Title: Quantum Field Theory for Cosmology - Lecture 10

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (Kempf)

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QFT for Cosmology, Achim Kempf, Lecture 10

Recall:

(because the Lagrangian framework treats space and time in the same way)

* Hamiltonian formulations are suitable for quantization.
* Lagrangian formulations are suitable to achieve general relativistic covariance.

⇝ **Strategy:**

SR, 1^{st}Q Hamiltonian formalism $\xrightarrow[\text{(equivalence)}]{\text{step 1: Legendre transform}}$ SR, 1^{st}Q. Lagrangian formalism

step 2 ↓ allow curvature

GR, 1^{st}Q Hamiltonian $\xleftarrow[\text{(equivalence)}]{\text{step 3: Legendre transform}}$ GR, 1^{st}Q Lagrangian

Recall:

(... framework treats space and time in the same way)

* Hamiltonian formulations are suitable for quantization.
* Lagrangian formulations are suitable to achieve general relativistic covariance.

→ **Strategy:**

SR, 1^{st}Q Hamiltonian formalism	step 1: Legendre transform (equivalence) →	SR, 1^{st}Q. Lagrangian formalism
		↓ step 2: allow curvature
GR, 1^{st}Q Hamiltonian formalism	← step 3: Legendre transform (equivalence)	GR, 1^{st}Q Lagrangian formalism
↓ step 4		↓
GR, 2^{nd}Q Hamiltonian formalism	← Dyson Schwinger eqns are same (equivalence) →	GR, 2^{nd}Q Lagrangian formalism (Path integral of QFT)

We already started step 1 :

$$H[\phi,\pi,t] \quad \underset{\pi(x,t) := \frac{\delta L}{\delta \beta(x,t)} \ (T^{-1})}{\overset{\beta(x,t) := \frac{\delta H}{\delta \pi(x,t)} \ (T)}{\rightleftarrows}} \quad L[\phi,\beta,t]$$

Proposition: These equations of motion are equivalent :

Hamiltonian eqns. of motion:

$$\dot\phi(x,t) = \frac{\delta H[\phi,\pi,t]}{\delta \pi(x,t)} \qquad (H1)$$

Lagrangian eqns. of motion:

$$\dot\phi(x,t) = \beta(x,t) \qquad (L1)$$

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$\dot\pi(x,t) \;\; \delta H[\phi,\pi,t] \quad (H2)$

$\delta L \quad = \frac{d}{} \; \delta L \quad (L2)$

The case "$\Longrightarrow$"

- Show L1: $\dot{\phi} \overset{(H1)}{=\!=} \frac{\delta H}{\delta \pi} \overset{(T)}{=\!=} \beta$ ✓

- Show L2:

$$\frac{d}{dt}\frac{\delta L(\phi,\beta,t)}{\delta \beta} \overset{(T^{-1})}{=\!=} \frac{d}{dt}\pi$$

$$\overset{(H2)}{=\!=} -\frac{\delta H(\phi,\pi,t)}{\delta \phi}$$

$$\underset{\text{of } L}{\overset{\text{by def.}}{=\!=}} -\frac{\delta}{\delta\phi}\left(\int \beta(\phi,\pi)\pi \, d^3x - L(\phi,\beta(\phi,\pi),t)\right)$$

$$= -\cancel{\frac{\delta\beta}{\delta\phi}\pi} + \frac{\delta L}{\delta\phi} + \cancel{\underbrace{\frac{\delta L}{\delta\beta}}_{\overset{(T)}{=}\pi}\frac{\delta\beta}{\delta\phi}}$$ ✓

Result so far:

- Legendre transform to Lagrangian formulation

$\Rightarrow$ Eqns of motion can be cast in the form L1, L2, i.e.:

(Notice: Only a time derivative, no occurrence of space derivatives?) $\rightarrow$

$$\frac{\delta L}{\delta \phi(x,t)} = \frac{d}{dt} \frac{\delta L}{\delta \beta(x,t)}, \quad \beta(x,t) = \dot{\phi}(x,t)$$

But: How is that advantageous? These equations still seem to treat time differently than space!

Analysis of L1, L2:

We notice: * The term $\frac{\delta L}{\delta \phi(x,t)}$ is the total derivative with respect to all occurences of ϕ in L, including occurences of $\frac{\partial}{\partial x_i}\phi(x,t)$ in L.

* Why? Because of the definition of $\frac{\delta}{\delta\phi}$:

$$\frac{\delta L}{\delta \phi(x,t)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(L\left[\left\{\phi(x',t)+\varepsilon\,\delta^3(x'-x)\right\}_{x'\in\mathbb{R}^3}\right] - L\left[\left\{\phi(x',t)\right\}_{x'\in\mathbb{R}^3}\right]\right)$$

E.g.: $F[u] := \int \sin(x)\left(\frac{d}{dx}u(x)\right)dx$ Is $\frac{\delta F}{\delta u(x)} = 0$? No:

$$= -\int \cos(x)\, u(x)\, dx$$ (We assume $u(x) \to 0$ at boundaries)

$\Longrightarrow$ L1, L2 will contain nontrivial

time and space derivatives.

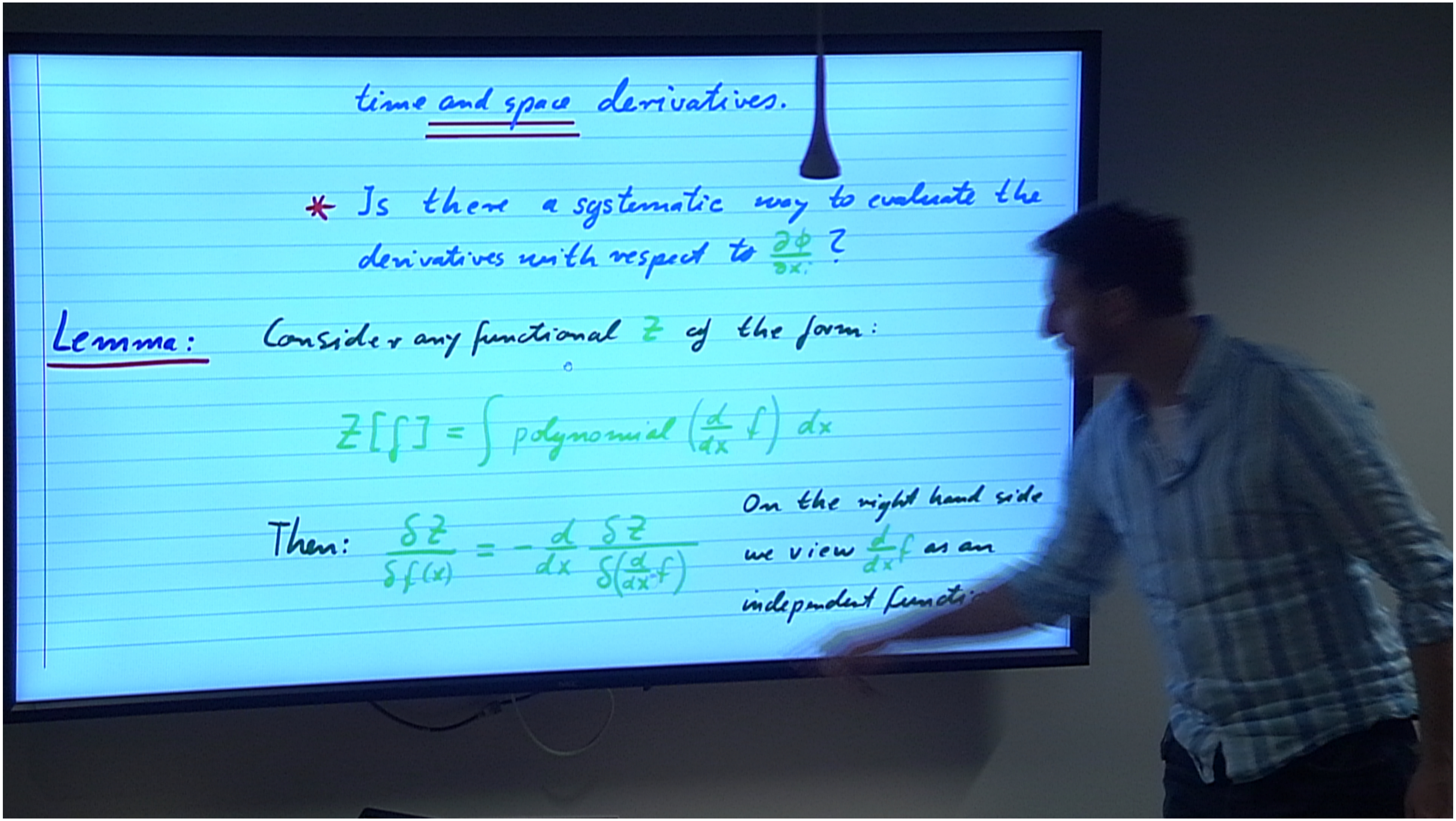
* Is there a systematic way to evaluate the derivatives with respect to $\frac{\partial \phi}{\partial x_i}$?

Lemma: Consider any functional Z of the form:

$$Z[f] = \int \text{polynomial}\left(\frac{d}{dx} f\right) dx$$

Then: $$\frac{\delta Z}{\delta f(x)} = -\frac{d}{dx} \frac{\delta Z}{\delta\left(\frac{d}{dx} f\right)}$$

On the right hand side we view $\frac{d}{dx}f$ as an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Example:

Notation: $\partial_x f(x) = \frac{d}{dx} f(x)$

$$Z[f] := \int_{\mathbb{R}} \left(\partial_{x'} f(x')\right)^2 dx'$$

- If we view $\partial_x f$ as an independent function, then we obtain of course:

$$\frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = 2\,\partial_x f(x)$$

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- Our lemma claims, therefore:

$$\delta Z[f] \quad \ldots \quad \delta Z[\partial_x f] \quad \ldots$$

$$Z[f] = \int_{\mathbb{R}} (\partial_x f)^2 \, dx$$

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- Our lemma claims, therefore:

$$\frac{\delta Z[f]}{\delta f(x)} = -\partial_x \frac{\delta Z[\partial_x f]}{\delta(\partial_x f(x))} = -2\, \partial_x \partial_x f(x)$$

Indeed:

$$\frac{\delta}{\delta f(x)} Z[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big[\int_{\mathbb{R}}\Big(\partial_{x'}\big(f(x') + \varepsilon\,\delta(x-x')\big)\Big)^2 dx'$$

$$- \int_{\mathbb{R}} \left(\partial_{x'} f(x')\right)^2 dx' \Big]$$

$$\overset{\lim \varepsilon\to 0}{=} 2\int_{\mathbb{R}} \left(\partial_{x'} f(x')\right)\left(\partial_{x'}\,\delta(x-x')\right) dx'$$

$$\underset{\text{parts}}{\overset{\text{int. by}}{=}} -2\int_{\mathbb{R}} \left(\partial^2_{x'} f(x')\right)\delta(x-x')\,dx' + \cancel{\text{boundary term}}$$

$$= -2\,\partial_x^2 f(x)$$

Indeed:

$$\frac{\delta}{\delta f(x)} Z[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\int_{\mathbb{R}} \left(\partial_{x'} \left(f(x') + \varepsilon \delta(x - x') \right) \right)^2 dx' - \int_{\mathbb{R}} \left(\partial_{x'} f(x') \right)^2 dx' \right]$$

$$\overset{\lim \varepsilon \to 0}{=} 2 \int_{\mathbb{R}} \left(\partial_{x'} f(x') \right) \left(\partial_{x'} \delta(x - x') \right) dx'$$

$$\underset{\text{parts}}{\overset{\text{int. by}}{=}} -2 \int_{\mathbb{R}} \left(\partial_{x'}^2 f(x') \right) \delta(x - x') \, dx' + \cancel{\text{boundary term}}$$

$$= -2 \, \partial_x^2 f(x) \quad \checkmark$$

Recall L2:

$$\frac{\delta L[\phi, \beta, t]}{\delta \phi(x,t)} = \frac{d}{dt} \frac{\delta L[\phi, \beta, t]}{\delta \beta(x,t)}$$

Use lemma:

$$\frac{\delta L[\phi, \beta, t]}{\delta \phi(x,t)} = \frac{\delta L[\phi, \partial_1\phi, \partial_2\phi, \partial_3\phi, \beta, t]}{\delta \phi(x,t)} - \sum_{j=1}^{3} \frac{\partial}{\partial x^j} \frac{\delta L[\phi, \partial_1\phi, \partial_2\phi, \partial_3\phi, \beta, t]}{\delta \left(\partial_j \phi(x,t)\right)}$$

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On the right hand side we view $\frac{d}{dx} f$ as an independent function.

$$\delta\,\phi(x,t) \qquad \sum_{j=1} \partial x^j \quad \delta\left(\partial_j\phi(x,t)\right) \qquad dt \qquad \delta\,\beta(x,t)$$

Recall also L1: $\beta(x,t) = \dot{\phi}(x,t)$

⇝ One is tempted to write:

$$\frac{\delta L[\phi, \partial_j\phi, t]}{\delta\,\phi(x,t)} \overset{?}{=} \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} \frac{\delta L[\phi, \partial_\mu\phi, t]}{\delta\left(\partial_\mu\phi(x,t)\right)} \qquad \text{with: } \partial_0 := \frac{d}{dt}$$

However:

Here, we must remember that here the true variable is ϕ, and that we can set $\beta = \dot{\phi}$ only after functional differentiation.

$$\frac{\delta L[\phi, \partial_j\phi, \beta, t]}{\delta\phi(x,t)} - \sum_{j=1}^{3} \frac{\partial}{\partial x^j} \frac{\delta L[\phi, \partial_j\phi, \beta, t]}{\delta(\partial_j\phi(x,t))} = \frac{d}{dt} \frac{\delta L[\phi, \partial_j\phi, \beta, t]}{\delta\beta(x,t)}$$

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The "Action functional":

Definition: $S[\phi] := \int_{\mathbb{R}} L[\phi, t]\, dt$

$S[\phi]$ is called the "action of the field evolution $\phi(x,t)$"

Then, the "Euler Lagrange field equations" are

$$\boxed{\frac{\delta S[\phi, \partial_\mu \phi]}{\delta \phi(x,t)} - \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} \frac{\delta S[\phi, \partial_\mu \phi]}{\delta(\partial_\mu \phi)} = 0}$$

or equivalently:

$\delta S[\phi]$ ...

"The action ..."

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or equivalently:

$$\boxed{\frac{\delta S[\phi]}{\delta \phi(x,t)} = 0}$$

"The action principle"

$$S[\phi] := \frac{1}{2}\int_{\mathbb{R}^4} (\partial_0\phi)^2 - \sum_{j=1}^{3}(\partial_j\phi)^2 - m^2\phi^2 \, d^4x$$

Using either the action principle or directly the Euler Lagrange field equations, one obtains the Klein Gordon equation (Exercise: verify):

$$\partial_0^2\phi - \Delta\phi + m^2\phi = 0, \text{ i.e., } (\Box + m^2)\phi(x,t) = 0$$

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- Definitions:

  * The action's integrand is called the "Lagrange density" $\mathcal{L}(x,t)$:

$$S[\phi] = \int_{\mathbb{R}^4} \mathcal{L}(x,t)\, d^4x$$

  * One often formally writes:

E.g., equation ($\mathcal{L}$) can be written as:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0$$

c.) One defines the **metric tensor** $g_{\mu\nu}(x,t)$. More about it soon. In special relativity in inertial rectangular coordinate system, we have:

$$g_{\mu\nu}(x,t) = \eta_{\mu\nu} := \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

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- Using these definitions, the K.G. action now reads:

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} g^{\mu\nu}\,\phi_{,\mu}\,\phi_{,\nu} - m^2\phi^2\, d^4x$$

the inverse matrix to $g_{\mu\nu}$. In special

We have now completed Step 1 :

SR, 1^{st}Q Hamiltonian formalism $\xrightarrow[\text{(equivalence)}]{\text{step 1: Legendre transform}}$ SR, 1^{st}Q. Lagrangian formalism

step 2 ↓ allow curvature

GR, 1^{st}Q Hamiltonian formalism $\xleftarrow[\text{(equivalence)}]{\text{step 3: Legendre transform}}$ GR, 1^{st}Q Lagrangian formalism

step 4 ↓

GR, 2^{nd}Q Hamiltonian formalism $\xleftrightarrow[\text{(equivalence)}]{\text{Dyson Schwinger eqns are same}}$ GR, 2^{nd} Q Lagrangian formalism (Path integral of QFT)

Step 2: How to allow for curvature of space-time?

Strategy:

A. Within special relativity, allow not just inertial rectangular coordinate systems but allow arbitrary coordinate systems.

B. Allow arbitrary coordinate systems and allow curvature.

Reconsider the K.G. action:

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 \, d^4x$$

If we change to arbitrary coordinates

$$x^\mu \to \tilde{x}^\mu = \tilde{x}^\mu(x)$$

then: $\phi(x) \to \tilde{\phi}(\tilde{x}) = \phi(x(\tilde{x}))$ (recall that $\sum_{\nu=0}^{3}$ is implied)

$$\frac{\partial}{\partial x^\mu}\phi(x) \to \frac{\partial}{\partial \tilde{x}^\mu}\tilde{\phi}(\tilde{x}) = \left(\frac{\partial}{\partial x^\nu}\phi(x(\tilde{x}))\right)\frac{\partial x^\nu}{\partial \tilde{x}^\mu}$$

Therefore, if we transform

$$g^{\mu\nu}(x) \to \tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} g^{\alpha\beta}(x(\tilde{x}))$$

then we have that this term in the action

$$g^{\mu\nu}(x)\, \phi_{,\mu}(x)\, \phi_{,\nu}(x)$$

is numerically the same in all coordinate systems:

$$g^{\mu\nu}(x) \left(\frac{\partial}{\partial x^\mu}\phi(x)\right)\left(\frac{\partial}{\partial x^\nu}\phi(x)\right) \to \tilde{g}^{\mu\nu}(\tilde{x})\left(\frac{\partial}{\partial \tilde{x}^\mu}\phi(\tilde{x})\right)\left(\frac{\partial}{\partial \tilde{x}^\nu}\phi(\tilde{x})\right)$$

$$= g^{\alpha\beta}(x) \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} \frac{\partial x^\rho}{\partial \tilde{x}^\mu} \frac{\partial x^\varepsilon}{\partial \tilde{x}^\nu} \left(\frac{\partial}{\partial x^\rho}\phi(x)\right)\left(\frac{\partial}{\partial x^\varepsilon}\phi(x)\right)$$

Recall:

As $x^\mu \to \tilde{x}^\mu(x)$ the integral measure changes by a Jacobian factor:

$$\int f(x)\, d^4x \longrightarrow \int \underbrace{\tilde{f}(\tilde{x})}_{= f(\tilde{x})} \underbrace{\det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right)}_{\text{a coordinate-dependent term!}} d^4\tilde{x}$$

A compensating term is needed:

How can we modify the action $S'[\phi]$ so that:

Solution:

Modify the action to include a "Volume factor":

$$S[\phi] := \frac{1}{2}\int_{\mathbb{R}^4}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right)\overbrace{\sqrt{-\det(g_{\mu\nu})}}^{\uparrow}\, d^4x$$

The volume factor:

* When $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ then $\sqrt{-\det(g)} = 1$ ✓

* Lemma: When $x^\mu \rightarrow \tilde{x}^\mu(x)$ then:

short for $\sqrt{-\det(g_{\mu\nu})}$

$$\sqrt{|g|} \longrightarrow \sqrt{|\tilde{g}|} = \det\left(\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right)\sqrt{|g|}$$

□ Therefore, we have now in special relativity that the action $S[\phi]$ of a field ϕ comes out the same number, independently of one's choice of coordinate system:

$$S[\phi] \longrightarrow \tilde{S}[\tilde{\phi}] = \int \tilde{\mathcal{L}} \sqrt{|\tilde{g}|}\, d^4\tilde{x}$$

$$= \int \mathcal{L} \det\left(\frac{\partial \tilde{x}}{\partial x}\right) \det\left(\frac{\partial x}{\partial \tilde{x}}\right) \sqrt{|g|}\, d^4x$$

$$= \int \mathcal{L} \det\left(\frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\alpha}{\partial x^\nu}\right) \sqrt{|g|}\, d^4x$$

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B. How to allow curvature?

* The trivial metric $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ can look very nontrivial in generic coordinate systems: $g_{\mu\nu}(x) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$

* But: Some metrics $g_{\mu\nu}(x)$ are not obtainable from the trivial metric by a coordinate change!

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* But: Some metrics $g_{\mu\nu}(x)$ are not obtainable from the trivial metric by a coordinate change!

$\rightarrow$ These metrics belong to spaces with we saw that in generic (i.e. arbitrarily chosen) coordinates $\tilde{x}^\mu = \tilde{x}^\mu(x)$, the metric tensor $\tilde{g}_{\mu\nu}(\tilde{x})$ is given by:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} \eta_{\alpha\beta} \qquad (c)$$

$\Rightarrow$ In special relativity, in arbitrary coordinates, the metric $g_{\mu\nu}$ is a position-dependent matrix of the form (c).

* We notice that $g_{\mu\nu}(x)$ is always symmetric $g_{\mu\nu}(x) = g_{\nu\mu}(x)$

Key Question:

Can any arbitrary function obeying $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ arise from

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{\mu\nu}$$

by changing coordinates according to $g_{\mu\nu}(\tilde{x}) = \frac{\partial x^{\alpha}(\tilde{x})}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}(\tilde{x})}{\partial \tilde{x}^{\nu}} \eta_{\alpha\beta}$?

Answer: No ! The others describe "curved" spacetimes.

A given spacetime can be described by any one of an equivalence class $[g]$ of metric functions $\{g_{\mu\nu}(x)\}$, which differ by a mere change of coordinates (i.e. which are related by a diffeomorphism).

Definition: Each equivalence class $[g]$ is called a Riemannian or Lorentzian Structure, depending on the signature of the metric.

How many Lorentzian or Riemannian structures are there?

Q: How many independent degrees of freedom D (i.e. independent functions) describe a spacetime fully?

A: In n dimensions, the metric g has n^2 component functions $g_{\mu\nu}(x)$.

Because of $g_{\mu\nu}(x) = g_{\nu\mu}(x)$, only $n(n+1)/2$ are independent.

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But we can choose n functions $\tilde{x}^{\mu}(x)$ in $\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^{\alpha}(\tilde{x})}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}(\tilde{x})}{\partial \tilde{x}^{\nu}} g_{\alpha\beta}$.

A: $D = \underbrace{n(n+1)/2}_{\text{\# of indep elements of a symmetric } n\times n \text{ matrix } g_{\mu\nu}} - \overbrace{n}^{\text{\# of change of coordinate functions } y^{\mu} = y^{\mu}(x)}$