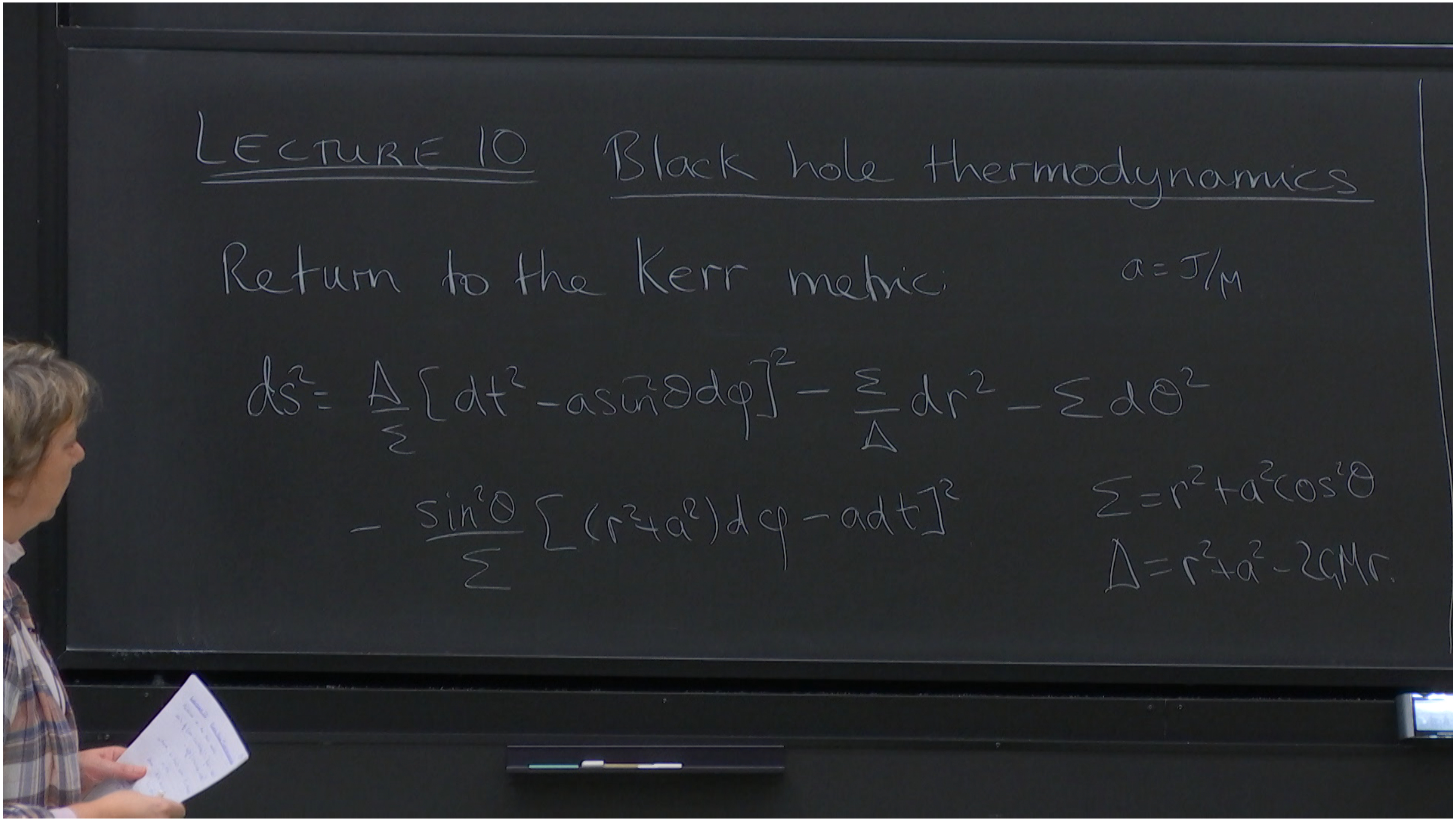
Title: PSI 2019/2020 - Gravitational Physics - Lecture 10

Speakers: Ruth Gregory

Collection: PSI 2019/2020 - Gravitational Physics

Date: January 17, 2020 - 10:15 AM

URL: http://pirsa.org/20010054


LECTURE 10
Black hole thermodynamics
Return to the Kerr metric:
$a = J/M$
$ds^2 = \frac{\Delta}{\Sigma}[dt^2 - a\sin^2\theta d\varphi]^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2$
$- \frac{\sin^2\theta}{\Sigma}[(r^2+a^2)d\varphi - a dt]^2$
$\Sigma = r^2 + a^2\cos^2\theta$
$\Delta = r^2 + a^2 - 2GMr.$

ics

Area $\mathcal{A} = 4\pi(r_+^2 + a^2)$

$$r_+ = GM + \sqrt{G^2M^2 - a^2}$$

$\cos^2\theta$

$-2GMr.$

Area $\mathcal{A} = 4\pi(r_+^2 + a^2)$

$$r_+ = GM + \sqrt{G^2M^2 - a^2}$$

What happens if we throw a particle into black hole?

Adding δm, δa - Area thm tells us $\delta \mathcal{A} \geq 0$

Area $\mathcal{A} = 4\pi(r_+^2 + a^2)$

$$r_+ = GM + \sqrt{G^2M^2 - a^2}$$

What happens if we throw a particle into black hole?

Adding δm, δa - Area thm tells us $\delta\mathcal{A} \geq 0$

$$\delta\mathcal{A} = 8\pi(r_+\delta r_+ + a\delta a)$$

$$= 8\pi r_+\left[\delta(GM) + \frac{GM\delta(GM) - a\delta a}{r_+ - GM}\right]$$

$$+ 8\pi a\delta a$$

$$= \frac{8\pi r_+^2 G\delta M}{r_+ - GM} - \frac{8\pi GMa\delta a}{r_+ - GM}$$

Now, $a = J/M$

$$\delta a = \frac{\delta J}{M} - \frac{J\,\delta M}{M^2}$$

& define $\Omega = \dfrac{a}{r_+^2 + a^2}$ angular velocity of horizon.

$$\delta A = \frac{8\pi}{r_+ - GM}\left[r_+^2\, G\,\delta M - \frac{G\cancel{M}a\,\delta J}{\cancel{M}} + G\cancel{M}a\,\frac{\overset{aM}{J}\,\delta M}{M^2}\right]$$

$$= \frac{8\pi}{r_+ - GM}\left[(r_+^2 + a^2)\, G\,\delta M - (r_+^2 + a^2)\, G\,\Omega\,\delta J \right]$$

$$\Rightarrow \quad \delta M = \frac{(r_+ - GM)}{8\pi G (r_+^2 + a^2)}\,\delta A + \Omega\,\delta J.$$

$$\delta M - \frac{G\cancel{M}a\,\delta J}{\cancel{M}} + G\cancel{M}a\,\frac{\overset{aM}{\overset{\downarrow}{J}}\,\delta M}{M^2}\Big]$$

$$a^2)\,G\,\delta M - (r_+^2 + a^2)\,G\,\Omega\,\delta J\Big]$$

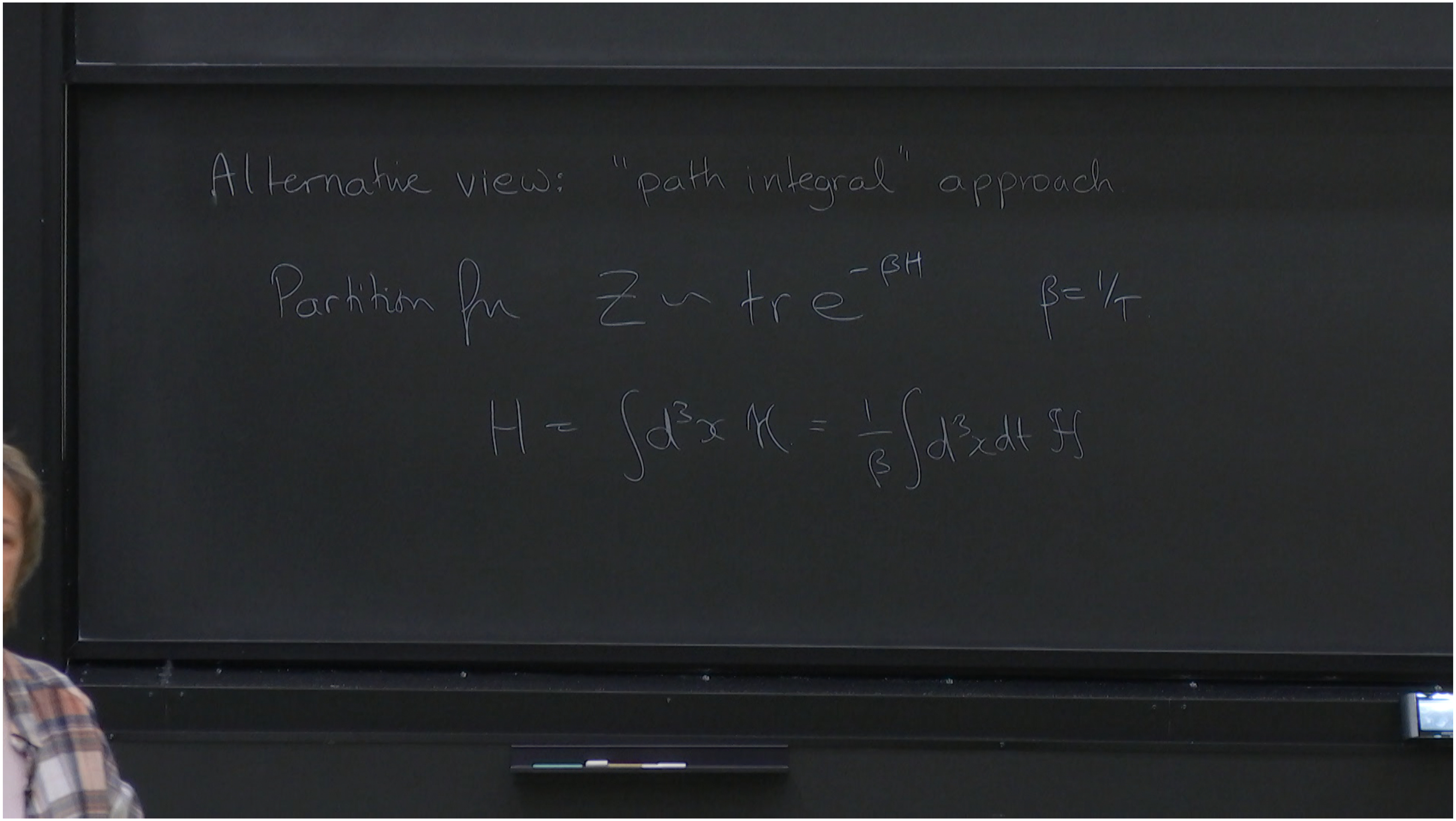
$$\delta A + \Omega\,\delta J.$$

cf. $dU = T\,dS - p\,dV + \mu_i\,dQ_i$

Suggests thermodynamic interpretation with

$$S = c\,A/G$$

$$T = \frac{(r_+ - GM)}{8\pi c\,(r_+^2 + a^2)}$$


Alternative view: "path integral" approach
Partition fn $Z \sim \mathrm{tr}\, e^{-\beta H}$ $\beta = 1/T$
$H = \int d^3x\, \mathcal{H} = \frac{1}{\beta}\int d^3x\, dt\, \mathcal{H}$

Alternative view: "path integral" approach

Partition fn $Z \sim \mathrm{tr}\, e^{-\beta H}$ $\beta = 1/T$

$$H = \int d^3x\, \mathcal{H} = \frac{1}{\beta}\int d^3x\, dt\, \mathcal{H}$$

Replace βH by I_E - Euclidean action.

Euclidean geometry?

$$t \to i\tau \qquad g_E = -g_L(i\tau)$$

$$\Rightarrow I_E = \int_{\mathcal{M}} \frac{R\sqrt{g}}{16\pi G}\, d^4x + \int_{\partial\mathcal{M}} \frac{K\sqrt{h}}{8\pi G}\, d^3x$$

Expect saddle points to dominate path integral, i.e. classical solns.

Euclidean geometry?

$t \to i\tau \qquad g_E = -g_L(i\tau)$

$$\Rightarrow I_E = \int_{\mathcal{M}} \frac{R\sqrt{g}}{16\pi G} d^4x + \int_{\partial\mathcal{M}} \frac{K\sqrt{h}}{8\pi G} d^3x$$

Expect saddle points to dominate path integral, i.e. classical solns with periodic τ.

$= -g_L(i\tau)$

$+ \int_{\partial M} \frac{K\sqrt{h}}{8\pi G} d^3x$

to dominate path

l solns with periodic τ.

Take SCH

$$ds^2 = \left(1 - \frac{2GM}{r}\right) d\tau^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 d\Omega_{II}^2$$

1st deal with β. Since $r = 2GM$ is nonsingular in Lorentzian geometry, it should be nonsing here.

Focus on $r \to 2GM$

$$\frac{dr^2}{1-\frac{2GM}{r}} \sim 8GM \, d\left(\sqrt{r-2GM}\right)^2$$

So look for a new coord

$$\rho^2 = \lambda(r-2GM)$$

Focus on $r \to 2GM$.

$$\frac{dr^2}{1-\frac{2GM}{r}} \sim 8GM\, d\left(\sqrt{r-2GM}\right)^2$$

So look for a new coord

$$\rho^2 = \lambda(r-2GM)$$

$$2\rho\, d\rho = \lambda\, dr.$$

$$4\lambda(r-2GM)\, d\rho^2 = \lambda^2 dr^2.$$

i.e. $\frac{2GM}{(r-2\ldots}$

$-2GM\big)^2$

rd

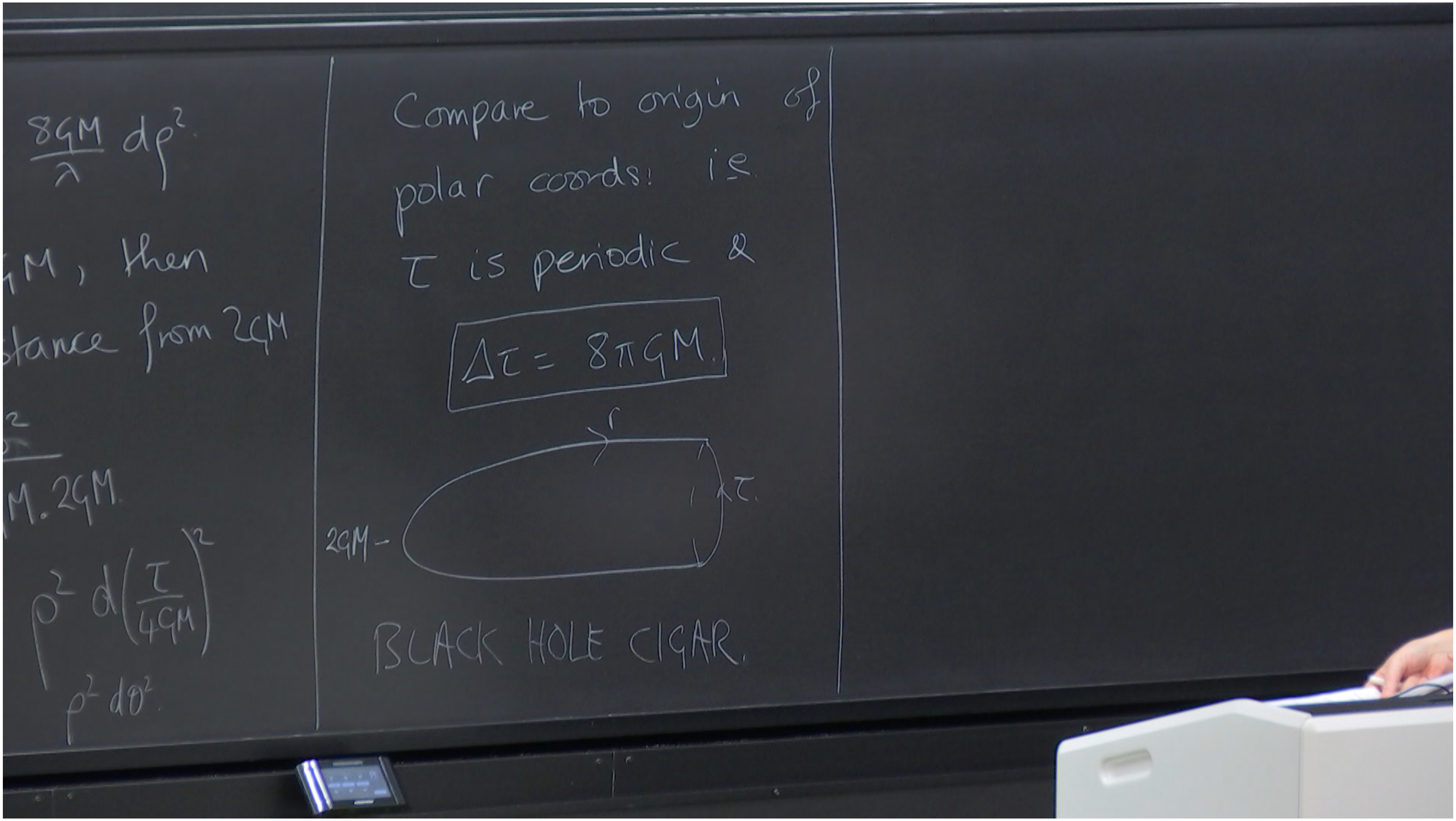
i.e. $\frac{2GM\,dr^2}{(r-2GM)} = \frac{8GM}{\lambda}\,d\rho^2$

So set $\lambda = 8GM$, then

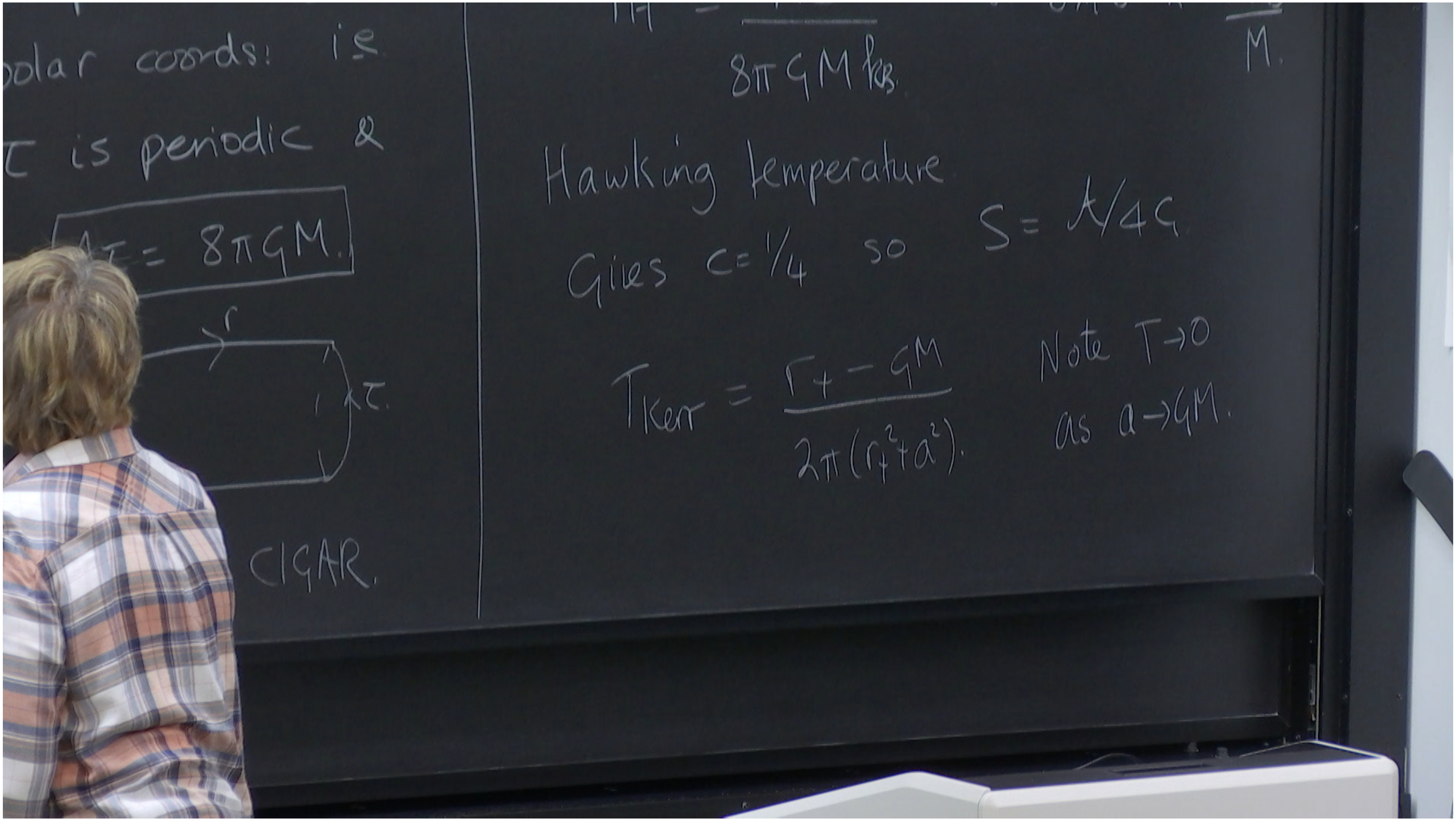
ρ is proper distance from $2GM$

$$\left(1-\frac{2GM}{r}\right) \sim \frac{\rho^2}{8GM \cdot 2GM}$$

$$\Rightarrow ds^2_{\tau,r} = d\rho^2 + \rho^2\, d\left(\frac{\tau}{4GM}\right)^2$$



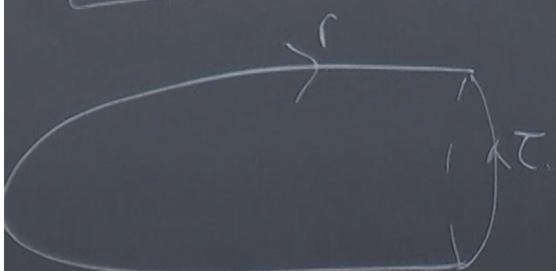
8GM/λ dρ²
GM, then
stance from 2GM
M.2GM
ρ² d(τ/4GM)²
ρ² dθ²
Compare to origin of
polar coords: i.e.
τ is periodic &
Δτ = 8πGM
r
τ
2GM
BLACK HOLE CIGAR



olar coords: ie.
τ is periodic &
= 8πGM.
CIGAR.
8πGM kB.
M.
Hawking temperature
Gives c = 1/4 so S = A/4G
TKerr = (r+ − GM) / 2π(r+² + a²).
Note T→0 as a→GM.

Compare to origin of
polar coords: i.e.
τ is periodic &

$$\boxed{\Delta\tau = 8\pi G M}$$



LACK HOLE CIGAR.

$$T_H = \frac{\hbar c^3}{8\pi G M k_B} \sim 6\times 10^{-8}\,K\,\frac{M_\odot}{M}.$$

Hawking temperature

Gives $c = 1/4$ so $S = A/4G$

$$T_{Kerr} = \frac{r_+ - GM}{2\pi(r_+^2 + a^2)}.$$

Note $T \to 0$ as $a \to GM$.

S from Z ?

$$S = \beta^2 \frac{\partial}{\partial \beta}\left[-\beta^{-1} \underbrace{\ln Z}_{I_E}\right]$$

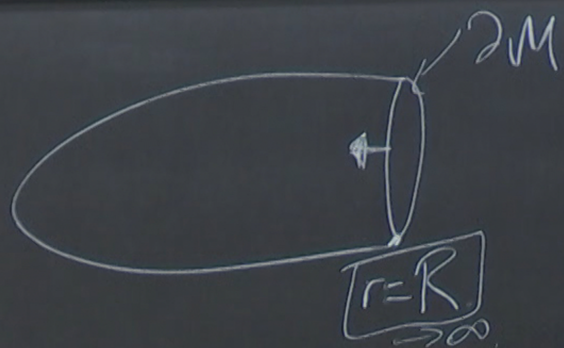
Need to compute I_{SCH}. Clearly bulk term vanishes as $R=0$, So everything is encoded in boundary term.

$\beta^{-1}\underbrace{\ln Z}_{I_E}]$

SCH. Clearly

s $R=0$,

ed in



$$ds^2_{\partial M} = \left(1-\frac{2GM}{R}\right)d\tau^2 + R^2 d\Omega_{II}^2$$

$$n = -\sqrt{1-\frac{2GM}{R}}\,\frac{\partial}{\partial r}.$$

$$K = D_a n^a = \frac{1}{r^2}\partial_r\left(r^2 n^r\right)$$

$$= \frac{2}{r} + \frac{GMn^r}{r(r-2GM)}.$$

$$\sqrt{h} = R^2 \sin\theta \sqrt{1-\frac{2GM}{R}}$$

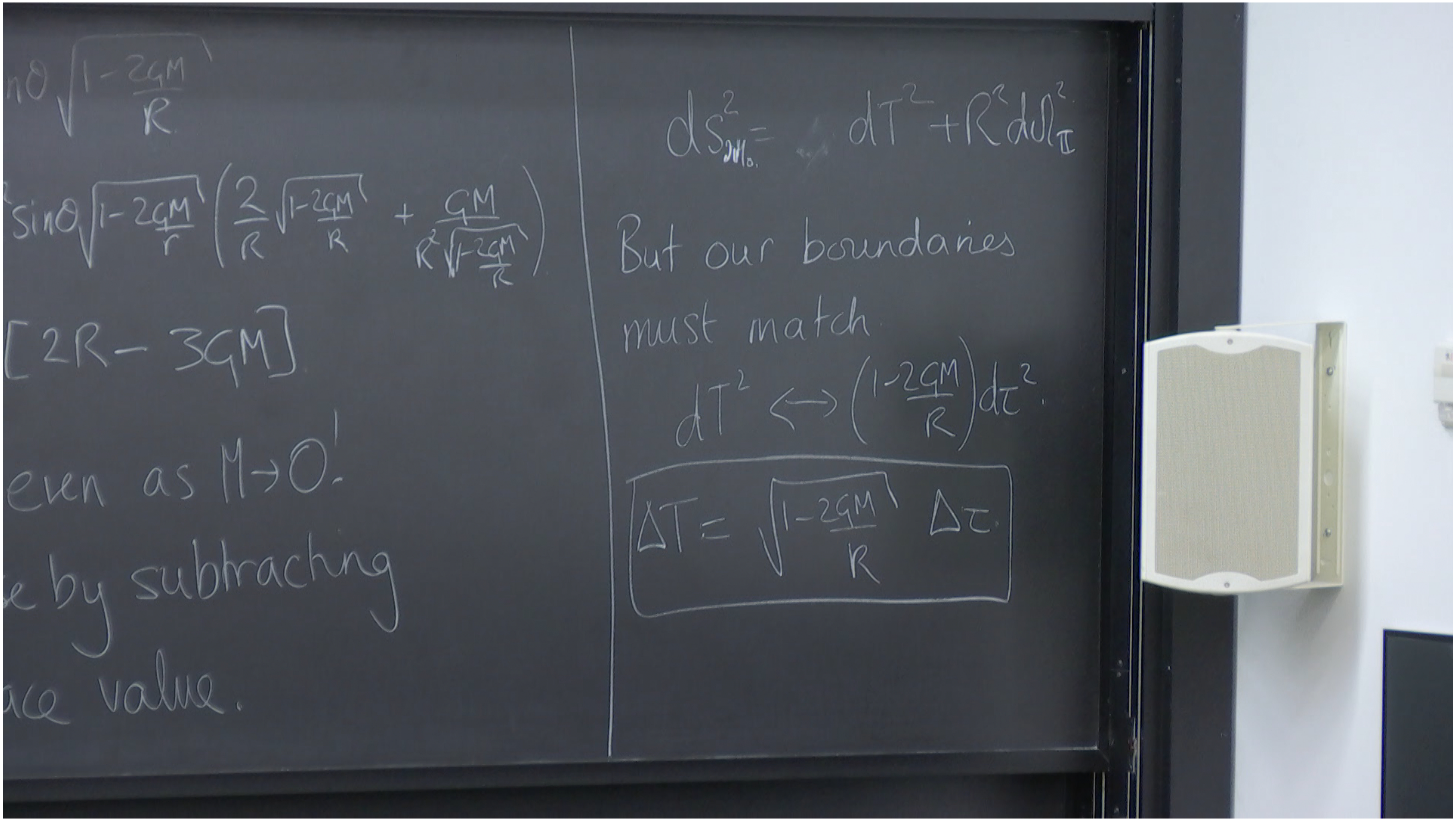
$$\sqrt{h} = R^2 \sin\theta \sqrt{1-\frac{2GM}{R}}$$

$$\int K\sqrt{h} = \int -R^2 \sin\theta \sqrt{1-\frac{2GM}{r}} \left(\frac{2}{R}\sqrt{1-\frac{2GM}{R}} + \frac{GM}{R^2\sqrt{1-\frac{2GM}{R}}} \right)$$

$$= -4\pi\beta \left[2R - 3GM \right]$$

divergent – even as $M \to 0$!

Renormalise by subtracting flat space value.



$n\theta \sqrt{1-\frac{2GM}{R}}$
$\sin\theta\sqrt{1-\frac{2GM}{r}}\left(\frac{2}{R}\sqrt{1-\frac{2GM}{R}}+\frac{GM}{R^2\sqrt{1-\frac{2GM}{R}}}\right)$
$[2R-3GM]$
even as $M\to 0$!
e by subtracting
ace value.
$ds^2_{\partial M_0} = dT^2 + R^2 d\Omega^2_{II}$
But our boundaries
must match.
$dT^2 \longleftrightarrow \left(1-\frac{2GM}{R}\right)d\tau^2$
$\Delta T = \sqrt{1-\frac{2GM}{R}}\,\Delta\tau$

Boundary term. $1.$ r $r(r-2GM)$.

$$n = -\frac{\partial}{\partial r} \Rightarrow K_0 = -\frac{2}{R}$$

$$\sqrt{h_0} = R^2 \sin\theta$$

$$\int K\sqrt{h_0} = -\frac{2}{R} \cdot 4\pi R^2 \cdot \beta_0$$

$$= -8\pi R \cdot \sqrt{1-\frac{2GM}{R}}\, \beta$$

$$= -8\pi\beta R\left[1-\frac{GM}{R}\right] + \cdots$$

$$= \frac{2r'}{r} + \frac{GMr'}{r(r-2GM)}.$$

flat space value.

Hence

$$I_{SCH} - I_0 = \frac{1}{8\pi G}\left[-4\pi\beta(2\cancel{R} - 3GM) + 8\pi\beta(\cancel{R} - GM) \right]$$

$$= \frac{\beta M}{2} \leftarrow \beta = 8\pi GM = 4\pi r_+$$

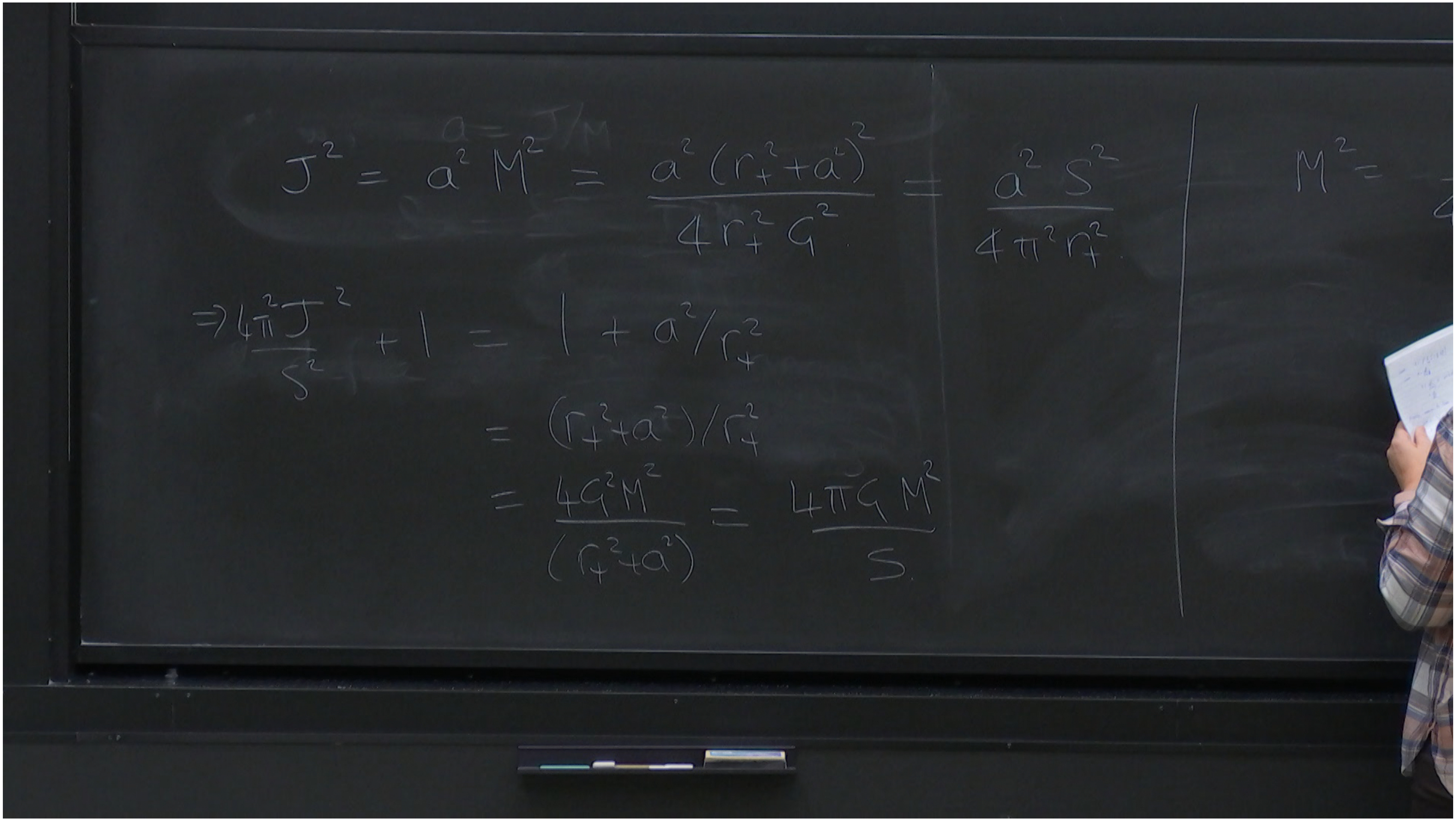
Check $S = \beta^2 \frac{\partial}{\partial\beta}\left[-\frac{1}{\cancel{\beta}} \cdot \frac{\cancel{\beta} M}{2} \right] = \frac{\beta^2}{16\pi G} = \frac{A}{4G}$

Finally, back to Kerr:

$$GM = \frac{r_+^2 + a^2}{2r_+} \qquad S = \frac{\pi(r_+^2 + a^2)}{G}$$

$$J = aM$$

$$\frac{A}{4G}$$


$J^2 = a^2 M^2 = \frac{a^2(r_+^2+a^2)^2}{4 r_+^2 G^2} = \frac{a^2 S^2}{4\pi^2 r_+^2}$
$M^2 =$
$\Rightarrow \frac{4\pi^2 J^2}{S^2} + 1 = 1 + a^2/r_+^2$
$= (r_+^2+a^2)/r_+^2$
$= \frac{4G^2M^2}{(r_+^2+a^2)} = \frac{4\pi G M^2}{S}$

$$\ldots)^2 = \frac{a^2 S^2}{4\pi^2 r_+^2}$$

$$\ldots\pi G M^2 / S$$

$$M^2 = \frac{S}{4\pi G}\left[1 + \frac{4\pi^2 J^2}{S^2}\right]$$

Christodoulou - Ruffini

$$\Omega = \left.\frac{\partial M}{\partial J}\right|_S = \frac{1}{2GM}\cdot\frac{2\pi J}{S}$$

$$= \frac{\cancel{\pi} a M}{\cancel{\pi}(r_+^2 + a^2)\cancel{M}}$$

$$\frac{S}{\ldots\pi G}\left[1+\frac{4\pi^2 J^2}{S^2}\right]$$

...odoulou - Ruffini

$$\left.\frac{\partial M}{\partial J}\right|_S = \frac{1}{2GM}\cdot\frac{2\pi J}{S} = \frac{\cancel{\pi} a \cancel{M}}{\cancel{\pi}(r_+^2+a^2)\cancel{M}} \quad \checkmark\checkmark$$

$$T = \left.\frac{\partial M}{\partial S}\right|_J = \frac{1}{8\pi GM}\left[1-\frac{4\pi^2 J^2}{S^2}\right]$$

$$= \frac{1}{8\pi GM}\left[1-\frac{a^2}{r_+^2}\right]$$

$$= \frac{1}{8\pi GM r_+^2}\left[r_+^2-(r_+^2-2GMr_+)\right]$$

$$= \frac{(r_+-GM)}{2\pi(r_+^2+a^2)} \quad \checkmark\checkmark$$

"Chemical" expressions for b.h. TD.