Title: Quantum Field Theory for Cosmology - Lecture 12

Speakers: Achim Kempf

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QFT for Cosmology, Achim Kempf, Lecture 12

Note Title

## We are now ready to $2^{nd}$ quantize:

SR, $1^{st}$Q Hamiltonian formalism $\xrightarrow[\text{(equivalence)}]{\text{step 1: Legendre transform}}$ SR, $1^{st}$Q. Lagrangian formalism

step 2 ↓ allow curvature

GR, $1^{st}$Q Hamiltonian formalism $\xleftarrow[\text{(equivalence)}]{\text{step 3: Legendre transform}}$ GR, $1^{st}$Q Lagrangian formalism

GR, $1^{st}$Q Hamiltonian formalism ↓ step 4 GR, $2^{nd}$Q

GR, $1^{st}$Q Lagrangian formalism ⇣ GR, $2^{nd}$ Q

SR, $1^{st}$Q
Hamiltonian
formalism

step 1
Legendre transform
(equivalence)

SR, $1^{st}$Q.
Lagrangian
formalism

step 2
allow
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GR, $1^{st}$Q
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GR, $1^{st}$Q
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step 4

GR, $2^{nd}$Q
Hamiltonian
formalism

Dyson Schwinger eqns are same
(equivalence)

GR, $2^{nd}$ Q
Lagrangian formalism
(Path integral of QFT)

# Solving the quantized theory is to solve:

1.) Commutation relations:

$$\left.\begin{aligned} [\hat{\phi}(x,t), \hat{\pi}(x',t)] &= i\hbar\,\delta^3(x-x') \\ [\hat{\phi}(x,t), \hat{\phi}(x',t)] &= 0 \\ [\hat{\pi}(x,t), \hat{\pi}(x',t)] &= 0 \end{aligned}\right\} \quad \text{(CCRs)}$$

2.) Hermiticity:

$$\hat{\phi}^{+}(x,t) = \hat{\phi}(x,t), \qquad \hat{\pi}^{+}(x,t) = \hat{\pi}(x,t) \qquad \text{(HC)}$$

## 3.) Equations of motion:

In the Heisenberg picture, they are formally unchanged:

$$\frac{d}{dt}\hat{f}(\phi,\pi) = \frac{1}{i\hbar}[\hat{f},\hat{H}] \qquad \text{for } \hat{f} = \hat{\phi},\ \hat{f} = \hat{\pi},\ \text{etc}$$

**Namely:**

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \qquad \text{(EoM1)}$$

and:

$$\hat{\pi}(x,t) = \sqrt{|g|}\, g^{0\nu}\frac{\partial}{\partial x^\nu}\hat{\phi}(x,t) \qquad \text{(EoM2)}$$

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# How to solve the CCR, HC and EoM equations?

The $a_k, a_k^+$ take care of the CCRs

**Recall:** the solution we obtained on Minkowski space:

$$\hat{\phi}(x,t) = \int_{\mathbb{R}^3} \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left( e^{-i\omega_k t + ikx}\, a_k + e^{i\omega_k t - ikx}\, a_k^+ \right) d^3k$$

Number-valued solutions to the K.G. equation

and $\hat{\pi}(x,t) = \dot{\hat{\phi}}(x,t)$

**Strategy:**

* ensure hermiticity, HC, by construction
* separate the CCR and EoM problems:

**Ansatz:**

$$\boxed{\begin{aligned} \hat{\phi}(x,t) &:= \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+ \\ \hat{\pi}(x,t) &:= \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t) \end{aligned}}$$

(k need not be a "momentum"!)

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- Here, we use the easy-to-construct operators that obey

$$[a_k , a^+_{k'}] = \delta_{k,k'}$$

- And, we use some number-valued functions $u_k(x,t)$ which are some yet-to-be-determined solutions to the first eqn. of motion, EoM1, i.e. to the Klein Gordon equation, called the **Mode Functions**.

- Try out the ansatz:

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*Number-valued solutions to the K.G. equation*

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(*k need not be a "momentum"!*)

- Try out the ansatz:

  * Hermiticity: ✓

    (HC) holds by construction.

  * The $1^{st}$ equation of motion: ✓

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^{\mu}}\, g^{\mu\nu}\sqrt{|g|}\,\frac{\partial}{\partial x^{\nu}} + m^2\right)\hat{\phi}(x,t) = 0 \qquad \text{(EoM1)}$$

* The 1st equation of motion: ✓

$$\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{|g|}\frac{\partial}{\partial x^\nu} + m^2\right)\hat{\phi}(x,t) = 0 \quad \text{(EoM1)}$$

This eqn holds because in our ansatz,

$$\hat{\phi}(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

the $a_k$ are constant operators while the functions $u_k(x,t)$ are assumed to solve (EoM1).

* The 2nd equation of motion: ✓

## Checking the CCRs:

$$[\hat\phi(x,t), \hat\pi(x',t)] \overset{!}{=} i\hbar\, \delta^3(x-x')$$

- Express $\hat\phi$ in terms of the ansatz:

$$\hat\phi(x,t) := \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+$$

- Express $\hat\pi$ in terms of the ansatz:

$$\hat\pi(x,t) := \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat\phi(x,t)$$

$$\Rightarrow \hat\pi(x,t) = \sqrt{|g|}\, g^{0\nu} \sum_k \left[ \left(\frac{\partial}{\partial x^\nu} u_k(x,t)\right) a_k + \left(\frac{\partial}{\partial x^\nu} u_k^*(x,t)\right) a_k^+ \right]$$

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## Now check CCR:

$$\left[\hat{\phi}(x',t),\ \hat{\pi}(x,t)\right]$$

$$= \left[\sum_k u_k(x',t)\, a_k + u_k^*(x',t)\, a_k^+,\ \sqrt{|g|}\, g^{0\nu} \sum_{k'} \left( \left(\frac{\partial}{\partial x^\nu} u_{k'}(x,t)\right) a_{k'} + \left(\frac{\partial}{\partial x^\nu} u_{k'}^*(x,t)\right) a_{k'}^+ \right)\right]$$

$$= \sqrt{|g|}\, g^{0\nu}(x,t) \sum_{k,k'} \left( u_k(x',t) \frac{\partial}{\partial x^\nu} u_{k'}^*(x,t) - u_k^*(x',t) \frac{\partial}{\partial x^\nu} u_{k'}(x,t) \right) \delta_{k,k'}$$

$$= \sqrt{|g|}\, g^{0\nu} \sum_k \left( u_k(x',t) \frac{\partial}{\partial x^\nu} u_k^*(x,t) - u_k^*(x',t) \frac{\partial}{\partial x^\nu} u_k(x,t) \right) \stackrel{!}{=} i\, \delta^3(\vec{x} - \vec{x}')$$

When do such $\{u_n(x,t)\}$ exist? I.e., when does the ansatz succeed?

<u>Proposition:</u> ▫ Assume spacetime is "globally hyperbolic",

i.e., that it possesses a foliation by Cauchy surfaces,

i.e., that it is topologically of the form:

$\mathbb{R} \times \mathcal{M}$

↑ any 3-dim differentiable manifold

▫ In this case, spacetime possesses no closed timelike curves (no travel into the past), i.e., initial conditions set on the Cauchy surfaces determine the solution everywhere.

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**Proof:**

- Consider the vector space, $V$, of all real-valued solutions of the Klein Gordon equation.

- We define a bi-linear form $( \, , \, )$ on $V$. For all $f, h \in V$:

$$(f,h) := \int_\Sigma d\Sigma_\mu \sqrt{g}\, g^{\mu\nu} \left( f \partial_\nu h - h \partial_\nu f \right)$$

$\Sigma$ ← any spacelike hypersurface i.e. set of points of equal time.

- Proposition: $(f,h)$ is independent of choice of $\Sigma$.

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- $(f,h)$ is a symplectic form, i.e.: $(f,h) = -(h,f)$.

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solutions of the Klein Gordon equation.

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☐ What can we do with $( \, , \, )$? No diagonalization?

## Theorem (Darboux):

For any nondegenerate symplectic form $( \, , \, )$, there exists a basis $\{V_n\}$ such that, in this basis, $( \, , \, )$ takes the matrix form:

$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & & \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & 0 \\ 0 & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \\ & & & \ddots \end{pmatrix}$$

i.e., such that $(V_{2n}, V_{2n+1}) = 1$, $(V_{2n+1}, V_{2n}) = -1$ and all other pairings vanish.

☐ Thus, if we expand $a, b \in V$ as: $\overset{V \ni}{a} = \overset{\mathbb{R} \ni}{a_n} \overset{V \ni}{V_n}$, $b = \overset{\mathbb{R} \ni}{b_n} V_n$

Then: $(a,b) = \sum^{\infty} a_{2n} b_{2n+1} - a_{2n+1} b_{2n}$

- Now assume we picked such a basis $\{v_n\}$ in $V$.
- **Recall:** $V$ = space of real-valued solutions of K.G. eqn.
- **Definition:**

$$\overline{V} := \text{space of complex-valued solutions of K.G. eqn.}$$

- We easily find a basis of $\overline{V}$, namely $\{u_n\} \cup \{u_n^*\}$ where:

$$u_n := \frac{1}{\sqrt{2}}\left(v_{2n} + i\, v_{2n+1}\right), \quad u_n^* = \frac{1}{\sqrt{2}}\left(v_{2n} - i\, v_{2n+1}\right)$$

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$$\langle f,h\rangle := i\int_\Sigma d\Sigma_\mu \sqrt{g}\, g^{\mu\nu}\left(f^*\partial_\nu h - (\partial_\nu f^*)h\right)$$

- Then, ( , ) yields:

$$\langle u_n, u_m\rangle = -\delta_{n,m}, \quad \langle u_n^*, u_m^*\rangle = +\delta_{n,m}, \quad \langle u_n, u_m^*\rangle = 0 \qquad (I)$$

Exercise: verify this.

Thus, $\langle\ ,\ \rangle$ is an indefinite inner product on $\overline{V}$: $\langle\ ,\ \rangle = \begin{pmatrix} \ddots & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & 1 \\ & & & & & \ddots \end{pmatrix}$

Proposition: A resolution of the identity on $\overline{V}$ is given by:

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Proposition: A resolution of the identity on $\bar{V}$ is given by:

$$[M^{\mu\nu}, P^{\rho}] = i\left(\eta^{\mu\rho}P^{\nu} - \eta^{\nu\rho}P^{\mu}\right).$$

$$[P^{\mu}, K^{\nu}] = -2i\left(\eta^{\mu\nu}D + M^{\mu\nu}\right).$$

$$[M^{\mu\nu}, K^{\rho}] = i\left(\eta^{\mu\rho}K^{\nu} - \eta^{\nu\rho}K^{\mu}\right).$$

$\Longrightarrow$ conformal algebra

$$\left( [D, P_{\mu}] = -iP_{\mu} \qquad [D, K_{\mu}] = +iK_{\mu} \right)$$

$$\mathbb{1} = \int dx\, |x\rangle\langle x|$$

Remark: One can also turn $\bar{V}$ into a Hilbert space, namely the Krein space: Let $P^+$ and $P^-$ be the projectors on the spaces spanned by the $u_n^*$ and the $u_n$ respectively.
Then, $\langle\langle f,g\rangle\rangle := \langle f, P^+ g\rangle - \langle f, P^- g\rangle$ is a positive definite inner product, and the Krein space $(\bar{V}, \langle\langle,\rangle\rangle)$ is a Hilbert space.

Proof:

Indeed, $\mathbb{1}|u_n\rangle = |u_n\rangle$ using $\langle u_n, u_n\rangle = -1$

$\mathbb{1}|u_n^*\rangle = |u_n^*\rangle$ using $\langle u_n^*, u_n^*\rangle = 1$.

so that for any $|f\rangle \in \bar{V}$ we have:

$$-\sum_n |u_n\rangle\langle u_n|f\rangle + |u_n^*\rangle\langle u_n^*|f\rangle = |f\rangle \qquad (P)$$

Writing this out, we will now show that it yields $(W)$, i.e.:

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Writing this out, we will now show that it yields $(W)$, i.e.:

$$\boxed{\sqrt{|g|}\, g^{0\nu} \sum_k \left( u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) = i\,\delta^3(x-x')}$$

Indeed, using

$$\langle f, h \rangle = i \int_\Sigma d\Sigma_\mu \sqrt{|g|}\, g^{\mu\nu} \left( f^* \partial_\nu h - (\partial_\nu f^*) h \right)$$

which reads, in a suitable coordinate system:

$$= i \int d^3x' \sqrt{|g(x')|}\, g^{0\nu}(x') \left( f^*(x') \partial_\nu h(x') - \partial_\nu f^*(x') h(x') \right)$$

we see from (P) that $\forall f \in \overline{V}$:

$$- \sum_n |u_n\rangle\langle u_n|f\rangle + |u_n^*\rangle\langle u_n^*|f\rangle = |f\rangle$$

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$$\sum_n u_n(x,t)\, i \int_\Sigma d^3x' \sqrt{g}\, g^{0\nu} \left( u_n^* \partial_\nu f - (\partial_\nu u_n^*) f \right)$$

$$- \sum_n u_n^*(x,t)\, i \int_\Sigma d^3x' \sqrt{g}\, g^{0\nu} \left( u_n \partial_\nu f - (\partial_\nu u_n) f \right) \;=\; f(x,t)$$

Now, interchanging $\sum_n$ and $\int_\Sigma$ yields $\forall f \in \overline{V}$:

$$i \int_\Sigma d^3x \sqrt{|g(x')|}\, g^{0\nu}(x') \sum_n \Big( u_n(x,t)\, u_n^*(x',t)\, \partial_{x',\nu} - u_n(x,t)\, \partial_{x',\nu} u_n^*(x',t)$$

$$- u_n^*(x,t)\, u_n(x',t)\, \partial_{x',\nu} + u_n^*(x,t)\, \partial_{x',\nu} u_n(x',t) \Big) f(x',t) \;=\; f(x,t) \qquad (*)$$

$$\sum_n u_n(x,t)\, i \int_\Sigma d^3x' \sqrt{g}\, g^{0\nu} \left( u_n^* \partial_\nu f - (\partial_\nu u_n^*) f \right)$$

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Now, interchanging $\sum_n$ and $\int_\Sigma$ yields $\forall f \in \overline{V}$:

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**Notice:**

On the left hand side of this equation $f$ is evaluated

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**Notice:**

On the left hand side of this equation, $f$ is evaluated for all $x'$ but only at the one time, say $t_0$, of $\Sigma$.

- Now choose an arbitrary function $g(x')$.

- Then there exists a solution $f(x';t)$ of the Klein Gordon equation obeying:

$$1)\qquad f(x',t_0) = g(x')$$

$$2)\qquad g^{0\nu}(x',t_0)\,\partial_{x'^\nu} f(x',t_0) = 0$$

(Because the 2nd order K.G. equation on a globally hyperbolic spacetime has a well-defined Cauchy problem)

- Therefore, (*) yields, for all choices of $g(x)$:

$$i\int_\Sigma d^3x'\,\sqrt{|g(x')|}\,g^{0\nu}(x')\sum_n\Big(-u_n(x,t)\,\partial_{x'^\nu}u_n^*(x',t) + u_n^*(x,t)\,\partial_{x'^\nu}u_n(x',t)\Big)\,g(x') = g(x)\quad\forall g(x)$$

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$$\Rightarrow \boxed{\sqrt{|g|}\, g^{0\nu}_{(x')} \sum_k \left( u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) = i\, \delta^3(x - x')} \quad (W) \checkmark$$

can always be solved with this ansatz on any globally hyperbolic spacetime:

$$\boxed{\begin{aligned}\hat{\phi}(x,t) &:= \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+ \\ \hat{\pi}(x,t) &:= \sqrt{|g|}\, g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t)\end{aligned}}$$

where $[a_k, a_{k'}^+] = \delta(k-k')$ and where the $u_k(X,t)$ are number-valued solutions to (KG) which also obey (R1):

$$\sqrt{|g|}\, g^{0\nu}_{(x')} \sum_k \left( u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) = i\, \delta^3(x-x')$$

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# Outlook:

Q: We showed that (W) ensures the CCRs for one time foliation $\mathbb{R} \times \mathcal{M}$.
What guarantees the CCRs for all foliations?

A: Stokes' theorem and unitarity.

Q: Are the $u_k(x,t)$ unique?

A: No! Math: Bogolubov transformations

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Q: Are the $u_k(x,t)$ unique?

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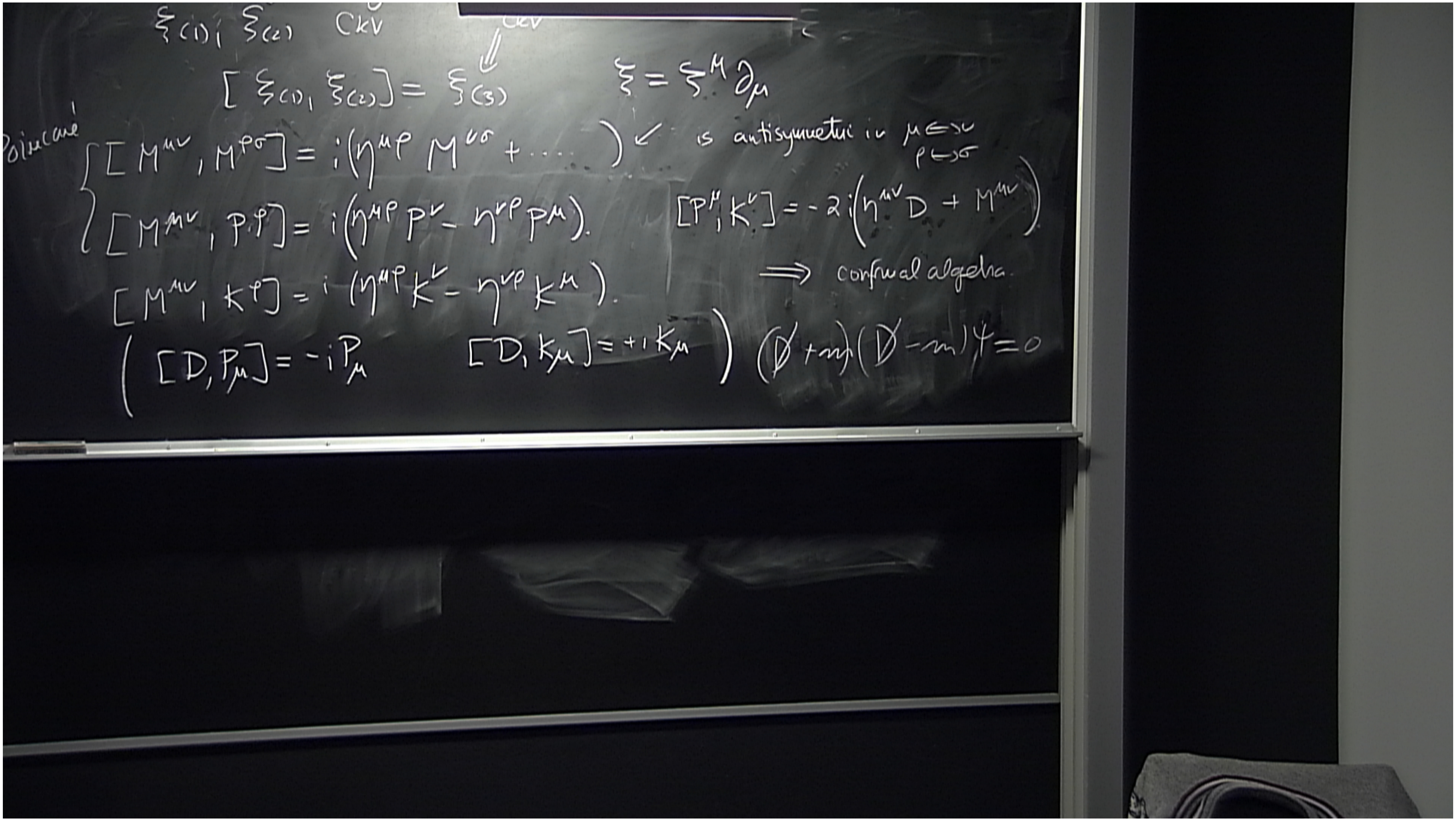
Q: What's the physics?

A: Vacuum ambiguity.


$\xi_{(1)}, \xi_{(2)}$ CKV
$[\xi_{(1)}, \xi_{(2)}] = \xi_{(3)}$
$\xi = \xi^\mu \partial_\mu$
Poincaré
$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho} M^{\nu\sigma} + \dots)$
is antisymmetric in $\mu \leftrightarrow \nu$, $\rho \leftrightarrow \sigma$
$[M^{\mu\nu}, P^\rho] = i(\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu)$.
$[P^\mu, K^\nu] = -2i(\eta^{\mu\nu} D + M^{\mu\nu})$
$[M^{\mu\nu}, K^\rho] = i(\eta^{\mu\rho} K^\nu - \eta^{\nu\rho} K^\mu)$.
$\Longrightarrow$ conformal algebra.
$([D, P_\mu] = -iP_\mu \quad [D, K_\mu] = +iK_\mu)$
$(\not\partial + m)(\not\partial - m)\psi = 0$