Title: Unrolled Quantum Groups and Vertex Operator Algebras

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Abstract: Connections between representation categories of quantum groups and vertex operator algebras (VOAs) have been studied since the 1990s starting with the pioneering work of Kazhdan and Lusztig. Recently, connections have been found between unrolled quantum groups and certain families of VOAs. In this talk, I will introduce unrolled quantum groups and describe their connections to the Singlet, Triplet, and Bp vertex operator algebras.

Let $q = e^{\frac{\pi i}{p}}$

Def'n: $\overline{U}_q^H(\mathfrak{sl}_2)$ is the $\mathbb{C}$-algebra generated by E, F, K, K^{-1}, and H, with rel'ns

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad KH = HK$$

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$E^p = F^p = 0$$

Mod

Modules:

simple $\begin{cases} V_\alpha \quad \alpha \in \mathbb{C} \text{ s.t. } \alpha + 1 - p \in \dot{\mathbb{C}} = (\mathbb{C} - \mathbb{Z}) \cup p\mathbb{Z} \\ \quad \text{Vermas, projective of dim } p \\ S_{i+kp}, \ i = 0, \ldots, p-2, k \in \mathbb{Z} \\ \quad \text{highest weight } i + kp, \text{ dim } i+1 \end{cases}$

P_{i+kp} · projective covers of S_{i+kp}

Defn: Let $L = \sqrt{2p}\,\mathbb{Z}$, $V_L = \bigoplus_{\lambda \in L} F_\lambda$,
and $e^{\gamma}(z)$, $\gamma \in L$ the associated fields
Set $Q = e_0^{-\sqrt{2/p}} \in L', V_{L'}$

Singlet $\mathcal{M}(p) = \mathrm{Ker}_{F_0}\, e_0^{-\sqrt{2/p}} = \mathrm{Ker}_{F_0}\, Q$

Modules: $F_\lambda \quad \lambda \in \mathbb{C}, \ \lambda \neq \alpha_{r,s} = \frac{1-r}{2}\sqrt{2p} + \frac{s-1}{2}\sqrt{\frac{2}{p}}, \ r \in \mathbb{Z}, \ 1 \leq s < p$

simple $M_{r,s}$, $r \in \mathbb{Z}$, $1 \leq s < p$

$P_{r,s}$ indecomposables

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Modules: $\mathcal{F}_\lambda$ $\lambda \in \mathbb{C}$, $\lambda \neq \alpha_{r,s} = \frac{1-r}{2}\sqrt{2p} + \frac{s-1}{2}\sqrt{2p}$ $r \in \mathbb{Z}$, $1 \le s < p$

simple: $M_{r,s}$, $r \in \mathbb{Z}$, $1 \le s \le p$

$P_{r,s}$ indecomposables

Let $\varphi : \mathrm{Rep}\, \overline{U}_q^{H}(sl_2)^{wt} \to \mathrm{Rep}_{\langle s \rangle} \mathcal{M}(p)$

be given by $V_\alpha \mapsto \mathcal{F}_{\frac{\alpha + p - 1}{\sqrt{2p}}}$, $S_{i+kp} \mapsto M_{1-k, i+1}$

$P_{i+kp} \mapsto P_{1-k, i+1}$

Prop: [CMR]: Suppose the fusion rules for simple $\mathcal{M}(p)$-modules are as conjectured by CM

Then, $\varphi(X \otimes Y) \cong \varphi(X) \otimes \varphi(Y)$ for X, Y simple

[CGR] Runkel: φ preserves Loewy diagrams $P_{i+kp} \leftrightarrow P_{i+kp}$

$(SG_1)^{U^+} \to \mathrm{Rep}_{\langle s \rangle} M(p)$

$, \; S_{1+kp} \mapsto M_{1-k,\, 1+1}$

$-k,\, 1+1$

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Alg Objects (notation)

Let $A \in \mathcal{C}$ (braided monoidal) be a conn. algebra object with modules in $\mathcal{C}$ denoted $\mathrm{Rep}A$

braiding on $\mathcal{C}$

$$\mathrm{Rep}^0 A := \{ (V, \mu_V) \in \mathrm{Rep}A \mid \mu_V \circ c_{V,A} \circ c_{A,V} = \mu_V \}$$

"local" $\quad A \otimes V \to V$

Def'n: The induction functor $\mathcal{F}: \mathcal{C} \to \mathrm{Rep}\, A$ is

product on A

$$\mathcal{F}(V,\mu_V) = (A\otimes V, \mu_{\mathcal{F}(V)}), \quad \mu_{\mathcal{F}(V)} = \mu \otimes id_V$$

Triplet (Brief) $W(p)$

The triplet can be realized as a simple current extension of $M(p)$

$$\varphi^{-1}(W(p)) = \Lambda_p$$

$$W(p) \in \mathrm{Rep}_{\langle s \rangle} M(p)^{\oplus} \xleftrightarrow{\varphi} \Lambda_p \in \mathrm{Rep}\, \overline{U}_i^H(sl_2)^{\oplus}_{\mathrm{wt}}$$

uet on A

<u>Exptect:</u> $\mathrm{Rep}^{fin}\,\mathcal{W}(p) \underset{\text{ribbon}}{\cong} \mathrm{Rep}^{0}\Lambda_p$

Br

[CGR]: Construct Quasi-Hopf modification $\overline{U_q^{(\Phi)}}(sl_2)$ of $\overline{U}_q(sl_2)$

s.t. $\mathrm{Rep}\,\overline{U_q^{(\Phi)}}(sl_2) \cong \mathrm{Rep}^{0}\Lambda_p$

$\cong \mathrm{Rep}^{\circ} \Lambda_p$
ribbon

asi-Hopf

$\ell_2)$ of $\overline{U}_q(s\ell_2)$

$\mathrm{Rep}^{\circ} \Lambda_p$

Braided Tensor Categories related to B_p VOAs

Auger, Creutzig, Kanade, Rupert

The B_p VOA can be realized as

$$B_p := \bigoplus_{k \in \mathbb{Z}} F_{\lambda p k} \boxtimes M_{1-k,1} \in \Big(\underbrace{\mathrm{Rep}\,\mathcal{H}}_{\text{Heisenberg}} \boxtimes \mathrm{Rep}_{\langle s \rangle} M(p)\Big)^{\oplus}$$

$$\varphi \updownarrow$$

$$A_p = \bigoplus_{k \in \mathbb{Z}} F_{\lambda p k} \boxtimes S_{kp} \in \Big(\mathrm{Rep}\,\mathcal{H} \boxtimes \mathrm{Rep}\,\overline{U}_q^{H}(s\ell_2)^{wt}\Big)^{\oplus}$$

<u>Prop</u> A_p is a comm alg object

...ed to B_p VOAs

...de, Rupert

...ed as

$$\ldots p\mathcal{H} \boxtimes \mathrm{Rep}_{\langle s \rangle} M(p)\Big)^{\oplus}$$

...isenberg

$$\mathrm{Rep}\,\mathcal{H} \boxtimes \mathrm{Rep}.\, \overline{U}_q^H(sl_2)^{\ldots}\Big)^{\oplus}$$

Th^m We have the following $\mathrm{Rep}^0 A_p$ modules

Simple:

$$E^V_{\gamma,\alpha} = \mathcal{F}(F_\gamma \boxtimes V_\alpha) \quad \alpha \in \mathbb{C} \quad \gamma\lambda_p + \tfrac{\alpha+p-1}{2} \in \mathbb{Z}$$

$$E^S_{\gamma,i,\ell} = \mathcal{F}(F_\gamma \boxtimes S_{i+p\ell}) \quad i = 0, \ldots, p-2, \quad \gamma\lambda_p + \tfrac{i+p\ell}{2} \in \mathbb{Z}$$

Proj:

$$Q^V_{\gamma,\alpha} = \mathcal{F}(F_\gamma \boxtimes V_\alpha) \quad \alpha \notin \mathbb{C} \quad ''$$

$$Q^P_{\gamma,i,\ell} = \mathcal{F}(F_\gamma \boxtimes P_{i+kp}) \quad ''$$

$$E^V_{\gamma,\alpha} \cong E^V_{\gamma+k\lambda_p,\alpha+pk}, \text{ etc.}$$

$\mathrm{Rep}^0 A_p$ is rigid, braided, monoidal

Rep A_p modules

$\lambda_p + \frac{\alpha + p - 1}{2} \in \mathbb{Z}$

$\gamma\lambda_p + \frac{1+p\ell}{2} \in \mathbb{Z}$

Modularity $\quad \psi \to E^{\vee}_{\gamma,\alpha}$

Let $\tilde{E}_{\gamma,\alpha} = \mathcal{F}(F_\gamma \boxtimes \frac{E_{\alpha+p-1}}{\sqrt{2p}})$: Re-parametrize $(\nu, \ell) = (\frac{2\alpha}{p}, \gamma\lambda_p + \frac{\alpha+p-1}{2})$

The modular group acts on charaters as

$$S(ch[\hat{E}_{\nu,\ell}]) = \sum_{\ell' \in \mathbb{Z}} \int_{-1}^{1} S^{\chi}_{(\nu,\ell)(\nu',\ell')}\, ch[\hat{E}_{(\nu',\ell')}]\, d\nu'$$

$\int_{\mathbb{C}}$

Let $S^{\infty}_{(\nu,\ell),(\nu',\ell')}$ be the closed Hopf-links associated to $E^{\vee}_{(\nu,\ell)}, E^{\vee}_{(\nu',\ell')}$

Prop:

$$\frac{S^{z^\infty}_{(\nu,\ell),(\nu',\ell')}}{S^{z^\infty}_{\mathbb{1},(\nu',\ell')}} = \frac{S^{\infty}_{(\nu,\ell)(\nu',\ell')}}{S^{\infty}_{\mathbb{1},(\nu',\ell')}}$$

Let p be odd and

$$\sigma^{s'}(W_s) = \mathcal{F}(F_\gamma \boxtimes M_{1-\ell,i+1}) \overset{\varphi}{\longleftrightarrow} E^{s}_{\gamma,i,\ell}$$

$$(s,s') = (i+1, -2\lambda_p\gamma - p\ell)$$

One can show that the alg of characters gen by $ch[\sigma^{s'}(W_s)]$ is gen by a subset

$$\Gamma_p := \{ ch[\sigma^{s'}(W_s)] \mid (s, s') \in \Lambda_p \}$$

$$\Lambda_p:\ 0 < s \leq p-1,\quad 0 \leq s' \leq p-1,\quad s + s' + 1 \in 2\mathbb{Z}$$

Γ_p is closed under modular transformations, and again

$$\frac{S^{x^{\epsilon}}_{(s,s'),(n,n')}}{S^{x^{\epsilon}}_{1,(n,n')}} = \frac{S^{\infty}_{(s,s'),(n,n')}}{S^{\infty}_{1,(n,n')}}$$

$$\Gamma_p \longleftrightarrow \underbrace{\left\{ E^{\circ}_{(s,s')} \,\middle|\, (s,s') \in \Lambda_p \right\}}$$

$\mathbb{Z}_+$ - basis of

$$\mathcal{G}^{ss}(\mathrm{Rep}^{\circ} A_p) = \mathcal{G}(\mathrm{Rep}^{\circ} A_p) / N\mathcal{G}(\mathrm{Rep}^{\circ} A_p)$$