

Title: Quantum Field Theory for Cosmology - Lecture 15

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (Kempf)

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QFT for Cosmology, Achim Kempf, Lecture 13

Note Title

Recall: The free Klein Gordon quantum field in a generic curved space-time must obey:

$$\dot{\phi}^+(x,t) = \hat{\phi}(x,t), \quad \dot{\pi}^+(x,t) = \hat{\pi}(x,t) \quad (\text{HC})$$

$$i\dot{\phi}(x,t) = [\hat{\phi}(x,t), \hat{H}(t)], \quad i\dot{\pi}(x,t) = [\hat{\pi}(x,t), \hat{H}(t)] \quad (\text{EoM})$$

which can be written in this form:

$$\left(\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\nu} \sqrt{g} g^{\nu\rho} \frac{\partial}{\partial x^\rho} + m^2 \right) \hat{\phi}(x,t) = 0, \quad \hat{\pi}(x,t) = \sqrt{g} g^{\nu\rho} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t) \quad (\text{EoM})$$

And: On all spacelike hypersurfaces, Σ , the CCRs must hold:

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which can be written in this form:

$$\left(\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} g^{\nu\lambda} \sqrt{g} \frac{\partial}{\partial x^\nu} + m^2 \right) \hat{\phi}(x,t) = 0, \quad \hat{\pi}(x,t) = \sqrt{g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t) \quad (\text{EoM})$$

And: On all spacelike hypersurfaces, Σ , the CCRs must hold:

$$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0, \quad [\hat{\pi}(x,t), \hat{\pi}(x',t)] = 0, \quad [\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar \delta^3(x-x') \quad (\text{CCR})$$

We want to show: The following ansatz for $\hat{\phi}(x,t)$ succeeds:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t) a_k + u_k^*(x,t) a_k^*, \text{ with } [a_k, a_k^+] = \delta_{k,k},$$

↑ number-valued
solutions to K.G. eqn.

at least if the spacetime is globally hyperbolic.

So far we showed:

- The HC and EoM obeyed at all time.
- In a fixed coordinate system, CCRs are obeyed $\forall t$ if $\{u_k\}$ obey $\forall t$:

$$\boxed{\sqrt{-g} g^{00} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x^0} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x^0} u_k(x',t) \right) = i \delta^3(x-x')} \quad (\text{W})$$

- Using Darboux's theorem, we showed that there exists

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- In a fixed coordinate system, CCRs are obeyed $\forall t$ if $\{u_b\}$ obey $\forall t$:

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- Using Darboux's theorem, we showed that there exists a set of solutions $\{u_b\}$ so that (W) holds at some time t_0 .

Conservation of the CCRs? This is implied by the self-adjointness of \hat{H} :

□ As always in quantum theory, the time evolution

operator $\hat{U}(t, t_0) = T e^{i \int_{t_0}^t \hat{H}(t') dt'}$ is unitary: $\hat{U}^\dagger = \hat{U}^{-1}$.

□ It allows one to express the time evolution of the observables, such as field operators, through:

$$\hat{\phi}(x, t) = \hat{U}(t, t_0) \hat{\phi}(x, t_0) \hat{U}^\dagger(t, t_0)$$

$$\hat{\pi}(x, t) = \hat{U}(t, t_0) \hat{\pi}(x, t_0) \hat{U}^\dagger(t, t_0)$$

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□ Thus: $[\hat{\phi}(x, t), \hat{\pi}(x', t')] = [\hat{U} \hat{\phi}(x, t_0) \hat{U}^{-1}, \hat{U} \hat{\pi}(x', t_0) \hat{U}^{-1}]$

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□ It allows one to express the time evolution of the observables, such as field operators, through:

$$\hat{\phi}(x, t) = \hat{U}(t, t_0) \hat{\phi}(x, t_0) \hat{U}^*(t, t_0)$$

$$\hat{\pi}(x, t) = \hat{U}(t, t_0) \hat{\pi}(x, t_0) \hat{U}^*(t, t_0)$$

□ Thus: $[\hat{\phi}(x, t), \hat{\pi}(x', t)] = [\hat{U} \hat{\phi}(x, t_0) \hat{U}^{-1}, \hat{U} \hat{\pi}(x', t_0) \hat{U}^{-1}]$

$$= \hat{U} [\hat{\phi}(x, t_0), \hat{\pi}(x', t_0)] \hat{U}^{-1}$$

$$= \hat{U} i \delta^3(x-x') \hat{U}^{-1} = i \delta^3(x-x')$$

Problem:

Is the quantization coordinate system independent?

Assume we solve the theory as above.

Now if we change coordinate system, and therefore the choices of $\{\Sigma\}$, would the CCRs still hold on every spacelike hypersurface Σ' ?

Proposition: Yes: if CCRs hold in one coordinate system, then they hold in all: The CCRs keep holding when deforming a Σ to a Σ' .

Proof: Rewrite the symplectic form (f, h) more abstractly:

Recall: $f, g \in V$ are solutions of KG eqn.

$$(f, h) := \int_{\Sigma} d\Sigma_r \sqrt{g} g^{\mu\nu} (f \partial_\nu h - h \partial_\nu f)$$

- $\int \downarrow$ a differential 3-form (Recall: Only 3-forms have 3-dim integrals)

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$$= \int_{\Sigma} j$$

a differential 3-form (Recall: Only 3-forms have 3-dim integrals)

Here, we defined the contravariant vector field

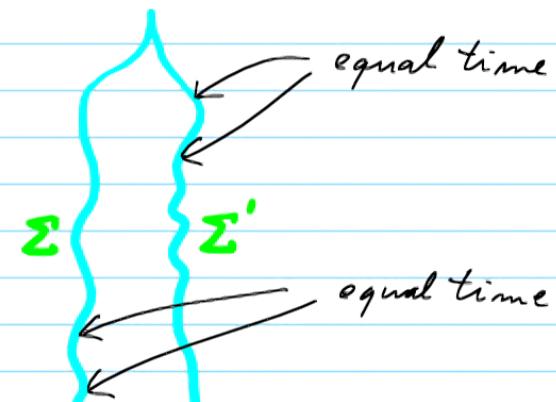
$$j^\nu(x,t) := g^{\mu\nu} (f \partial_\mu h - h \partial_\mu f)$$

and from it the differential 3-form:

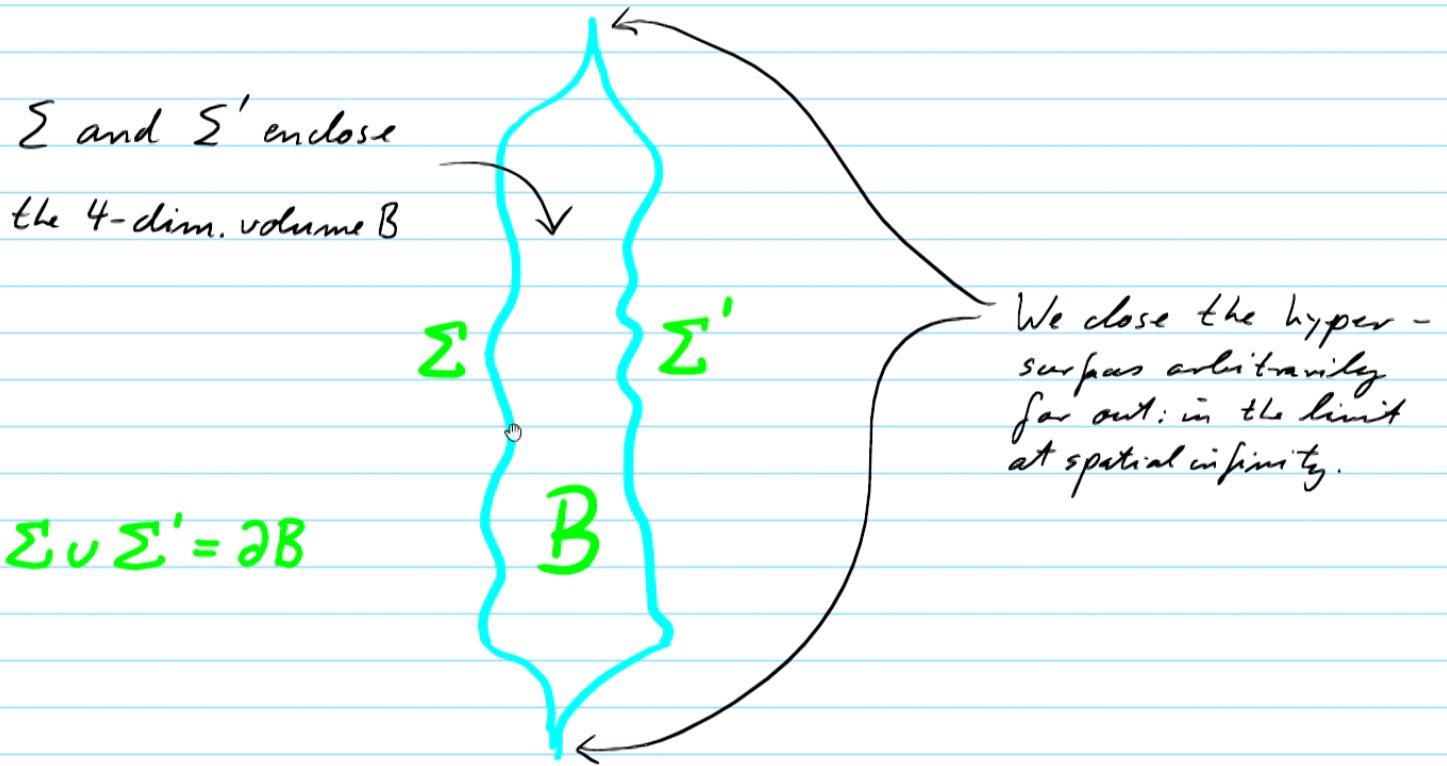
$$\tilde{j} := i_j \Omega$$

↑ inner derivation
3-form ↑ ↓ Volume 4-form $\sqrt{g} \Omega d^4x$

We need to show that the value of the symplectic form stays the same when deforming Σ :



□ Now integrate over both Σ and Σ' :



□ Use Stokes' theorem:

$$\int \tilde{j} = \int d\tilde{j}$$

□ Notice:

If we can show $d\tilde{j} = 0$ we are done!

That's because then:

$$0 = \int_{\Sigma \cup \Sigma'} \tilde{j} = \int_{\Sigma} \tilde{j} + \int_{\Sigma'} \tilde{j} = - \int_{\Sigma} \tilde{j} + \int_{\Sigma'} \tilde{j}$$



Both \tilde{j} pointing
out of B , i.e one
to the future one to
the past.

Both \tilde{j} future
pointing.

$\Rightarrow \int_S \tilde{j}$ is indeed indep. of choice of Σ ,

□ Indeed:

$$d\tilde{j} = d(i_j \Omega) = \text{div}_n j = (\sqrt{g} g^{\mu\nu} j^\nu)_{,\mu} d^4x$$

Hence:

$$(\sqrt{g} g^{\mu\nu} j^\nu)_{,\mu} \stackrel{(\text{by definition of } j)}{=} (\sqrt{g} g^{\mu\nu} (f \partial_\nu h - h \partial_\nu f))_{,\mu}$$

$$= \cancel{\sqrt{g} g^{\mu\nu} \partial_\nu h \partial_\mu f} + f \cancel{(\sqrt{g} g^{\mu\nu} \partial_\nu h)_{,\mu}}^{= -m^2 h \sqrt{g}}$$

$$- \cancel{\sqrt{g} g^{\mu\nu} \partial_\mu h \partial_\nu f} + \cancel{h (\sqrt{g} g^{\mu\nu} \partial_\nu f)_{,\mu}}^{= -m^2 f \sqrt{g}}$$

Recall:

$(\Box + m^2) \phi = 0$ reads:

$$\frac{1}{\sqrt{g}} (\sqrt{g} g^{\mu\nu} \partial_\mu \phi)_{,\nu} + m^2 \phi = 0$$

and the f and h are

$$\text{and } f = -m^2 K \phi \approx 1$$

→ We finally proved that, for globally hyperbolic spacetimes, there always exist mode functions $\{u_k(x, t)\}$ so that our ansatz for $\hat{\phi}$ and $\hat{\Pi}$ also obeys the CCRs at all time and indeed $\forall \Sigma$:

$$\sqrt{|g|} g^{00} \int \left(u_k(x, t) \frac{\partial}{\partial x^0} u_k^*(x, t) - u_k^*(x, t) \frac{\partial}{\partial x^0} u_k(x, t) \right) d^3k = i\delta^3(x-x') \quad (W)$$

Example: For Minkowski space, we had found this solution for the noninteracting Klein-Gordon field:

$$\hat{\phi}(x, t) = \int_{\mathbb{R}^3} \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{-i\omega_k t + ikx} + a_k^* e^{i\omega_k t - ikx} \right) d^3k$$

$-i\omega_k t + ikx$

$$\sqrt{g} g^{00} \int \left(u_k(x,t) \frac{\partial}{\partial x^0} u_k^*(x,t) - u_k^*(x,t) \frac{\partial}{\partial x^0} u_k(x,t) \right) d^3k = i \delta^3(x-x') \quad (W)$$

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We read off: $u_k(x,t) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k t + ikx}$

Now: Verify the CCR condition, (w):

□ Here: $\nabla' g^{\mu\nu} = 1$ and $g^{\mu\nu} = \delta_{\mu\nu}$.

□ Thus, the LHS of Eq. (w) reads:

$$\int u_k(x,t) \frac{\partial}{\partial x^\mu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\mu} u_k(x',t) d^3k$$

$$= \frac{1}{(2\pi)^3} \int \frac{1}{2\omega_k} \left[e^{-i\omega_k t + i k x} (i\omega_k) e^{i\omega_k t - i k x'} - e^{i\omega_k t - i k x} (-i\omega_k) e^{-i\omega_k t + i k x'} \right] d^3k$$

Summary so far:

□ To solve the QFT of a free KG field on curved spacetime is to solve the HC, EoM and CCRs.

□ Make solution ansatz:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t) \alpha_k + u_k^*(x,t) \alpha_k^* \quad (\text{A})$$

or integral, e.g., if no IR cutoff

□ We showed that at least if spacetime is globally hyperbolic:

□ There exists a set of solutions of the KG eqn, $\{u_k\}$, so that ansatz (A) solves HC, EoM and CCR for all time.

Q: Does there exist only one such set of solutions?

A: No, there exist many other such sets of solutions: $\{\bar{u}_k\}, \{\tilde{u}_k\} \dots$

How to see this non-uniqueness?

□ Recall symplectic form for $f, h \in V$:

$$(f, h) := \int_{\Sigma} d\Sigma, \sqrt{g} g^{\mu\nu} (f \partial_{\mu} h - h \partial_{\mu} f)$$

□ Darboux: There exists a basis $\{v_m\}$ of V in which the form (\cdot, \cdot) reads:

$$\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \\ 0 & & & \ddots \end{pmatrix}$$

□ However: Darboux bases are not unique!

□ Example: 2-dim. solution subspace.

Assume: $v_1, v_2 \in V$ are a Darboux basis, so

so that (\cdot, \cdot) reads $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Q: Can we change basis, $v_i := Q \bar{v}_i$, so that (\cdot, \cdot) keeps that matrix form? Is there a matrix Q so that

$$Q^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ?$$

$$\left. \begin{array}{l} \text{Since } (v, w) = v^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} w \text{ we require} \\ v^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} w = \bar{v}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{w} \quad \forall \bar{v}, \bar{w} \\ (\beta \bar{v})^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\beta \bar{w}) = \bar{v}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{w} \quad \forall \bar{v}, \bar{w} \\ \text{i.e.: } \beta^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array} \right\} \rightarrow$$

A: Yes, any change of basis $Q = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

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A: Yes, any change of basis $Q = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

with $ad - bc = 1$ will do. (Exercise: check)

□ More generally: Also different boxes $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ can get mixed!



Exercise: □ Recall that with respect to the hermitian bi-linear form $\langle \cdot, \cdot \rangle$ of Lecture 12, the mode functions $\{u_n\}$ obey:

$$\begin{aligned}\langle u_n, u_m \rangle &= \delta_{nm} \\ \langle u_n^*, u_m^* \rangle &= -\delta_{nm} \\ \langle u_n, u_m^* \rangle &= 0 = \langle u_n^*, u_m \rangle\end{aligned}\quad \left.\right\} (*)$$

□ Now consider an invertible change of basis (in the space of complex-number-valued solutions of the K.B. eqn) to new mode functions:

$$\bar{u}_n := \sum_m (A_{nm} u_m + B_{nm} u_m^*)$$

□ Show that for the $\{\bar{u}_n\}$ to qualify as mode functions, i.e., for them to obey $(*)$, i.e., $\langle \bar{u}_n, \bar{u}_m \rangle = \delta_{nm}$ etc, A, B must obey:

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$$A^* A - B^t B^* = 1 \text{ and } A^* B - B^t A^* = 0$$



Non-uniqueness of the solution (A):

□ Clearly, this means that there are infinitely many solutions of the form of (A) to HC, EoM and CCRs:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t) \alpha_k + u_k^*(x,t) \alpha_k^*$$

$$\hat{\phi}(x,t) := \sum_k \bar{u}_k(x,t) \bar{\alpha}_k + \bar{u}_k^*(x,t) \bar{\alpha}_k^*$$

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□ Correspondingly, we obtain different Fock bases:

Either: $\alpha_k |0\rangle = 0 \quad |n_k\rangle := \frac{1}{\sqrt{n!}} (\alpha_k^*)^n |0\rangle$

$$\hat{\phi}(x,t) := \sum_k \bar{u}_k(x,t) \bar{a}_k + \bar{u}_k^*(x,t) \bar{a}_k^* \quad \text{etc, etc...}$$

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Or: $\bar{a}_k |\bar{0}\rangle = 0 \quad |\bar{n}_k\rangle := \frac{1}{\sqrt{n!}} (\bar{a}_k^*)^n |\bar{0}\rangle \quad \text{etc, etc...}$

□ Q: Do these solutions of the QFT

○ describe different physics, or

o do they differ by a mere change of basis in Fock space and so describe the same physics?

□ A: It depends!

1) Assume first we can impose IR and UV cutoffs with negligible consequences

* This means we truncate to a finite (though large) number of independent mode oscillators, a_n, a_n^* .

* Then, the following theorem implies that all solutions to HC, EOM, CCR differ merely by a change of basis:



Theorem (Stone and von Neumann):

* Assume in a Hilbert space, \mathcal{K} , the operators \hat{x}_i, \hat{p}_j obey:

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \quad [\hat{x}_i, \hat{x}_j] = 0 = [\hat{p}_i, \hat{p}_j] \quad i, j \in \{1, \dots, N\}$$

* Assume that in a Hilbert space $\tilde{\mathcal{K}}$ other operators \tilde{x}_i, \tilde{p}_j also obey:

$$[\tilde{x}_i, \tilde{p}_j] = i\delta_{ij} \quad [\tilde{x}_i, \tilde{x}_j] = 0 = [\tilde{p}_i, \tilde{p}_j] \quad i, j \in \{1, \dots, N\}$$

* Assume that the representations are irreducible (i.e., no invariant subspace)

We can assume that $\mathcal{K}' = \mathcal{K}$ because all separable Hilbert spaces (the usual: with countable bases) are unitarily equivalent.

Then: there exists a unitary operator \hat{U} so that:

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Then: there exists a unitary operator \hat{U} so that:

$$\hat{x}_i = U \hat{x}_i U^+ \quad \hat{p}_i = U \hat{p}_i U^+ \quad (\text{i.e., a change of basis})$$

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 The pathological cases can be avoided by requiring representations of the CCRs of the (bounded and therefore better behaved) operators:

$$e^{i\alpha_i \hat{x}_i}, e^{i\beta_i \hat{p}_i}$$

* Application to QM and to UV&IR regularized QFT:

Consider $\hat{x}_n := \frac{1}{\sqrt{2}} (a_n + a_n^+)$, $\hat{p}_n := \frac{-i}{\sqrt{2}} (a_n - a_n^+)$

and $\bar{x}_n := \frac{1}{\sqrt{2}} (\bar{a}_n + \bar{a}_n^+)$, $\bar{p}_n := \frac{-i}{\sqrt{2}} (\bar{a}_n - \bar{a}_n^+)$ etc.

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The theorem of Stone & v. Neumann implies that

$$\hat{A} = \hat{A}_1 \dots \hat{A}_n$$

2.) Consider now the possibility that we cannot truncate to a finite number of degrees of freedom.

Q: When would this happen?

A: E.g., phase transitions formally need systems with an infinite number of degrees of freedom.

Then: The QFT can have unitarily non-equivalent solutions, that differ physically: different "phases".

Underlying math of non-equivalent representations?

Assume $\langle a | b \rangle = d$ with $0 < d < 1$, i.e., not \perp

Then $(\underbrace{ka|ka|ka|...|a|}_{N})(\overbrace{|b\rangle |b\rangle |b\rangle ... |b\rangle}^N) = d^N$, i.e., not \perp

But for $N = \infty$ have $|a\rangle |a\rangle ... |a\rangle \perp |b\rangle |b\rangle ... |b\rangle$, so that then

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But for $N=\infty$ have $|a\rangle|a\rangle|\dots|a\rangle \perp |b\rangle|b\rangle|\dots|b\rangle$, so that then can no longer use $|a\rangle|a\rangle|\dots|a\rangle$ to help linearly combine, e.g., $|b\rangle|b\rangle|\dots|b\rangle$.

From now on: We will assume IR & UV cutoffs are possible and that Stone v. Neumann therefore applies.

Therefore:

□ No matter which set of suitable mode functions

$$\{u_n(x,t)\} \text{ or } \{\bar{u}_n(x,t)\} \text{ or } \{\bar{\bar{u}}_n(x,t)\}, \dots$$

we choose, we obtain the same solution

$$\begin{aligned}\hat{\phi}(x,t) &= \sum_k u_k(x,t) \alpha_k + u_k^*(x,t) \alpha_k^* \\ &= \sum_k \bar{u}_k(x,t) \bar{\alpha}_k + \bar{u}_k^*(x,t) \bar{\alpha}_k^* \\ &= \sum_k \bar{\bar{u}}_k(x,t) \bar{\bar{\alpha}}_k + \bar{\bar{u}}_k^*(x,t) \bar{\bar{\alpha}}_k^* = \dots\end{aligned}$$

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$$\begin{aligned}\hat{\phi}(x,t) &= \sum_k u_k(x,t) \alpha_k + u_k^*(x,t) \alpha_k^* \\ &= \sum_k \bar{u}_k(x,t) \bar{\alpha}_k + \bar{u}_k^*(x,t) \bar{\alpha}_k^* \\ &= \sum_k \bar{\bar{u}}_k(x,t) \bar{\bar{\alpha}}_k + \bar{\bar{u}}_k^*(x,t) \bar{\bar{\alpha}}_k^* = \dots\end{aligned}$$

with their Fock bases being different bases in the same Hilbert space.



□ For example, using the $\{u_n\}$, we are led to span the Hilbert space \mathcal{H} using this ON basis:

$$|0\rangle \text{ where } a_k |0\rangle = 0 \quad \forall k$$

$$a_k^+ |0\rangle, \frac{1}{\sqrt{n!}} (a_k^+)^n |0\rangle$$

$$\frac{1}{\sqrt{\dots}} (a_{k_1}^+)^{i_1} \cdots (a_{k_r}^+)^{i_r} |0\rangle, \text{ etc ...}$$

□ Or, using other mode functions, say $\{\bar{u}_n\}$, we may span the same Hilbert space, \mathcal{H} , using this ON basis:

$$|\bar{0}\rangle \text{ where } \bar{a}_k |\bar{0}\rangle = 0 \quad \forall k$$

$$\bar{a}_k^+ |\bar{0}\rangle, \frac{1}{\sqrt{n!}} (\bar{a}_k^+)^n |\bar{0}\rangle$$



|0> |1> |2> |3> |4>

Does the choice of mode functions matter?

- In principle, it does not:

Any state of the system, say $|4\rangle$, can be expanded in each basis.

- In practice, however:

It is convenient, whenever we know which state is the no-particle (i.e., vacuum) state, say $|0\rangle$, to choose the mode functions $\{u_k\}$ such that the corresponding $|0\rangle$ is $|0\rangle$, i.e., such that

$$|0\rangle = |0\rangle, \text{ i.e., such that } a_k |0\rangle = 0$$

- - - - -

Any state of the system, say $|F\rangle$, can be expanded in each basis.

□ In practice, however:

It is convenient, whenever we know which state is the no-particle (i.e., vacuum) state, say $|\Omega\rangle$, to choose the mode functions $\{u_n\}$ such that the corresponding $|0\rangle$ is $|\Omega\rangle$, i.e., such that

$$|0\rangle = |\Omega\rangle, \text{ i.e., such that } a_n |\Omega\rangle = 0$$

Then, conveniently, states like $\frac{1}{\sqrt{n!}} (a_n^*)^n |0\rangle$ are the multi-particle states.



Outlook: (only a rough sketch)

- Say we know the system's state, $|\Psi\rangle$, is the vacuum initially.
- \rightsquigarrow We choose $\{u_n\}$ appropriately, so that $|0\rangle_{in} = |\Psi\rangle$.
- After some evolution (e.g. the universe expands) the vacuum state may be a different state, say $|X\rangle$.
- \rightsquigarrow We choose $\{\bar{u}_n\}$ appropriately, so that $|\bar{0}\rangle_{out} = |X\rangle$
- At late times, since we work in the Heisenberg picture, the system is still in the state $|0_{in}\rangle$, but this is then an excited state!
 - \rightsquigarrow Description of particle production due to cosmic expansion.

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□ Recall: We had an analogous situation with driven harmonic oscillators!



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- Recall: We had an analogous situation with driven harmonic oscillators!