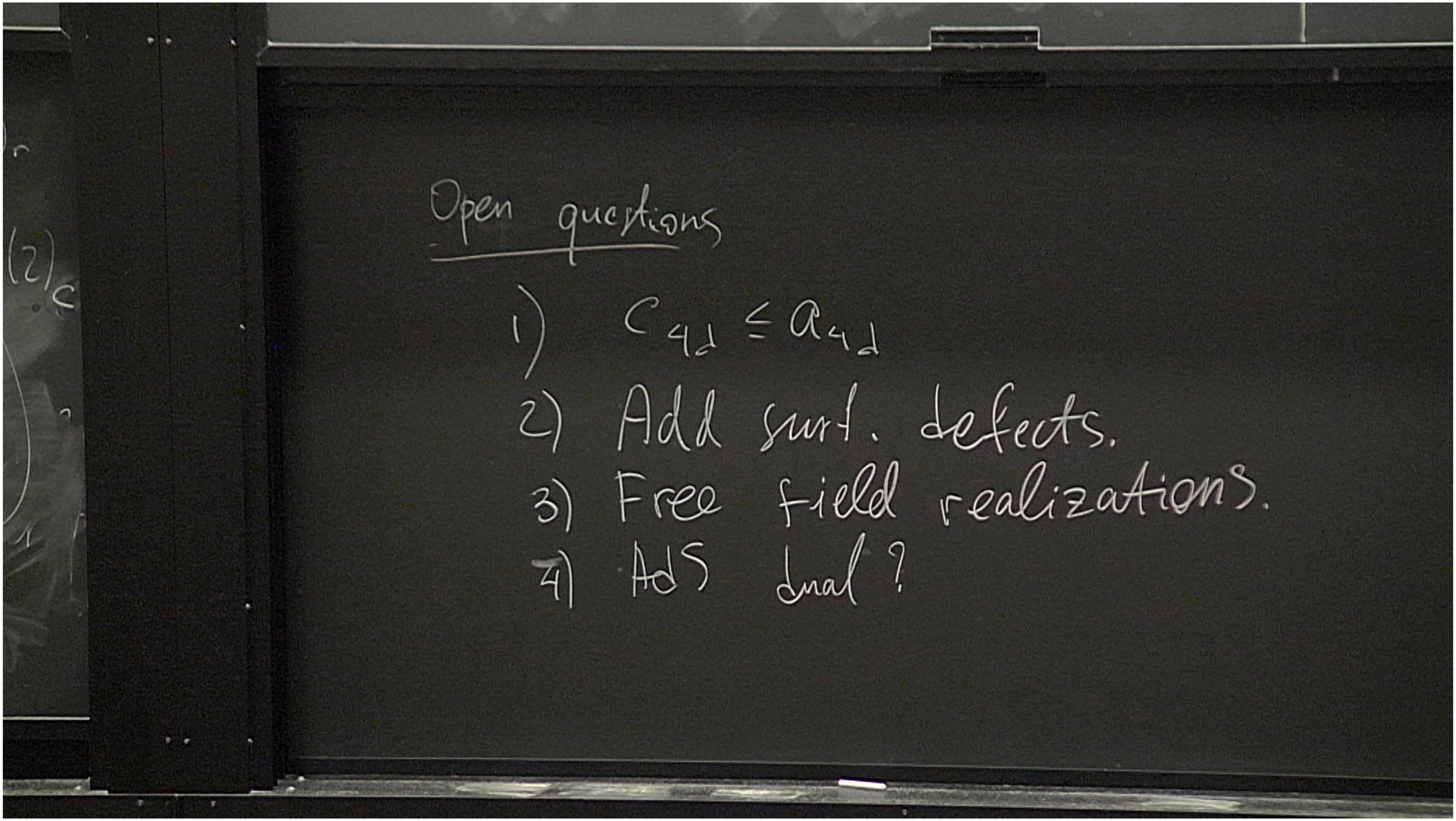
$$SU(2)_R \times U(1)_r \longrightarrow SU(2)_H \times SU(2)_C$$

$$\downarrow$$

3d: $T[SU(2)] \otimes$ (n−1 free twisted hypers)

$$\downarrow$$

SQED, $N=2$


Open questions
1) $c_{4d} \leq a_{4d}$
2) Add surf. defects.
3) Free field realizations.
4) AdS dual?