Title: PSI 2019/2020 - QFT III - Lecture 4

Speakers:

Collection: PSI 2019/2020 - QFT III

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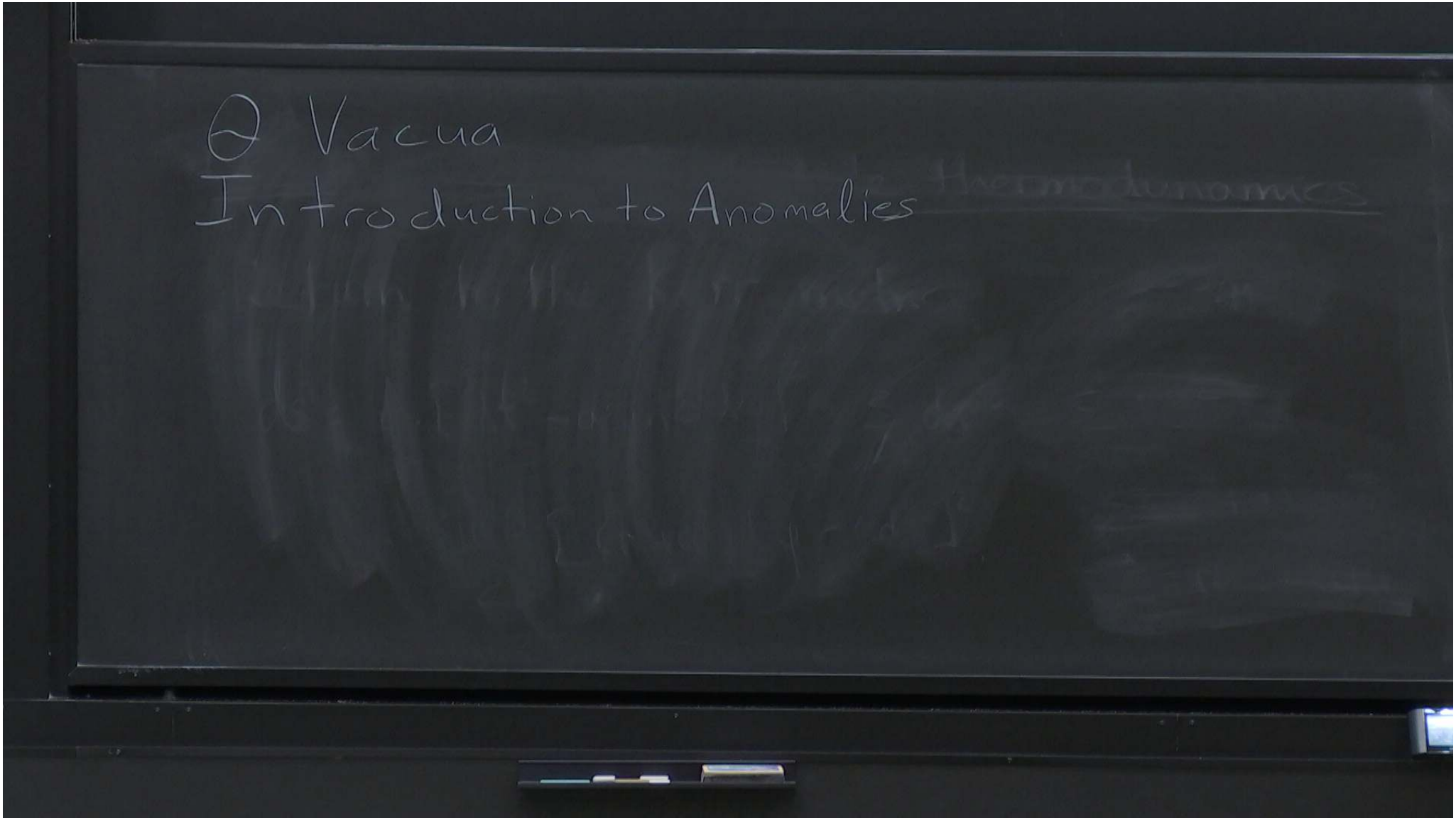
θ Vacua
Introduction to Anomalies

θ Vacua

Introduction to Anomalies

$$S_E = \frac{1}{2g^2}\int d^4x \, \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu}$$

finite action $\rightarrow$ $F_{\mu\nu} \xrightarrow[\rho\to\infty]{} 0$ faster than $\frac{1}{\rho^2}$

$U = \Theta(\rho^0) \rightarrow U(x) = U(\hat{x})$ ← function of angles only

$F_{\mu\nu} = \Theta(\frac{1}{\rho^3}) \rightarrow A_\mu = \Theta(\frac{1}{\rho^2}) + iU\partial_\mu U^{-1}$

$$A_\mu \to h A_\mu h^{-1} + i h \partial_\mu h^{-1}$$

$$= i(hU) A_\mu (hU)^{-1} + \mathcal{O}\left(\frac{1}{\rho^2}\right)$$

If $hU = 1$ as $\rho \to \infty$

generally not possible

Simpler case: $U(1)$ in $D=2$

$F_{\mu\nu} =$

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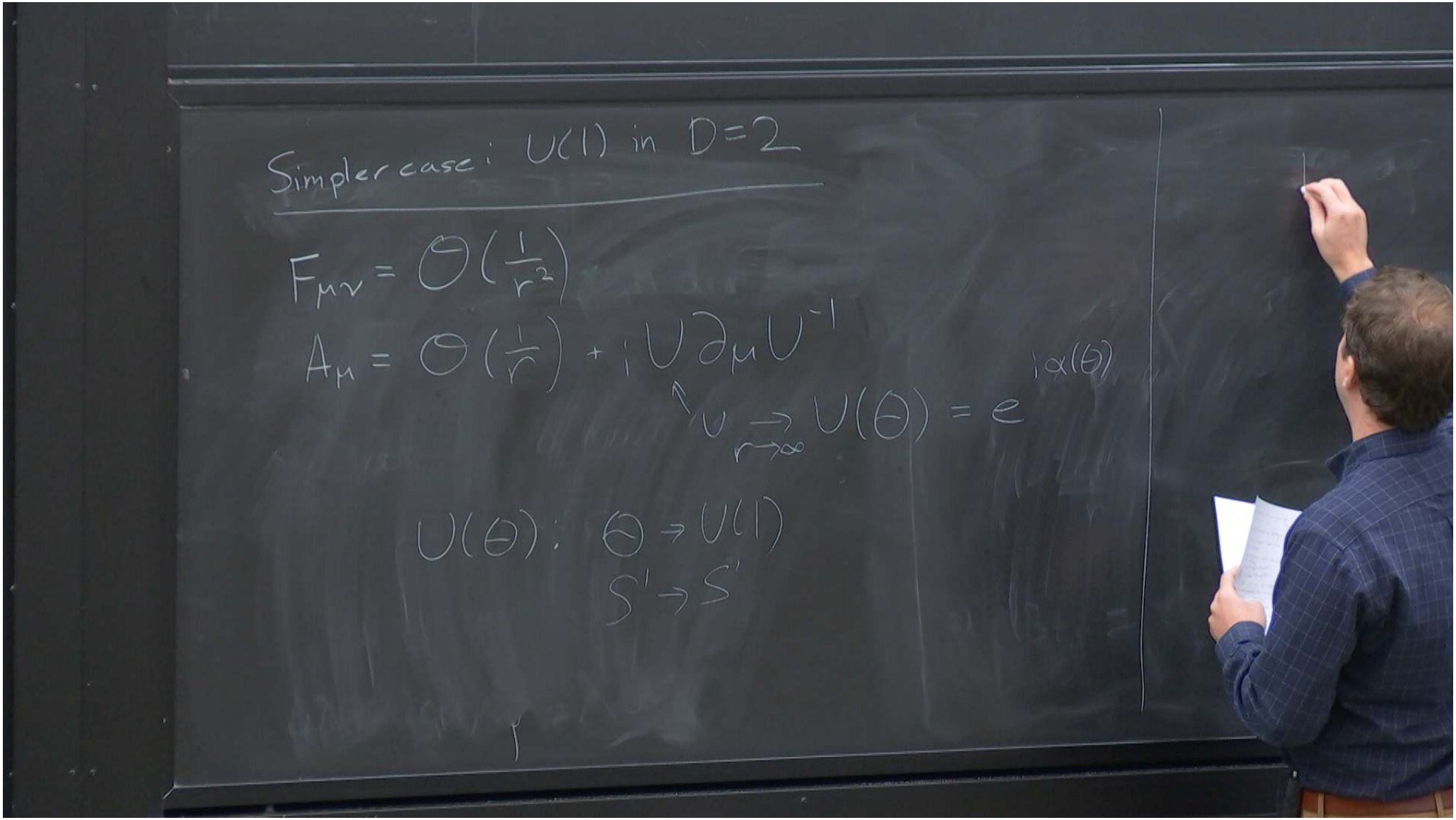
Introduction to Anomalies

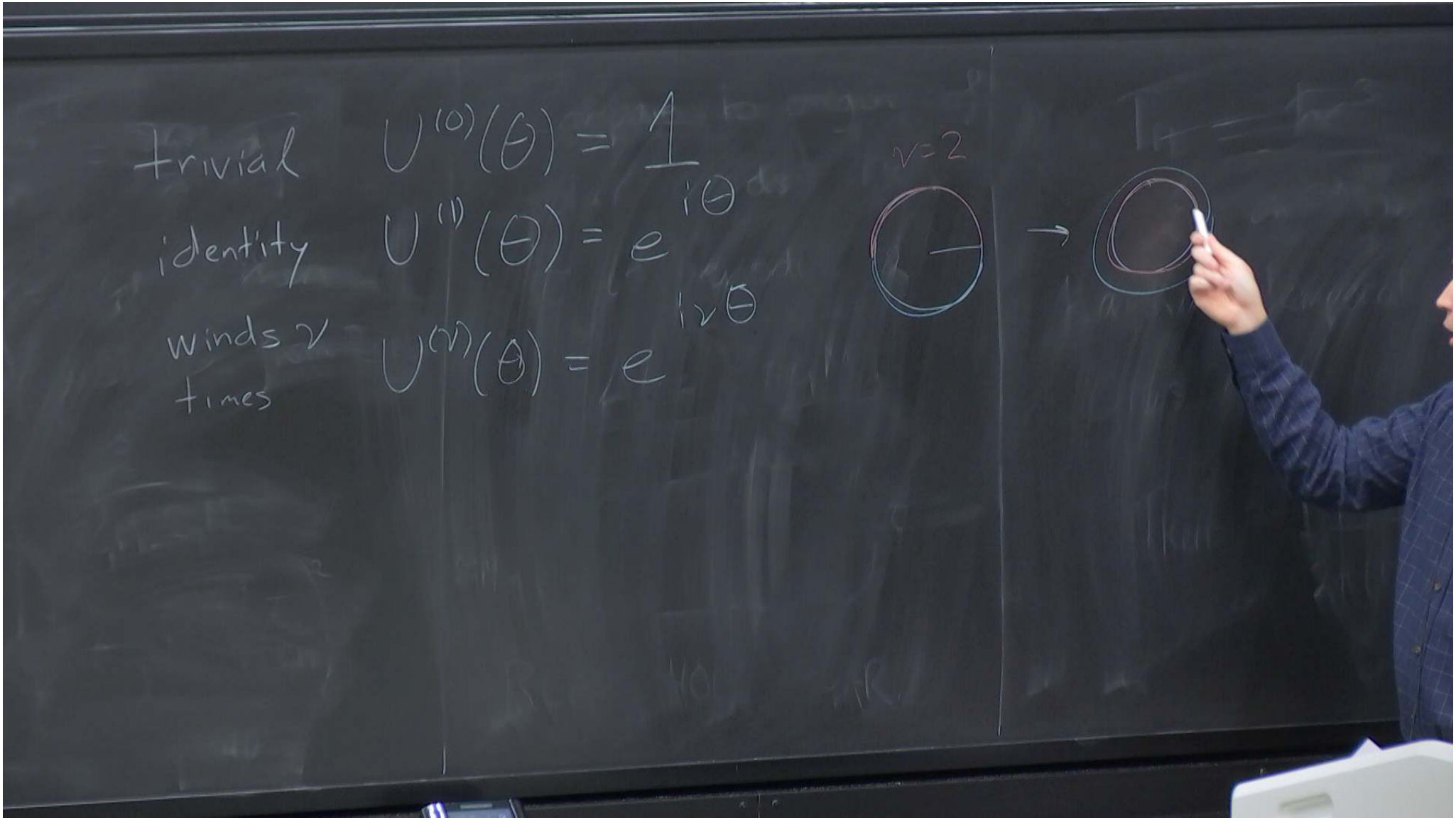
$$S_E = \frac{1}{2g^2}\int d^4x \, \cancel{tr} \, F_{\mu\nu} F^{\mu\nu}$$

finite action $\rightarrow$ $F_{\mu\nu} \xrightarrow[\rho\to\infty]{} 0$ faster than $\frac{1}{\rho^2}$

$U = \Theta(\rho^0) \rightarrow U(x) = U(\hat{x})$ (function of angles only)

$F_{\mu\nu} = \Theta(\frac{1}{\rho^3}) \rightarrow A_\mu = \Theta(\frac{1}{\rho^2}) + iU\partial_\mu U^{-1}$

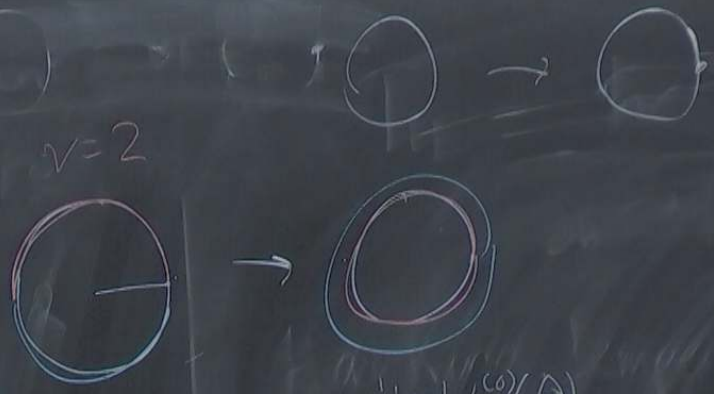

Simpler case: U(1) in D=2
$F_{\mu\nu} = \mathcal{O}(\frac{1}{r^2})$
$A_\mu = \mathcal{O}(\frac{1}{r}) + iU\partial_\mu U^{-1}$
$U \xrightarrow[r\to\infty]{} U(\theta) = e^{i\alpha(\theta)}$
$U(\theta)$: $S^1 \to U(1)$
$S^1 \to S^1$

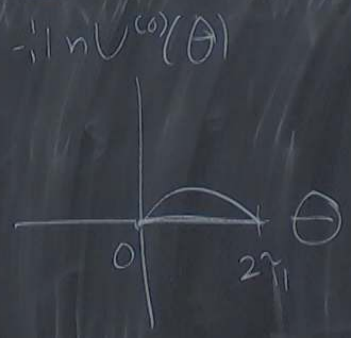

trivial
$U^{(0)}(\theta) = 1$
identity
$U^{(1)}(\theta) = e^{i\theta}$
winds ν times
$U^{(\nu)}(\theta) = e^{i\nu\theta}$
$\nu = 2$

trivial $U^{(0)}(\theta) = 1$

identity $U^{(1)}(\theta) = e^{i\theta}$

winds ν times $U^{(\nu)}(\theta) = e^{i\nu\theta}$ $\nu \in \mathbb{Z}$

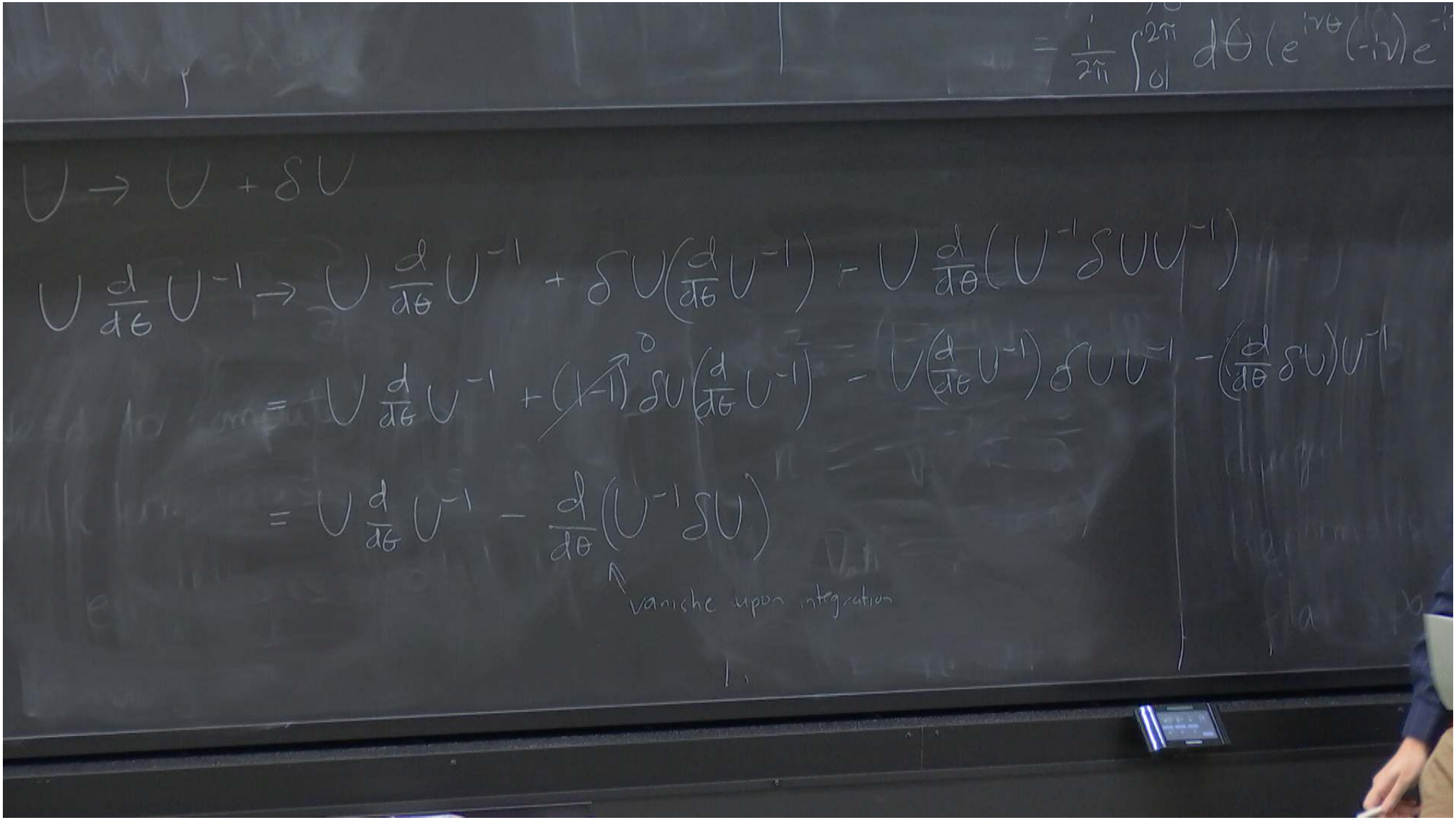




any map $f: S^1 \to S^1$ continuously deformable to exactly one of these maps

$$\nu = \frac{1}{2\pi i} \int_0^{2\pi} d\theta \left(U \frac{d}{d\theta} U^{-1} \right)$$

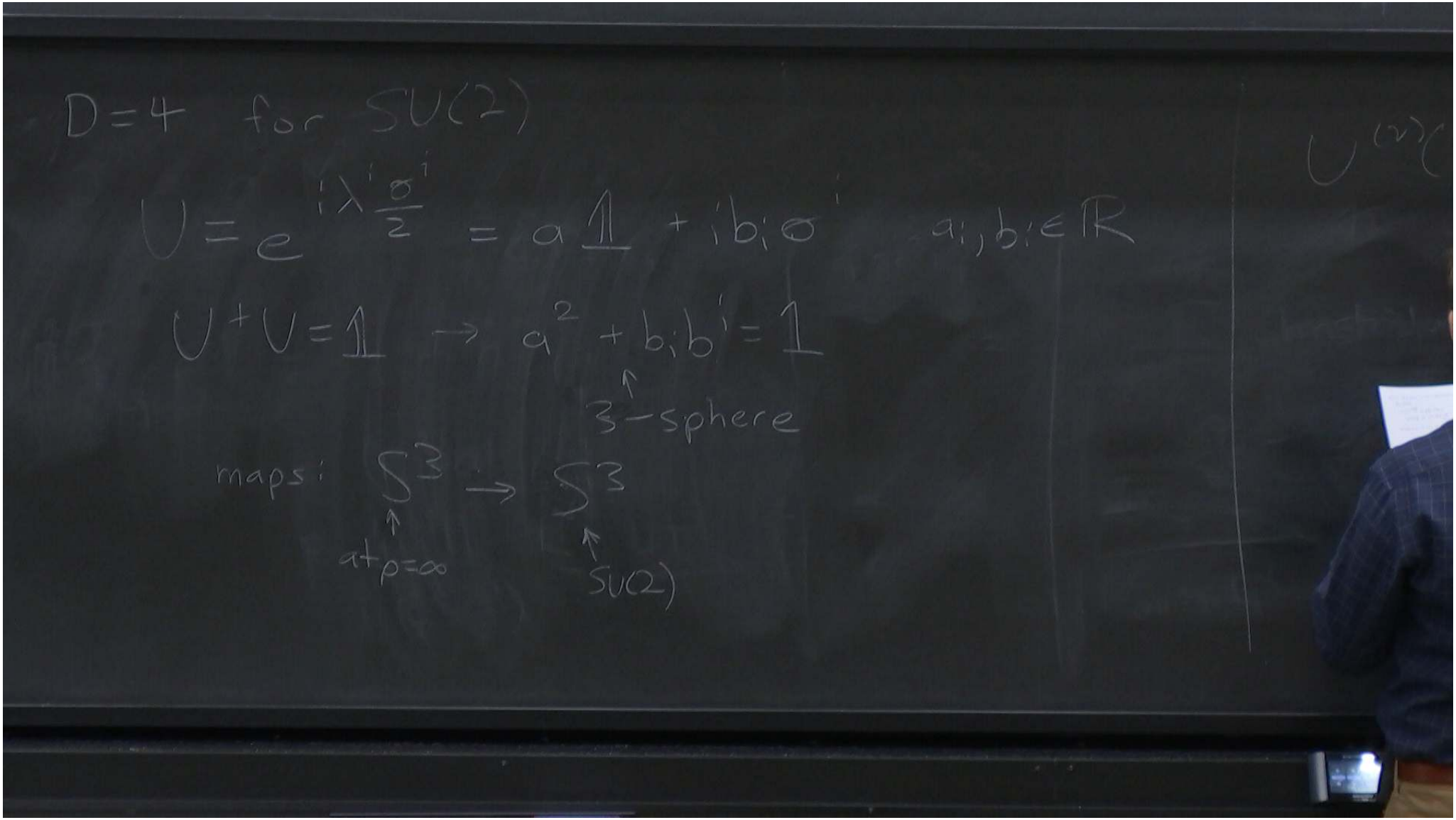
$$= \frac{1}{2\pi i} \int_0^{2\pi} d\theta \left(e^{i\nu\theta} (-i\nu) e^{-i\nu\theta} \right)$$


$= \frac{1}{2\pi i} \int_0^{2\pi i} d\theta \, (e^{i\nu\theta}(-i\nu) e^{-i}$
$U \to U + \delta U$
$U \frac{d}{d\theta} U^{-1} \to U \frac{d}{d\theta} U^{-1} + \delta U \left(\frac{d}{d\theta} U^{-1}\right) - U \frac{d}{d\theta}(U^{-1} \delta U U^{-1})$
$= U \frac{d}{d\theta} U^{-1} + (1-1)^{0} \delta U \left(\frac{d}{d\theta} U^{-1}\right) - \left(\frac{d}{d\theta} U^{-1}\right) \delta U U^{-1} - \left(\frac{d}{d\theta} \delta U\right) U^{-1}$
$= U \frac{d}{d\theta} U^{-1} - \frac{d}{d\theta}(U^{-1} \delta U)$
vanishe upon integration

$$U^{(\nu)}(\theta) = U^{(\nu_1)}(\theta)\, U^{(\nu_2)}(\theta)$$

$\nu = \nu_1 + \nu_2$ → instantons change winding number

$$\nu = \frac{i}{4\pi} \int d^2x \, \underbrace{\epsilon_{\mu\nu} F^{\mu\nu}}_{\text{chiral anomaly}}$$


D=4 for SU(2)
$U = e^{i\lambda^i \frac{\sigma^i}{2}} = a\,\mathbb{1} + i b_i \sigma^i \quad a_i, b_i \in \mathbb{R}$
$U^+ U = \mathbb{1} \rightarrow a^2 + b_i b^i = 1$
3-sphere
maps: $S^3 \rightarrow S^3$
at $\rho = \infty$
SU(2)

$U^{(\nu)}(\hat{x}) = (\hat{x} + i\hat{x}\cdot\vec{\sigma})^{\nu}$ standard maps

$$\nu = -\frac{1}{24\pi^2}\int d\Omega\, \mathrm{Tr}\left[\epsilon^{ijk}(U\partial_i U^{-1})(U\partial_j U^{-1})(U\partial_k U^{-1})\right]$$

$$= \frac{1}{16\pi^2}\int d^4x\, \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\,F^{\mu\nu}F^{\rho\sigma}$$

chiral anomaly in $D=4$

$U(1)$ no analog of winding number

$f: S^3 \to S^1$ all deformable to trivial map

Thm: $f: S^3 \to G$ deformable to $f: S^3 \to S^3$

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{-in\theta} |n\rangle$$

$|\theta\rangle$: energy eigenstate

$|n\rangle$: definite ν states

$$\lim_{T\to\infty} \langle\theta'| e^{-HT} |\theta\rangle = \lim_{T\to\infty} \sum_{n,m} \langle n| e^{-HT} |m\rangle e^{n\theta' - m\theta}$$

$$= \lim_{T\to\infty} \sum_{n} e^{in(\theta'-\theta)} \sum_{\nu} \int [DA]\, e^{i\nu\theta} e^{-S[A]}$$

$\nu = n - m$

$\int [DA]\, e^{i\nu\theta} e^{-S[A]}$: extra term

$\sum_n e^{in(\theta'-\theta)} = \delta(\theta'-\theta)$