Title: Unrolled Quantum Groups and Vertex Operator Algebras

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Abstract: Connections between representation categories of quantum groups and vertex operator algebras (VOAs) have been studied since the 1990s starting with the pioneering work of Kazhdan and Lusztig. Recently, connections have been found between unrolled quantum groups and certain families of VOAs. In this talk, I will introduce unrolled quantum groups and describe their connections to the Singlet, Triplet, and Bp vertex operator algebras.

Let $q = e^{\frac{\pi i}{p}}$

Def'n: $\overline{U}_q^H(\mathfrak{sl}_2)$ is the $\mathbb{C}$-algebra generated by E, F, K, K^{-1}, and H, with rel'ns

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad KH = HK$$

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$E^p = F^p = 0$$

Modules:

simple: V_α $\alpha\in\mathbb{C}$ s.t. $\alpha+1-p\in\dot{\mathbb{C}}=(\mathbb{C}-\mathbb{Z})\cup p\mathbb{Z}$

Vermas, projective of dim p

S_{i+kp}, $i=0,\ldots,p-2$, $k\in\mathbb{Z}$

highest weight $i+kp$, dim $i+1$

P_{i+kp} . projective covers of S_{i+kp}

Defn: Let $L=\sqrt{2p}\,\mathbb{Z}$, $V_L=\bigoplus_{\lambda\in L}F_\lambda$,

and $e^\gamma(z)$, $\gamma\in L$ the associated fields

Set $Q=e_0^{-\sqrt{2/p}}\in L', V_{L'}$

Singlet $\mathcal{M}(p)=\mathrm{Ker}_{F_0}\,e_0^{-\sqrt{2/p}}=\mathrm{Ker}_{F_0}\,Q$

Modules: F_λ $\lambda\in\mathbb{C}$, $\lambda\neq\alpha_{r,s}=\frac{1-r}{2}\sqrt{2p}+\frac{s-1}{2}\sqrt{2/p}$, $r\in\mathbb{Z}$, $1\le s<p$

simple: $M_{r,s}$, $r\in\mathbb{Z}$, $1\le s\le p$

$P_{r,s}$ indecomposables

Defn: Let $L = \sqrt{2p}\,\mathbb{Z}$, $V_L = \bigoplus_{\lambda \in L} F_\lambda$

and $e^{\gamma}(z)$, $\gamma \in L$ the associated fields

Set $Q = e_0^{-\sqrt{2/p}} \in L', V_{L'}$

Singlet $\mathcal{M}(p) = \mathrm{Ker}_{F_0}\, e_0^{-\sqrt{2/p}} = \mathrm{Ker}_{F_0}\, Q$

Modules: $\mathcal{F}_\lambda$ $\lambda \in \mathbb{C}$, $\lambda \neq \alpha_{r,s} = \frac{1-r}{2}\sqrt{2p} + \frac{s-1}{2}\sqrt{\frac{2}{p}}$, $r \in \mathbb{Z}$, $1 \leq s < p$

simple $M_{r,s}$, $r \in \mathbb{Z}$, $1 \leq s < p$

$P_{r,s}$ indecomposables

Let $\varphi : \mathrm{Rep}\,\bar{U}_i^{H}(sl_2)^{wt} \Longrightarrow \mathrm{Rep}_{\langle s \rangle} \mathcal{M}(p)$

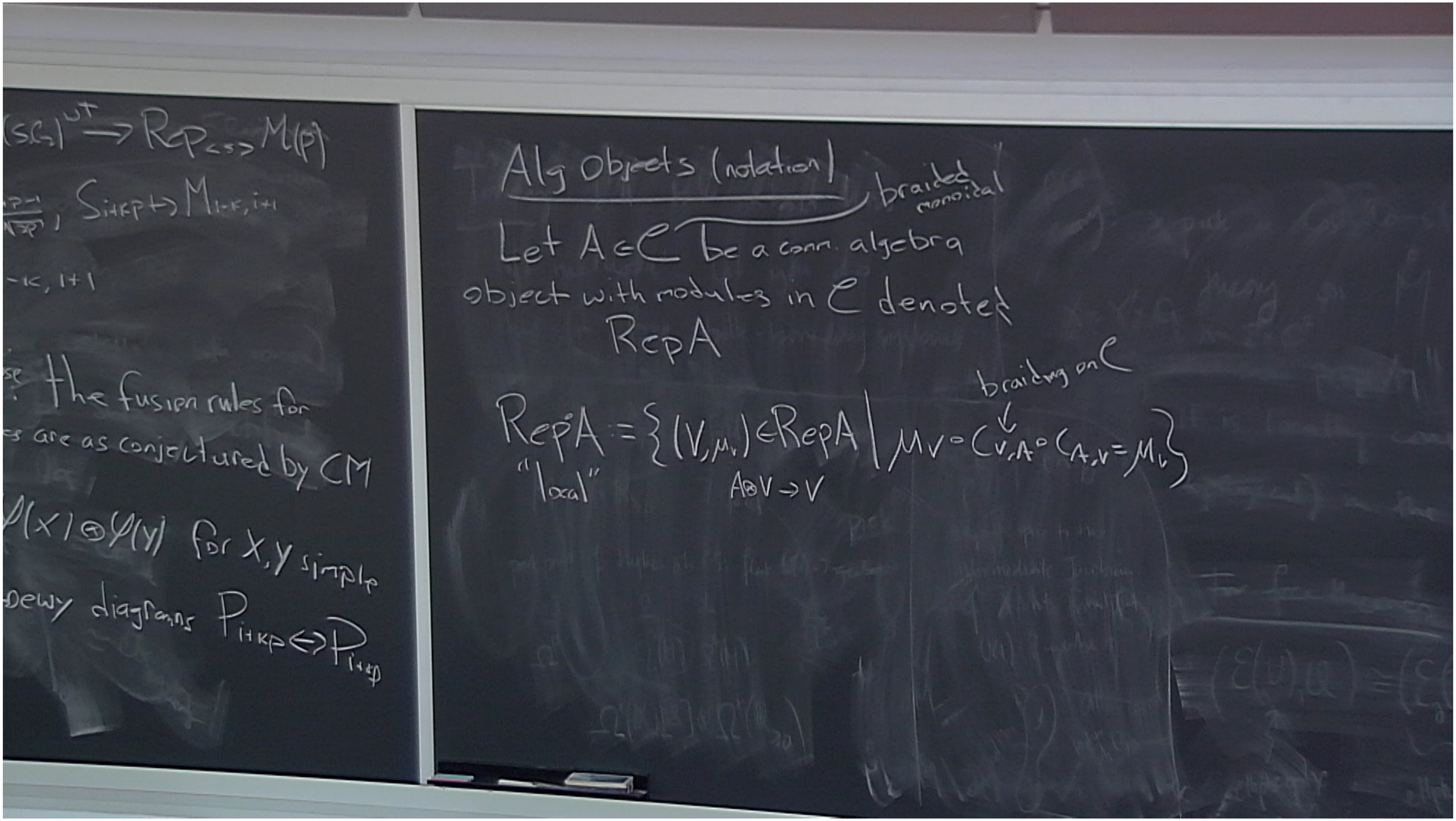
be given by $V_\alpha \mapsto \mathcal{F}_{\frac{\alpha + p - 1}{\sqrt{2p}}}$, $S_{i+kp} \mapsto M_{1-k,\, i+1}$

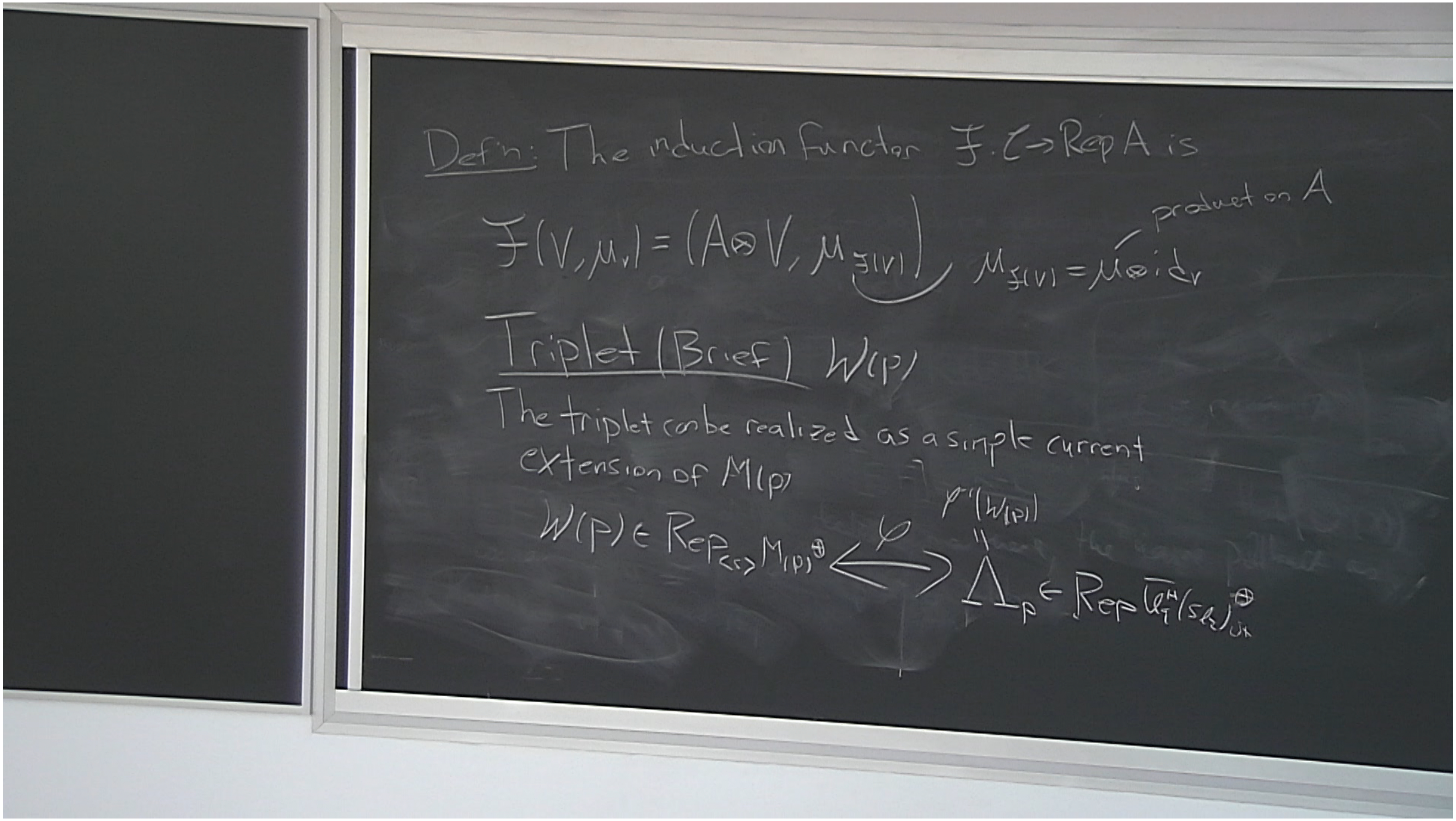
$P_{i+kp} \mapsto P_{1-k,\, i+1}$

Prop: [CMR]: Suppose the fusion rules for simple $\mathcal{M}(p)$-modules are as conjectured by CM

Then, $\varphi(X \otimes Y) \cong \varphi(X) \otimes \varphi(Y)$ for X, Y simple

[CGR] Runkel $\cdot$ φ preserves Loewy diagrams $P_{i+kp} \Longleftrightarrow P_{i+kp}$


Alg Objects (notation)
braided monoidal
Let A ∈ C be a conn. algebra
object with modules in C denoted
RepA
braiding on C
RepA = {(V, μ_V) ∈ RepA | μ_V ∘ c_{V,A} ∘ c_{A,V} = μ_V}
"local"
A⊗V → V
the fusion rules for
are as conjectured by CM
for x, y simple
diagrams


Def'n: The induction functor $\mathcal{F}: \mathcal{C} \to \operatorname{Rep} A$ is
product on A
$\mathcal{F}(V, \mu_V) = (A \otimes V, \mu_{\mathcal{F}(V)})$
$\mu_{\mathcal{F}(V)} = \mu \otimes id_V$
Triplet (Brief) $\mathcal{W}(p)$
The triplet can be realized as a simple current extension of $\mathcal{M}(p)$
$\mathcal{W}(p) \in \operatorname{Rep}_{\langle c \rangle} \mathcal{M}(p)^{\oplus} \xleftrightarrow{\varphi} \Lambda_p \in \operatorname{Rep} \overline{\mathcal{U}}_i^H(sl_2)^{\oplus}_{\text{ux}}$
$\varphi^{-1}(\mathcal{W}(p)) = \Lambda_p$

...uct on A

<u>Exptect:</u> $\mathrm{Rep}^{fin}\, W(p) \underset{\text{ribbon}}{\cong} \mathrm{Rep}^{0} \Lambda_p$

[CGR]: Construct Quasi-Hopf modification $\overline{U_q^{(\Phi)}}(sl_2)$ of $\bar{U}_q(sl_2)$

s.t. $\mathrm{Rep}\, \overline{U_q^{(\Phi)}}(sl_2) \cong \mathrm{Rep}^{0} \Lambda_p$

Br...

$\cong \mathrm{Rep}^0 \Lambda_p$
ribbon

-si-Hopf

$\mathfrak{sl}_2)$ of $\overline{U}_q(\mathfrak{sl}_2)$

$\mathrm{Rep}^0 \Lambda_p$

Braided Tensor Categories related to B_p VOAs

Auger, Creutzig, Kanade, Rupert

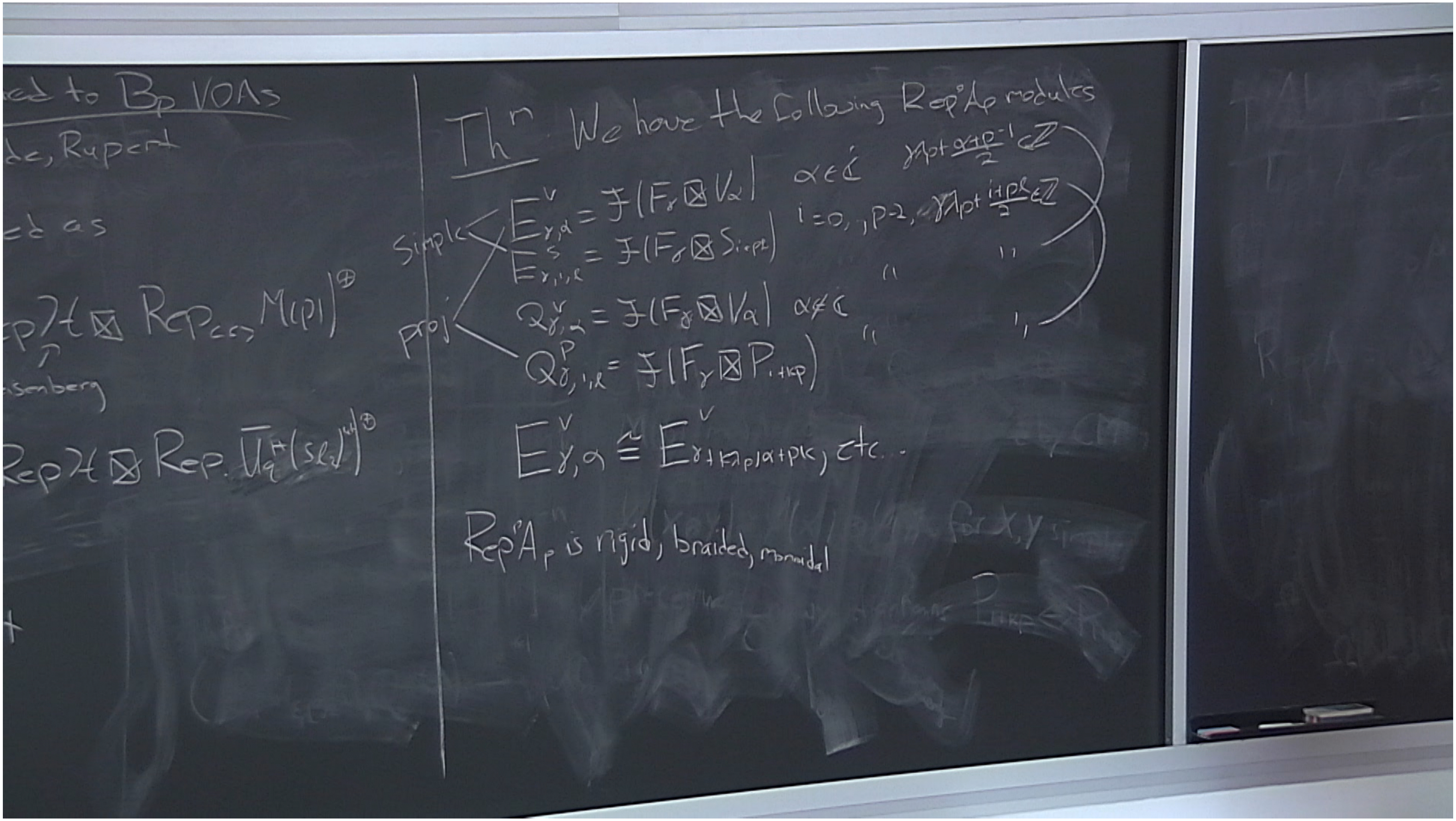
The B_p VOA can be realized as

$$B_p := \bigoplus_{k\in\mathbb{Z}} F_{\lambda_p k} \boxtimes M_{1-k,1} \in \big(\underset{\text{Heisenberg}}{\mathrm{Rep}\,\mathcal{H}} \boxtimes \mathrm{Rep}_{\langle s\rangle} M(p)\big)^{\oplus}$$

$$\varphi \updownarrow$$

$$A_p = \bigoplus_{k\in\mathbb{Z}} F_{\lambda_p k} \boxtimes S_{kp} \in \big(\mathrm{Rep}\,\mathcal{H} \boxtimes \mathrm{Rep}.\,\overline{U}_q^H(\mathfrak{sl}_2)^{wt}\big)^{\oplus}$$

Prop A_p is a comm alg object


to B_p VOAs
Rupert
Heisenberg
$\mathrm{Rep}\,\mathcal{H} \boxtimes \mathrm{Rep}\,\overline{U}_q^H(sl_2)$
Th^m. We have the following $\mathrm{Rep}^0 A_p$ modules
simple
proj
$E^v_{\gamma,\alpha} = \mathcal{F}(F_\gamma \boxtimes V_\alpha)$ $\alpha \in \mathbb{C}$
$E^s_{\gamma,i,\ell} = \mathcal{F}(F_\gamma \boxtimes S_{i+p\ell})$ $i = 0, \ldots, p-2$
$Q^v_{\gamma,\alpha} = \mathcal{F}(F_\gamma \boxtimes V_\alpha)$ $\alpha \notin \mathbb{C}$
$Q^P_{\gamma,i,\ell} = \mathcal{F}(F_\gamma \boxtimes P_{i+\ell p})$
$E^v_{\gamma,\alpha} \cong E^v_{\gamma+k\lambda_p,\alpha+pk}$, etc.
$\mathrm{Rep}^0 A_p$ is rigid, braided, monoidal

Rep$^\circ$ A_p modules

$\lambda_p + \frac{\alpha+p-1}{2} \in \mathbb{Z}$

$\gamma\lambda_p + \frac{1+p\ell}{2} \in \mathbb{Z}$

Modularity

$\varphi \mapsto \check{E}_{\gamma,\alpha}$

Let $\tilde{E}_{\gamma,\alpha} = \mathfrak{F}(F_\gamma \boxtimes \frac{E_{\alpha+p-1}}{\sqrt{2p}})$

Re-parametrize $(\nu,\ell) = (\frac{2\alpha}{?}, \gamma\lambda_p + \frac{\alpha+p-1}{2})$

The modular group acts on charaters as

$$S(\mathrm{ch}[\hat{E}_{\nu,\ell}]) = \sum_{\ell' \in \mathbb{Z}} \int_{-1}^{1} S^{\chi}_{(\nu,\ell)(\nu',\ell')} \, \mathrm{ch}[\hat{E}_{(\nu',\ell')}] \, d\nu'$$

$\mathbb{C}$

Let $S^{\infty}_{(\nu,\ell),(\nu',\ell')}$ be the closed Hopf-links associated to $\check{E}_{(\nu,\ell)}, \check{E}_{(\nu',\ell')}$

Prop: $$\frac{S^{z}_{(\nu,\ell),(\nu';\ell')}}{S^{z}_{\mathbb{1},(\nu';\ell')}} = \frac{S^{\infty}_{(\nu,\ell)(\nu';\ell')}}{S^{\infty}_{\mathbb{1},(\nu';\ell')}}$$

Let p be odd and

$$\sigma^{s'}(W_s) = \mathcal{F}(F_\gamma \boxtimes M_{1-\ell,i+1}) \overset{\varphi}{\longleftrightarrow} E^{s}_{\gamma,i+1,\ell}$$

$$(s,s') = (i+1, -2\lambda_p\gamma - p\ell)$$

One can show that the alg of characters gen by $ch[\sigma^{s'}(W_s)]$ is gen by a subset

$$\Gamma_p := \{ ch[\sigma^{s'}(W_s)] \mid (s,s') \in \Lambda_p \}$$

Λ_p: $0 < s \leq p-1$, $0 \leq s' \leq p-1$, $s+s'-1 \in 2\mathbb{Z}$

Γ_p is closed under modular transformations, and again

$$\frac{S^{z^-}_{(s,s'),(n,n')}}{S^{z^-}_{1,(n,n')}} = \frac{S^{\infty}_{(s,s'),(n,n')}}{S^{\infty}_{1,(n,n')}}$$

the alg

$\sigma^{s'}(w_s)$

$(s,s') \in \Lambda_p\}$

$0 < s \leq p-1$

$0 \leq s' \leq p-1$

$s + s' + 1 \in 2\mathbb{Z}$

odular

again

$(s,s'),(n,n')$

$\mathbb{1},(n,n')$

$$\Gamma_p \longleftrightarrow \underbrace{\{E_{(s,s')} \mid (s,s') \in \Lambda_p\}}$$

$\mathbb{Z}_+$ - basis of

$$\mathcal{G}^{ss}(\mathrm{Rep}^\circ A_p) = \mathcal{G}(\mathrm{Rep}^\circ A_p)/N\mathcal{G}(\mathrm{Rep}^\circ A_p)$$

Th^n

Simple E, E

proj Q, Q

E

$\mathrm{Rep}^\circ A_p$