Title: Quantum Field Theory for Cosmology - Lecture 16

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (Kempf)

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QFT for Cosmology, Achim Kempf, Lecture 14

Note Title

Quantum field theory on FRW spacetimes.

Observations:

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On scales > 1 GLy:

- □ The universe is spatially very flat

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On scales > 1 GLy:

- The universe is spatially very flat
- The cosmic expansion is very isotropic.

Friedmann Robertson Walker (FRW) spacetimes:

- Simplifying approximation:

Spacetime is modeled as having

- no spatial curvature at all.

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Friedmann Robertson Walker (FRW) spacetimes:

- Simplifying approximation:

Spacetime is modeled as having

- no spatial curvature at all.
- entirely isotropic expansion

Remark: It is known that the Einstein equations allow for highly nontrivial evolutions of non-isotropic spacetimes, see, e.g., the text by Wainwright & Ellis.

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Spacetime is modeled as having

- no spatial curvature at all.
- entirely isotropic expansion

Remark: It is known that the Einstein equations allow for highly nontrivial evolutions of non-isotropic spacetimes, see, e.g., the text by Wainwright & Ellis.

There are even solutions that only temporarily get very close to flatness. The Einstein eqns are nonlinear!

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With these assumptions, we choose convenient coordinates:

* Time coordinate t:

Definition: The motion of galaxies due to the cosmic expansion is called the Hubble flow.

Definition: The peculiar velocity is the "small" extra random velocity that galaxies can possess relative to the general Hubble flow.

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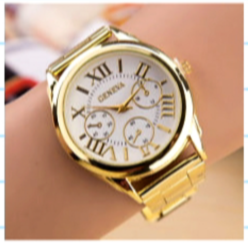

(to a good approximation, you can use your wrist watch on earth)

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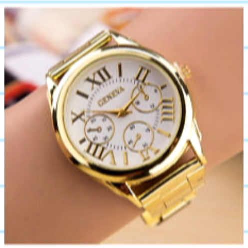

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* Space coordinates:

It is convenient to use "comoving coordinates", x_1, x_2, x_3:

- At one time, t_0, (say today) we set up an ordinary rectangular coordinate system.
- Then, we let our spatial coordinate system shrink or grow to past or future, to match the Hubble flow.

Advantages:

- In the comoving coordinate system, galaxies have constant coordinates, except for possible peculiar motion.

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- At one time, t_0, (say today) we set up an ordinary rectangular coordinate system.
- Then, we let our spatial coordinate system shrink or grow to past or future, to match the Hubble flow.

Advantages:

- In the comoving coordinate system, galaxies have constant coordinates, except for possible peculiar motion.
- Waves keep their wavelengths numerically constant even while they get physically stretched.

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* The metric:

Recall that $ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$

is the invariant 4-distance.

In our coordinates, $g_{\mu\nu}(x)$ must read:

because we use wrist watch "proper" time

$$g_{\mu\nu}(t,\vec{x}) = \begin{pmatrix} 1 & & & \\ & -a^2(t) & & \\ & & -a^2(t) & \\ & & & -a^2(t) \end{pmatrix}$$

because our coordinate system's

* Dynamics of $a(t)$:

The function $a(t)$ is determined by **all** equations of motion:

1. Calculate the energy momentum tensor $T_{\mu\nu}(t,\vec{x})$ contributions of at least the most important fields, say $\varphi_i(x,t)$.

2. Solve, simultaneously:

   * The equations of motion for the fields φ_i
   * The Einstein equation for $g_{\mu\nu}$, while setting $g_{\mu\nu}(t,x) = \begin{pmatrix} 1 & & & \\ & -a^2 & & \\ & & -a^2 & \\ & & & -a^2 \end{pmatrix}$:

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* The Einstein equation for $g_{\mu\nu}$, while setting $g_{\mu\nu}(t,x) = \begin{pmatrix} 1 & & & \\ & -a^2 & & \\ & & -a^2 & \\ & & & -a^2 \end{pmatrix}$:

$$R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x) = 8\pi G\, T_{\mu\nu}(x)$$

* Semi-classical approximation

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* Semi-classical approximation

We can solve these classically, but not quantum mechanically:

Can quantize only φ_i, not $g_{\mu\nu}$.

$\Rightarrow$ need to "make quantum $T_{\mu\nu}(t,\vec{x})$ classical" for Einstein eqn!

$\rightsquigarrow$ One uses: $\bar{T}_{\mu\nu}(x) := \langle \Omega | T_{\mu\nu}(t,\vec{x}) | \Omega \rangle$

Problem: Energy & Momentum are naturally nonlocal because of uncertainty principle.

Remark: $\dot{a}(t)$ is related to curvature between space & time.

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$$\boxed{\text{For now, we will assume that the expansion's scale factor function } a(t) \text{ is given.}}$$

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Convenient Definition: The conformal time coordinate, η.

- Recall that:

$$g_{\mu\nu}(t,\vec{x}) = \begin{pmatrix} 1 & & & 0 \\ & -a^2(t) & & \\ & & -a^2(t) & \\ 0 & & & -a^2(t) \end{pmatrix}$$

- It would be convenient if $g_{\mu\nu}$ were proportional to $\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$.

- This can be achieved by choosing a new time coordinate η, so that time also has a prefactor a^2, i.e., so that:

$$(dt)^2 = a^2(t)(d\eta)^2$$

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$$g_{\mu\nu}(t,\vec{x}) = \begin{pmatrix} \ddots & & \\ & -a^2(t) & \\ 0 & & -a^2(t) \end{pmatrix}$$

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$$(\Delta t)^2 = a^2(t_0)(\Delta \eta)^2$$

- To this end, we need: $a \, d\eta = dt$

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i.e.: $\frac{d\eta}{dt} = \frac{1}{a}$

and therefore $\eta(t) = \int_{t_0}^{t} \frac{1}{a(t')}\, dt'$

t_0 ← yields arbitrary integration constant.

- The variable η is called the "conformal time".

(..because it shows that the FRW spacetime is equivalent to Minkowski space up to time-dependent conformal, i.e., angle-preserving, i.e. scale-factor-only transformations)

- Using conformal time and comoving spatial coordinates the metric reads:

do not mix up

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- This also implies:

$$g^{\mu\nu}(\eta,\vec{x}) = a^{-2}(\eta)\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = a^{-2}(\eta)\,\eta^{\mu\nu}$$

Recall: $g^{\mu\nu} g_{\nu\xi} = \delta^{\mu}{}_{\nu}$, i.e., $g_{\mu\nu}$ and $g^{\mu\nu}$ are inverse to another.

- We easily obtain the integral measure needed for the action:

The Klein Gordon field in FRW spacetimes

- Neglecting a potential $V(\phi)$ for now, we obtain the action of the "free K.G. field on the FRW background":

$$S_{KG} = \int \left(\frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2 \right) \sqrt{|g|}\, d^4x$$

$$\overset{\text{here}}{=} \int \left(\frac{1}{2} a^{-2}(\eta)\, \eta^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2 \right) a^4\, d\eta\, d^3x$$

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Thus, from the general Euler Lagrange equation

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\phi(x) = 0$$

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$$\left(\frac{1}{a^4(\eta)}\frac{\partial}{\partial x^\mu}\eta^{\mu\nu}a^2\frac{\partial}{\partial x^\nu}+m^2\right)\phi(x)=0$$

$a' = \frac{da}{d\eta}$

$$\left(\frac{1}{a^4(\eta)}\eta^{\mu\nu}a^2(\eta)\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}+\frac{1}{a^4(\eta)}2a'a\frac{\partial}{\partial x^0}+m^2\right)\phi(x)=0$$

$$\boxed{\phi''(\eta,\vec{x})+2\frac{a'(\eta)}{a(\eta)}\phi'(\eta,\vec{x})-\Delta\phi(\eta,\vec{x})+a^2(\eta)m^2\phi(\eta,\vec{x})=0}$$

This is the K.G. eqn. in FRW spacetimes!

Problem: the equation above has this general form:

$$\phi''+\times\!\!\times\,\phi'+\times\!\!\times\,\phi=0$$

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$$\boxed{\phi''(\eta,\vec{x}) + 2\frac{a'(\eta)}{a(\eta)}\phi'(\eta,\vec{x}) - \Delta\phi(\eta,\vec{x}) + a^2(\eta)\,m^2\phi(\eta,\vec{x}) = 0}$$

This is the K.G. eqn. in FRW spacetimes!

Problem: the equation above has this general form:

$$\phi'' + \cancel{\times}\,\phi' + \cancel{\times}\,\phi = 0$$

a time-dependent friction-like term that is entirely new.

a term that also occurs in the usual harmonic oscillator. Notice though that it is now time-dependent.

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Strategy: Use a new, re-scaled, field variable χ:

We try to change from $\phi(\eta,\vec{x})$ to a new field variable, say $\chi(\eta,\vec{x})$, so that the equation of motion for χ has no "friction"-type term.

This simple ansatz succeeds:

$$\chi(\eta,\vec{x}) := a(\eta)\,\phi(\eta,\vec{x})$$

Namely:

we have: $\phi' = \frac{\partial}{\partial\eta}\frac{1}{a}\chi = -\frac{a'}{a^2}\chi + \frac{1}{a}\chi'$

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we try to change from $\varphi(\eta,x)$ to a new field variable, say $\chi(\eta,\vec{x})$, so that the equation of motion for χ has no "friction"-type term.

This simple ansatz succeeds:

$$\chi(\eta,\vec{x}) := a(\eta)\,\phi(\eta,\vec{x})$$

Namely:

we have: $\phi' = \frac{\partial}{\partial\eta}\frac{1}{a}\chi = -\frac{a'}{a^2}\chi + \frac{1}{a}\chi'$

and: $\phi_{,i} = \frac{\partial}{\partial x^i}\frac{1}{a(\eta)}\chi(\eta,\vec{x}) = \frac{1}{a}\chi_{,i}$ for $i=1,2,3$

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Using these, the action in terms of χ becomes:

$$S_{KG} = \int \frac{1}{2}\left(\chi'^2 - \sum_{i=1}^{3} \chi_{,i}^2 - \underbrace{\left(m^2 a^2 - \frac{a''}{a}\right)}\chi^2\right) d\eta\, d^3x$$

Note that this term is like a time-dependent mass term $m_{eff}^2(\eta)$

Exercise: verify

Equation of motion:

* Do

$$\frac{\delta S'}{\delta \phi(\eta,\vec{x})} = 0 \quad \text{and} \quad \frac{\delta S'}{\delta \chi(\eta,\vec{x})} = 0$$

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Equation of motion:

* Do

$$\frac{\delta S'}{\delta \phi(\eta,\vec{x})}=0 \quad \text{and} \quad \frac{\delta S'}{\delta \chi(\eta,\vec{x})}=0$$

yield equivalent equations of motion?

* Yes, because:

$$0=\frac{\delta S'}{\delta \phi}=\frac{\delta S}{\delta \chi}\frac{\delta \chi}{\delta \phi}$$

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* Yes, because:

$$0 = \frac{\delta S}{\delta \phi} = \frac{\delta S}{\delta \chi}\frac{\delta \chi}{\delta \phi}$$

if $\delta S/\delta\chi$ vanishes then also $\delta S/\delta\phi$ vanishes.

* Thus, we may calculate the equation of motion directly in terms of χ from $S[\chi]$, to obtain:

Exercise: verify!

$$\chi'' - \Delta\chi + \left(m^2 a^2 - \frac{a''}{a}\right)\chi = 0 \qquad (\text{EoM!})$$

Remark:

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$\partial\varphi \qquad \partial\chi \quad \partial\varphi$

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Exercise: verify!

$$\chi'' - \Delta\chi + \left(m^2a^2 - \frac{a''}{a}\right)\chi = 0 \qquad (\text{EoM 1})$$

Remark:

We could have obtained this equation of motion directly from that of ϕ by change of variable. But finding the action for χ was still worthwhile, namely to get the conjugate to χ!

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* We need the canonically conjugate field

$$\pi^{(\chi)}(\eta,\vec{x})$$

to the field $\chi(\eta,\vec{x})$, i.e., the Legendre transform of χ':

* To this end, we consider the Lagrangian:

$$L = \int \frac{1}{2}\left(\chi'^2 - \sum_{i=1}^{3} \chi_{,i}^2 - \left(m^2 a^2 - \frac{a''}{a}\right)\chi\right) d^3x$$

* Thus, the Legendre transformed variable reads:

$$\pi^{(\chi)}(\eta,\vec{x}) := \frac{\delta L}{\delta \chi'(\eta,\vec{x})} = \chi'(\eta,\vec{x}) \qquad \text{(EoM 2)}$$

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* Which is the field that is conjugate to ϕ ?

$$S_{K.G.} = \int \left(\frac{1}{2} a^{-2}(\eta)\, \eta^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2 \right) a^4 \, d\eta \, d^3x$$

$\Rightarrow$ The field $\pi^{(\phi)}$ which is conjugate to ϕ reads:

$$\pi^{(\phi)} := \frac{\delta \mathcal{L}}{\delta \phi'} = a^2 \phi'$$

* Compare:

$$\pi^{(x)} = x'$$

$$= (a\phi)'$$

$$= a\phi' + a'\phi$$

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Quantization:

$$[\hat\phi(\eta,\vec x),\hat\pi^{(\phi)}(\eta,\vec x')] = i\delta^3(\vec x-\vec x')$$

$$[\hat\phi(\eta,\vec x),\hat\phi(\eta,\vec x')] = 0$$

$$[\hat\pi^{(\phi)}(\eta,\vec x),\hat\pi^{(\phi)}(\eta,\vec x')] = 0$$

Proposition:

In terms of the fields $\hat{x} := a\hat\phi$, $\hat\pi^{(x)} := \hat{x}'$, these commutation relations become:

$$[\hat x(\eta,\vec x),\hat\pi^{(x)}(\eta,\vec x')] = i\delta^3(\vec x-\vec x')$$

$$[\hat x(\eta,\vec x),\hat x(\eta,\vec x')] = 0$$

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$$= [\hat{\phi}(\eta, \vec{x}), \hat{\pi}^{(\phi)}(\eta, \vec{x}')]$$

$$= i\delta^3(\vec{x} - \vec{x}')$$

Thus, the change from ϕ to x is fairly trivial.

Notice, however:

$$L \xrightarrow[\phi' \text{ replaced by } \pi^{\phi}]{L.T.} H^{(\phi)} := \int \phi' \pi^{(\phi)} d^3x - L$$

$$L \xrightarrow[x' \text{ replaced by } \pi^{x}]{L.T.} H^{(\pi)} := \int x' \pi^{(x)} d^3x - L$$

they have no reason to be the same!

$$= \left[\hat{\phi}(\eta,\vec{x}),\ \hat{\pi}^{(\phi)}(\eta,\vec{x}')\right]$$

$$= i\,\delta^3(\vec{x}-\vec{x}')$$

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Explicitly:

* From $\hat{x} = a\hat{\phi}$ and $i\hat{\phi}' = [\hat{\phi}, \hat{H}^{(\phi)}]$ we obtain:

$$i\left(\frac{1}{a}\hat{x}\right)' = \frac{1}{a}[\hat{x}, \hat{H}^{(\phi)}]$$

$$\Rightarrow \quad i\frac{1}{a}\hat{x}' - i\frac{a'}{a^2}\hat{x} = \frac{1}{a}[\hat{x}, \hat{H}^{(\phi)}]$$

$$\Rightarrow \quad i\hat{x}' = [\hat{x}, \hat{H}^{(\phi)}] + i\frac{a'}{a}\hat{x}$$

* But we also have:

$$i\hat{x}' = [\hat{x}, \hat{H}^{(x)}]$$

→ We must have: $\hat{H}^{(x)} \neq \hat{H}^{(\phi)}$

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$$i\left(\frac{1}{a}\hat{x}\right)' = \frac{1}{a}[\hat{x}, \hat{H}^{(\phi)}]$$

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$\Rightarrow$ We must have: $\quad \hat{H}^{(x)} \neq \hat{H}^{(\phi)}$

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$$T^{(\phi)}_{\mu\nu}(\eta,\vec{x}) = \frac{2}{\sqrt{|g|}}\frac{\delta S}{\delta g^{\mu\nu}} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}m^2\phi^2\right]$$

- Consider $T_{00}(\eta,\vec{x})$, which is called the "**energy density**":

Note: In differential geometry, there is also another use of the term "density": For every tensor, say $Q_{\mu\nu}$, there is a so-called "tensor density" $\tilde{Q}_{\mu\nu}$, defined as $\tilde{Q}_{\mu\nu} := Q_{\mu\nu}\sqrt{g}$, which absorbs the obligatory measure factor in integrations.

$$T_{00}(\eta,\vec{x}) = a^{-4}\frac{1}{2}\pi^{(\phi)^2} + \frac{1}{2}\sum_{i=1}^{3}\phi_{,i}^{\ 2} + \frac{a^2}{2}m^2\phi^2 \qquad (T)$$

- **Exercises:**

a) Verify (T).

b) Calculate $H^{(\phi)}$. Notice that $H^{(\phi)}$ is not a scalar.

c) Show that $H^{(\phi)}(\eta) = \int_{\mathbb{R}^3} T^0_{\ 0}(\eta,\vec{x})\sqrt{g}\, d^3x$.

d) Calculate $H^{(a)}(\eta)$.

Proof: Only the first CCR is nontrivial to check:

$$[\hat{x}(\eta,\vec{x}), \hat{\pi}^{(x)}(\eta,\vec{x}')] = [a(\eta)\hat{\phi}(\eta,\vec{x}), \frac{1}{a(\eta)}\hat{\pi}^{(\phi)}(\eta,\vec{x}') + a'(\eta)\hat{\phi}(\eta,\vec{x}')]$$

$$= [\hat{\phi}(\eta,\vec{x}), \hat{\pi}^{(\phi)}(\eta,\vec{x}')]$$

$$= i\delta^3(\vec{x}-\vec{x}')$$

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L.T. x' replaced by π^{x}

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