Title: PSI 2019/2020 - Gravitational Physics - Lecture 9

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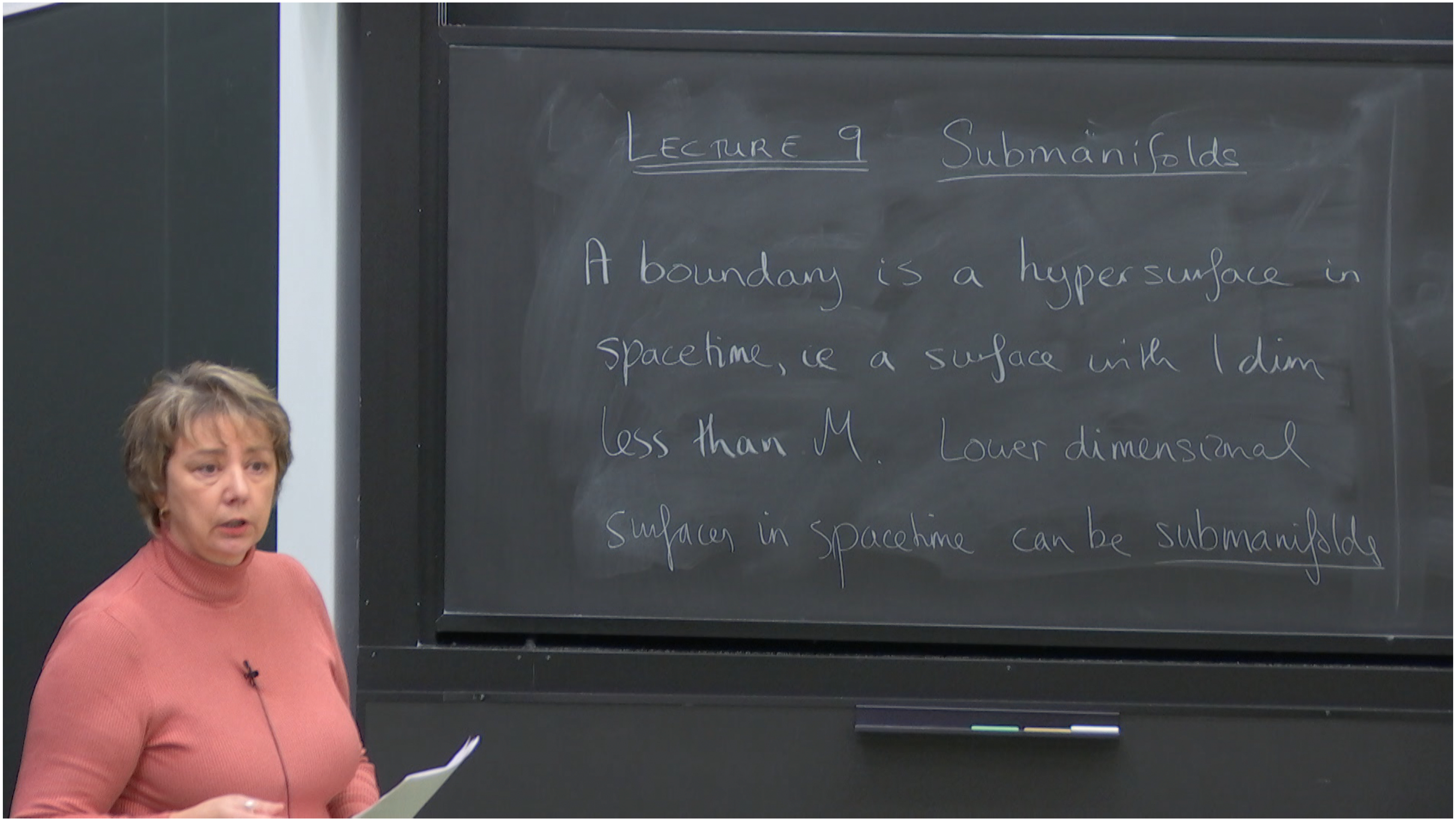
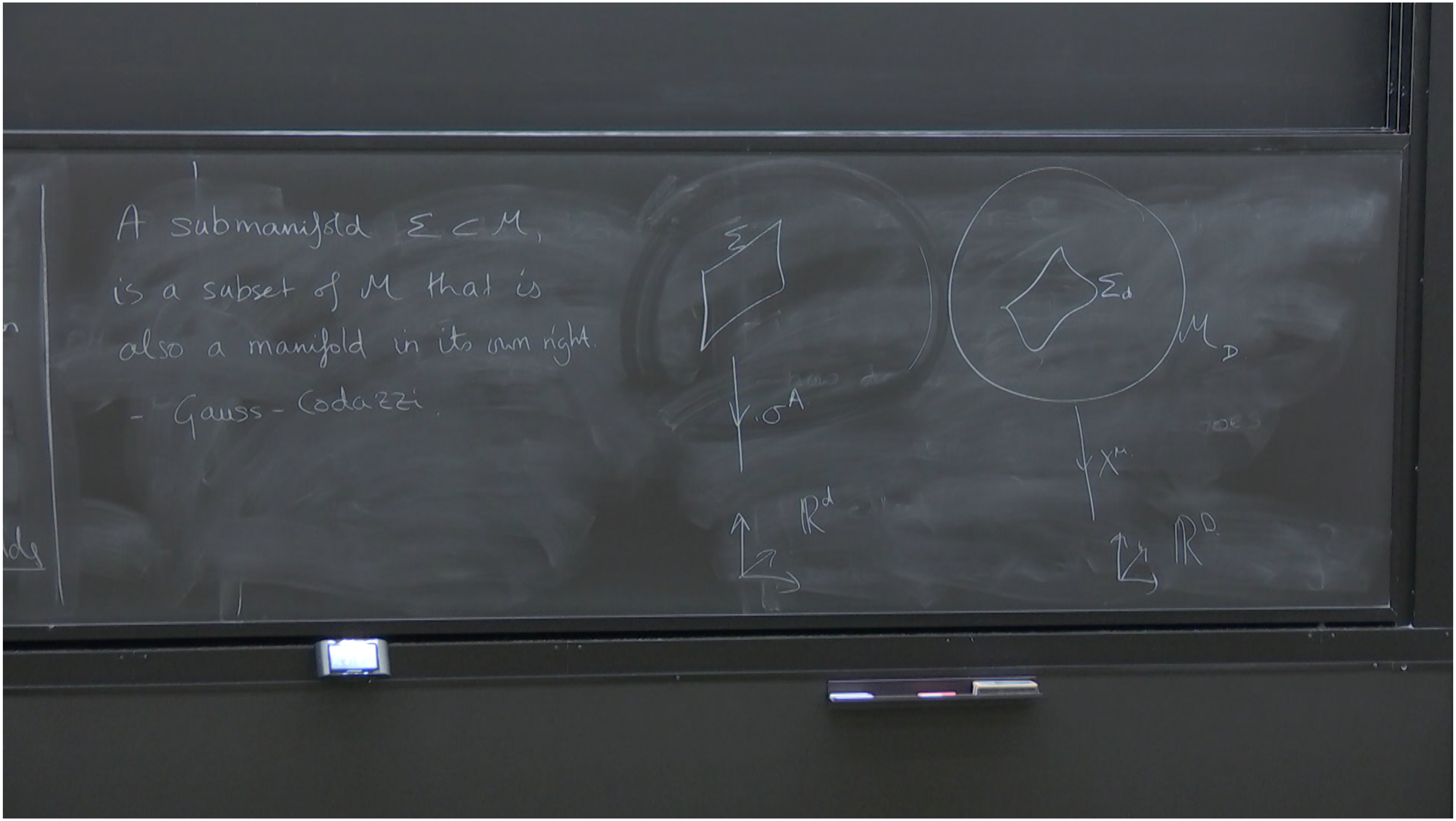
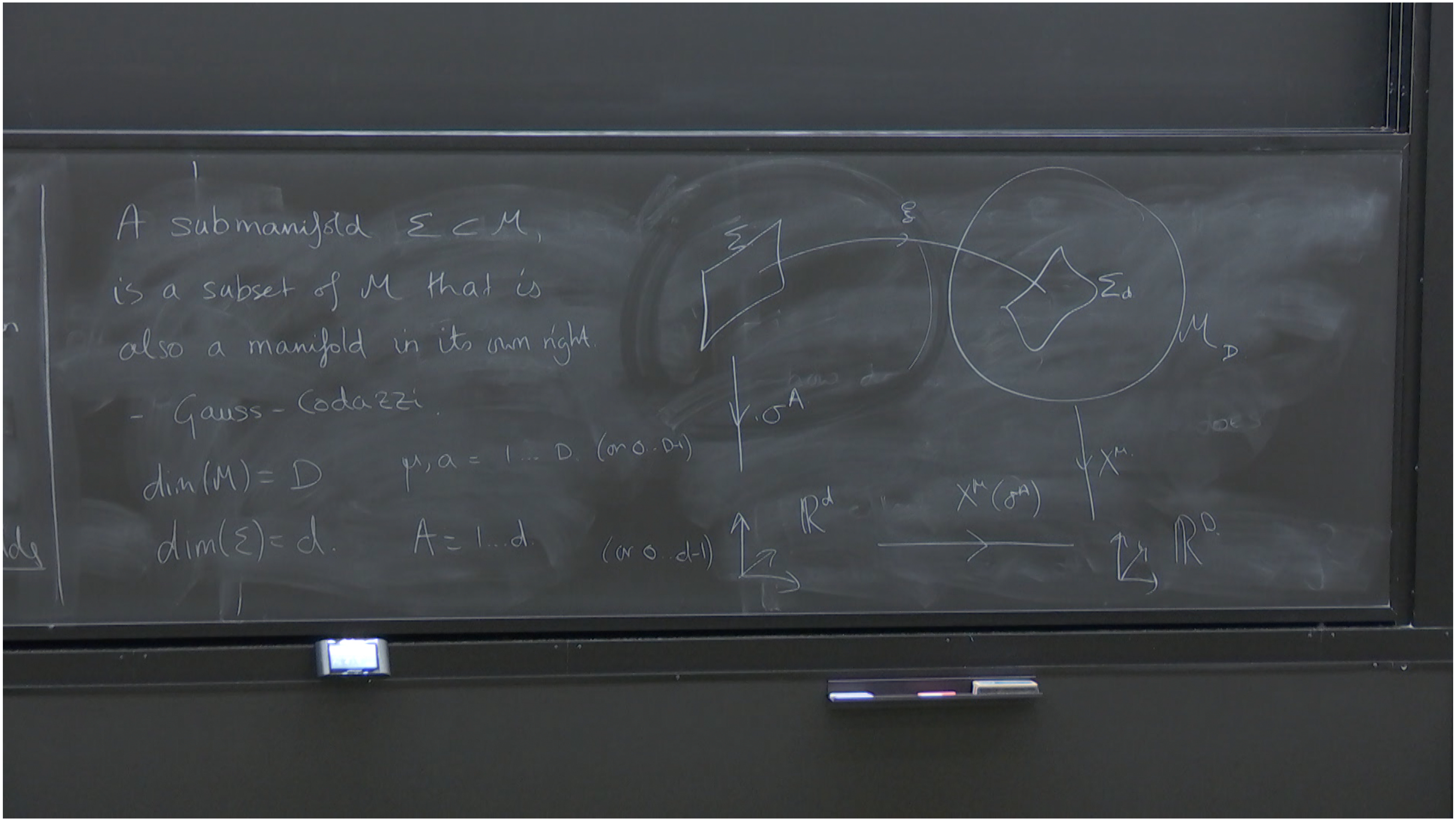
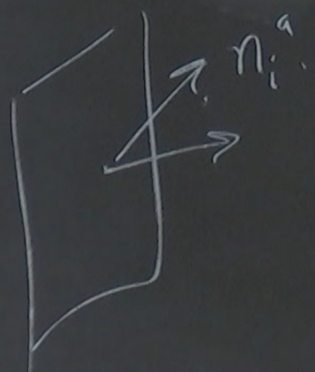
LECTURE 9 Submanifolds


LECTURE 9 Submanifolds
A boundary is a hypersurface in
spacetime, ie a surface with 1 dim
less than M. Lower dimensional
surfaces in spacetime can be submanifolds


A submanifold $\Sigma \subset \mathcal{M}$,
is a subset of $\mathcal{M}$ that is
also a manifold in its own right.
- Gauss - Codazzi
Σ
σ^A
$\mathbb{R}^d$
Σ_d
$\mathcal{M}_D$
X^μ
$\mathbb{R}^D$


A submanifold $\Sigma \subset M$,
is a subset of M that is
also a manifold in its own right.
- Gauss - Codazzi
$\dim(M) = D$
$\mu, a = 1 \ldots D$. (or $0 \ldots D-1$)
$\dim(\Sigma) = d$.
$A = 1 \ldots d$.
(or $0 \ldots d-1$)
Σ
ξ
Σ_d
M_D
σ^A
X^μ
$\mathbb{R}^d$
$X^\mu(\sigma^A)$
$\mathbb{R}^D$

<u>Defn</u> The <u>codimension</u> of Σ in M is $n = D - d$, & $\exists$ n lin. indep. normal vectors to Σ.



$$\underline{n}_i \in T_p(M) \qquad (p \in \Sigma)$$

$$\text{s.t.} \quad \underline{n}_i \, \sigma^A = 0 \quad \forall \, \sigma^A$$

Σ in M

lin. indep.

$(p \in \Sigma)$

σ^A

or $n_{;\mu} \frac{\partial X^\mu}{\partial \sigma^A} = 0$.

e.g. S^2: $x^2+y^2+z^2=1$

$X^\mu \leftrightarrow (x,y,z)$ $\sigma^A \leftrightarrow \theta, \varphi$

$n_\mu = \frac{\partial}{\partial r} = \frac{x^i}{r}\frac{\partial}{\partial x^i}$

$X^i(\sigma^A) = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$

$n_\mu \frac{\partial X^\mu}{\partial \sigma^A} =$

$A=\varphi$: sin

$= 0$

or $n_{i,\mu} \frac{\partial X^\mu}{\partial \sigma^A} = 0.$

e.g. S^2: $x^2+y^2+z^2=1$

$X^\mu \leftrightarrow (x,y,z) \qquad \sigma^A \leftrightarrow \theta, \varphi$

$$n_\mu = \frac{\partial}{\partial r} = \frac{x^i}{r}\frac{\partial}{\partial x^i}$$

$$X^i(\sigma^A) = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

$$n_\mu \frac{\partial X^\mu}{\partial \sigma^A} = \frac{x_i}{r}\frac{\partial x^i}{\partial \sigma^A}$$

$A=\varphi$: $\sin\theta\cos\varphi \frac{\partial}{\partial\varphi}\sin\theta\cos\varphi + \sin\theta\sin\varphi\frac{\partial}{\partial\varphi}\sin\theta\sin\varphi$

$= 0.$

or $n_{i,\mu} \dfrac{\partial X^\mu}{\partial \sigma^A} = 0.$

e.g. $S^2 : x^2 + y^2 + z^2 = 1$

$X^\mu \leftrightarrow (x, y, z) \qquad \sigma^A \leftrightarrow \theta, \phi$

$n_\mu = \dfrac{\partial}{\partial r} = \dfrac{x^i}{r} \dfrac{\partial}{\partial x^i}$

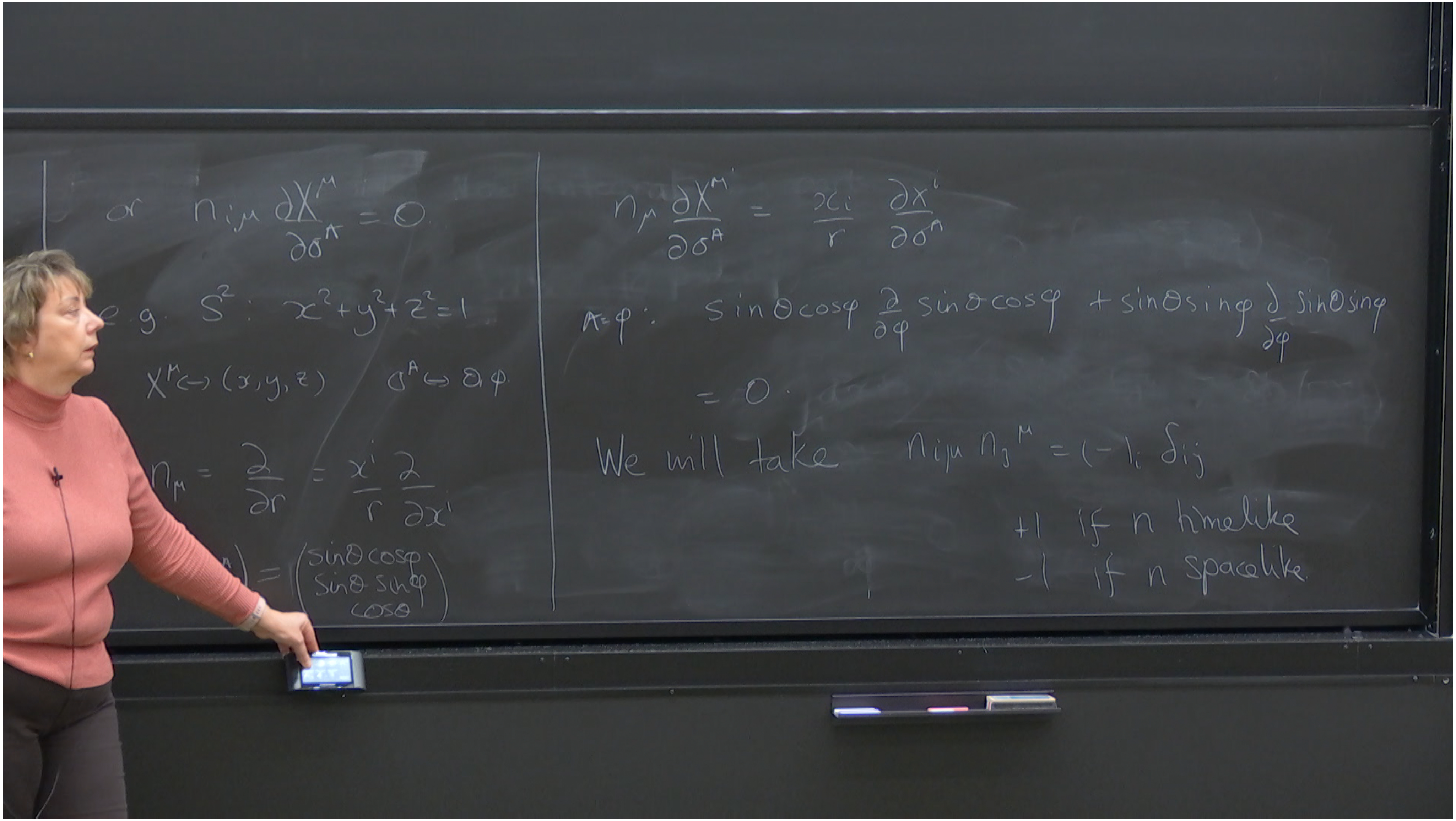
$X^i(\sigma^A) = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}$

$n_\mu \dfrac{\partial X^\mu}{\partial \sigma^A} = \dfrac{x_i}{r} \dfrac{\partial x^i}{\partial \sigma^A}$

$A = \varphi$: $\sin\theta\cos\varphi \dfrac{\partial}{\partial\varphi} \sin\theta\cos\varphi + \sin\theta\sin\varphi \dfrac{\partial}{\partial\varphi} \sin\theta\sin\varphi$

$= 0.$

We will take $n_{i\mu} n_j{}^\mu = (-1)_i \delta_{ij}$


or $n_{i,\mu} \frac{\partial X^\mu}{\partial \sigma^A} = 0$.
e.g. S^2: $x^2+y^2+z^2=1$
$X^\mu \leftrightarrow (x,y,z)$ $\sigma^A \leftrightarrow \theta, \phi$
$n_\mu = \frac{\partial}{\partial r} = \frac{x^i}{r}\frac{\partial}{\partial x^i}$
$= \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$
$n_\mu \frac{\partial X^\mu}{\partial \sigma^A} = \frac{x_i}{r} \frac{\partial x^i}{\partial \sigma^A}$
$A=\varphi$: $\sin\theta\cos\varphi \frac{\partial}{\partial\varphi} \sin\theta\cos\varphi + \sin\theta\sin\varphi \frac{\partial}{\partial\varphi} \sin\theta\sin\varphi$
$= 0$.
We will take $n_{i\mu} n_j^{\ \mu} = (-1)_i \delta_{ij}$
+1 if n timelike
-1 if n spacelike

WLOG take n_i^a to be spacelike

Defn The 1st fundamental form or induced metric of Σ is

$$h_{ab} = g_{ab} + n_{ia} n_{ib}$$

WLOG take n_i^a to be spacelike.

Defn The 1st fundamental form or induced metric of Σ is

$$h_{ab} = g_{ab} + n_{ia} n_{ib}$$

& is the metric Σ inherits from $\mathcal{M}$.

$X^\mu(\sigma^A) = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$

celike

l form or

is

n_{ib}

s from $\mathcal{M}$.

$+ n_{ia} n_{ib}$ kills off part of g

normal to Σ. Note $h_{ab} \in T_p^*(\mathcal{M}) \otimes T_p^*(\mathcal{M})$

Σ is also a manifold, so we have a pull-back: $\varphi^*: T_p^*(\mathcal{M}) \to T_p^*(\Sigma)$.

$$\omega_\mu \longmapsto \omega_A = \frac{\partial X^\mu}{\partial \sigma^A} \omega_\mu$$

Pull back metric: $\gamma_{AB} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X^\nu}{\partial \sigma^B}$ INTRINSIC METRIC

$\otimes T_p^*(\mathcal{M})$

we

$T_p^*(\Sigma)$.

$\omega_A = \frac{\partial X^\mu}{\partial \sigma^A} \omega_\mu$

$\frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X^\nu}{\partial \sigma^B}$ INTRINSIC METRIC

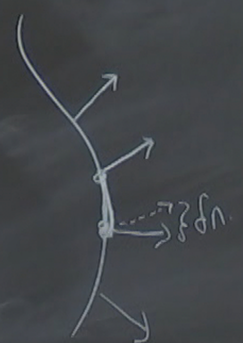
e.g. worldline

$$\gamma_{\tau\tau} = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}.$$

$$= \left(\frac{ds}{d\tau}\right)^2$$

$X^\mu(\tau)$

Defn The <u>2nd fundamental form</u> or <u>extrinsic curvature</u> of Σ measures how Σ curves in $\mathcal{M}$.



$$K_{i\mu\nu} = h_\mu{}^\sigma h_\nu{}^\lambda \nabla_\sigma n_{i\lambda}$$

or
$$K_{iAB} = X^\mu_{,A} X^\nu_{,B} \nabla_\mu n_{i\nu}.$$
$$= - n_{i\mu} D_A\left(\frac{\partial X^\mu}{\partial \sigma^B}\right)$$

where D_A is the connection inherited by Σ:

$$D_\mu V_\nu = h_\mu{}^\lambda h_\nu{}^\sigma \nabla_\lambda V_\sigma$$

$$(V^\nu n_\nu = 0)$$

e.g. Cylinder: $\{x^2+y^2=a^2\} \subset \mathbb{R}^3$

$$\underline{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

$$h_{ab} = \delta_{ab} - n_a n_b = \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$K_{ab} = \Gamma^r_{ab}$ in cyl polars.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3 \times 3 \text{ matrix})$$

Intrinsic coords: $\{\theta, z\}$.

$$\gamma_{AB} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad K_{AB} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

Defn the normal fundamental forms

$$\beta_{\mu ij} = n_{i\nu} \nabla_\mu n_j^{\nu}.$$

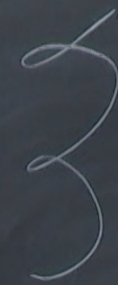
are the connection on the normal bundle of Σ.

damental forms

n_j^{ν}.

normal bundle

e.g. helix



$$(\cos\theta, \sin\theta, \theta)$$

$$\dot{X}^a = (-\sin\theta, \cos\theta, 1)$$

$$n_{1a} = (\cos\theta, \sin\theta, 0)$$

$$n_{2a} = (-\sin\theta, \cos\theta, -1)$$

$\cos\theta = x/\rho$

$\sin\theta = y/\rho$

Defn the normal fundamental forms

$$\beta_{\mu ij} = n_{i\nu} \nabla_\mu n_j^{\,\nu}.$$

are the connection on the normal bundle of Σ.

$$\beta_{2i} = \cos\theta(-\sin\theta)_{,i} + \sin\theta(\cos\theta)_{,i}$$

$$= \begin{pmatrix} \frac{x}{r}\left(\frac{yx}{r^3}\right) + \frac{y}{r}\left(\frac{1}{r} - \frac{x^2}{r^3}\right) \\ \frac{x}{r}\left(-\frac{1}{r} + \frac{y^2}{r^3}\right) + \frac{y}{r}\left(-\frac{xy}{r^3}\right) \end{pmatrix}.$$

e.g. helix

$$(\cos\theta, \sin\theta, \theta)$$

$$\dot{X}^a = (-\sin\theta, \cos\theta, 1)$$

$$n_a = (\cos\theta, \sin\theta, 0)$$

$\cos\theta = x/\rho$

$\sin\theta = y/\rho$

$$n_{2a} = (-\sin\theta, \cos\theta, -1)$$

Defn the <u>normal fundamental forms</u>

$$\beta_{\mu ij} = n_{i\nu} \nabla_\mu n_j^{\ \nu}.$$

are the connection on the normal bundle of Σ.

$$\boxed{\begin{aligned}\beta_{i} &= \cos\theta(-\sin\theta)_{,i} + \sin\theta(\cos\theta)_{,i} \\ &= \begin{pmatrix} \frac{x}{r}\left(\frac{yx}{r^3}\right) + \frac{y}{r}\left(\frac{1}{r} - \frac{x^2}{r^3}\right) \\ \frac{x}{r}\left(-\frac{1}{r} + \frac{y^2}{r^3}\right) + \frac{y}{r}\left(-\frac{xy}{r^3}\right) \end{pmatrix}\end{aligned}}$$

$$= \begin{pmatrix} y/r^2 \\ x/r^2 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}.$$

e.g. helix

$$(\cos\theta, \sin\theta, \theta)$$

$$\dot{X}^a = (-\sin\theta, \cos\theta, 1)$$

$$n_a = (\cos\theta, \sin\theta, 0)$$

$$n_{2a} = (-\sin\theta, \cos\theta, -1)$$

$\cos\theta = x/\rho$

$\sin\theta = y/\rho$

We can decompose the curvature on Σ as a combination of ambient manifold curvature & embedding curvature.

LECTURE 9 Submanifolds

Boundary term: $-\frac{1}{16\pi G}\int d^3x\sqrt{g}\; n_a\left(\nabla^a \delta g^{-1} - \nabla_b \delta g^{ab}\right)$

Now take $\Sigma = \partial M$, codim 1

n_a extends geodesically into bulk (M).

$n_\mu n^\mu = -1$

$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$

$\nabla_n n^\mu = 0$

$g_{\mu\nu} + n_\mu n_\nu$

$n^\mu = 0$

Then

$$\delta(n^\mu n^\nu g_{\mu\nu}) = 0 \Rightarrow 2 n_\mu \delta n^\mu = n_\mu n_\nu \delta g^{\mu\nu}$$

& $$\delta(n^\mu h_{\mu\nu}) = 0 \Rightarrow h_{\mu\nu} \delta n^\mu = n_\mu \delta h^{\mu\sigma} g_{\sigma\nu}$$

but $$h^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu$$

Then

$$\delta(n^\mu n^\nu g_{\mu\nu}) = 0 \Rightarrow 2 n_\mu \delta n^\mu = n_\mu n_\nu \delta g^{\mu\nu}$$

$$\& \quad \delta(n^\mu h_{\mu\nu}) = 0 \Rightarrow h_{\mu\nu} \delta n^\mu = n_\mu \delta h^{\mu\sigma} g_{\sigma\nu}$$

does

$$\text{but} \quad h^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu$$

$$\Rightarrow (g_{\mu\nu} + \cancel{n_\mu n_\nu}) \delta n^\mu = n_\mu (\delta g^{\mu\sigma} + \cancel{\delta n^\mu n^\sigma} + n^\mu \delta n^\sigma) g_{\sigma\nu}$$

$\partial x'$

$$\begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}$$

+1 if n timelike
-1 if n spacelike

Then

$$\delta(n^\mu n^\nu g_{\mu\nu}) = 0 \Rightarrow 2 n_\mu \delta n^\mu = n_\mu n_\nu \delta g^{\mu\nu}$$

& $$\delta(n^\mu h_{\mu\nu}) = 0 \Rightarrow h_{\mu\nu} \delta n^\mu = n_\mu \delta h^{\mu\sigma} g_{\sigma\nu}$$

but $$h^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu$$

$$\Rightarrow (g_{\mu\nu} + \cancel{n_\mu n_\nu}) \delta n^\mu = n_\mu (\delta g^{\mu\sigma} + \cancel{\delta n^\mu n^\sigma} + n^\mu \delta n^\sigma) g_{\sigma\nu}.$$

$$\delta n^\mu = \frac{1}{2} n_\nu \delta g^{\mu\nu}.$$

Want to add boundary term to action to cancel integration by parts:

$$\delta K = \delta(\nabla_\mu n^\mu) = \nabla_\mu \delta n^\mu + \delta\Gamma^\mu_{\mu\lambda} n^\lambda$$

$$= \frac{1}{2}\nabla_\mu(n_\nu \delta g^{\mu\nu}) - \frac{1}{2} n^\lambda \nabla_\lambda \delta g$$

$$= \frac{1}{2}\left[n_\nu \nabla_\mu \delta g^{\mu\nu} - \nabla_n \delta g\right] + \frac{1}{2} K_{\mu\nu}\delta g^{\mu\nu}$$

Want to add boundary term to action to cancel integration by parts:

$$\delta K = \delta(\nabla_\mu n^\mu) = \nabla_\mu \delta n^\mu + \delta \Gamma^\mu_{\mu\lambda} n^\lambda$$

$$= \frac{1}{2} \nabla_\mu (n_\nu \delta g^{\mu\nu}) - \frac{1}{2} n^\lambda \nabla_\lambda \delta g^{-1}$$

$$= \underbrace{\frac{1}{2}\left[n_\nu \nabla_\mu \delta g^{\mu\nu} - \nabla_n \delta g^{-1} \right]}_{\text{Boundary}} + \underbrace{\frac{1}{2} K_{\mu\nu} \delta g^{\mu\nu}}_{\text{E-L term}}$$

dary
term

$\delta g^{\mu\nu}$

Take $\int \sqrt{h}\, K \, d^3x$, then $\sqrt{g} = \sqrt{h}$. $|n^2| = 1$.

Then $\delta \int \sqrt{h}\, K\, d^3x = \frac{1}{2} \int \Big[\sqrt{h} \left[n_a \nabla_b \delta g^{ab} - \nabla_n \delta g^{-1} \right]$

$+ \sqrt{h} \left(K_{ab} - \frac{1}{2} K h_{ab} \right) \delta g^{ab} \Big] d^3x$

dary
term

$\delta g^{\mu\nu}$

Take $\int \sqrt{h}\, K \, d^3x$, then $\sqrt{g} = \sqrt{h}$. $|n^2| = 1$.

$$\text{then } \delta \int \sqrt{h}\, K \, d^3x = \frac{1}{2} \int \Big[\sqrt{h} \left[n_a \nabla_b \delta g^{ab} - \nabla_n \delta g^{-1} \right] + \sqrt{h} \left(K_{ab} - \frac{1}{2} K h_{ab} \right) \delta g^{ab} \Big] d^3x$$

So add $S_{GH} = -\frac{1}{8\pi G} \int_{\partial M} K \sqrt{h}\, d^3x$ to action.

GIBBONS-HAWKING

If only interested in bulk ∂M non-dynamical,

$\delta g^{ab} = 0$ on ∂M & $\delta(S_{EH} + S_{GH}) = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\, G_{ab}\,\delta g^{ab}$

Could have dynamical boundary - get additional boundary e.o.m.

$$K_{ab} - \frac{1}{2} K h_{ab} = 8\pi G\, S_{ab}$$

boundary energy momentum

If only interested in bulk ∂M non-dynamical,

$\delta g^{ab} = 0$ on ∂M & $\delta(S_{EH} + S_{GH}) = -\frac{1}{16\pi G}\int d^4x \sqrt{g}\, G_{ab}\, \delta g^{ab}$

Could have dynamical boundary: - get additional boundary e.o.m

$$K_{ab} - \frac{1}{2} K h_{ab} = 8\pi G S_{ab}$$

boundary energy momentum.

→ ISRAEL Junction Cons.